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introduction to fibre bundles

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Introduction to FIBRE BUNDLES

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PREFACE

This is a set of lecture notes for a graduate course given at Brown University. The purpose in writing these notes was to bring together some basic material on fibre bundles previously scattered in books and papers.

The first section of Chapter 1 contains the definitions of the classical groups and the definition of homogeneous space. In sections 2 and 3, cellular decompositions of the groups are given and the Pontryagin rings are calculated. The results of sections 2 and 3 are not used until the last section of Chapter 3, so the reader can, if he chooses, proceed from section 1 directly into Chapter 2. The material in Chapter 1 can be found in [Steenrod 1] for an application of the results of Chapter 1 to the existence of vector fields on spheres.

The material in Chapter 2 and in the first three sections of Chapter 3 can be found in [Steenrod 2] and [Husemoller]. The final two sections of Chapter 3 contain a sketch of Milgram's construction of the universal classifying space of a topological group. The reader is referred to [Steenrod 3] and [Milgram] for additional details.

I want to express my appreciation to the Mathematics Department at Brown University for the opportunity to teach the course and for their support in having the notes typed.

Richard D. Porter

CONTENTS

Chapter 1

THE CLASSIC GROUPS

1. Homogeneous Spaces and the Classic Groups

Definition of Quaternions. The algebra Q of quaternions is the four dimensional vector space over the Reals with basis {1,i,j,k} with multiplication defined by setting.

$$i^2 = j^2 = k^2 = -1$$
$$ij = -ji = k$$

by making 1 a two sided unit and by requiring that the multiplication be associative. The conjugate of the quaternion $x = a + ib + jc + kd$ is $\bar{x} = a - jb - jc - kd$ and the norm of a quaternion x is $\bar{x}x$, denoted by $||x||$.

It is easily seen that the quaternions satisfy:

$$ik = -ki = -j$$
$$jk = -kj = i$$
$$||x|| = a^2 + b^2 + c^2 + d^2 \quad \text{so} \quad \bar{x}\cdot x = x\cdot\bar{x}$$
$$||x|| = 0 \quad \text{iff} \quad x = 0$$
$$\text{if } x \neq 0, \text{ then } \frac{\bar{x}}{||x||} \text{ is a two sided inverse of } x$$
$$\text{and } ||x||\cdot||y|| = ||xy||.$$

The Quaternions form an associative (but not commutative) algebra of dimension 4 over the reals. Each nonzero Quaternion has a two sided inverse.

The Quaternions can also be viewed as a two dimensional vector space over the complex numbers $\mathbb{C}$ with basis $\{1,j\}$, multiplication defined by requiring associativity, by requiring that 1 be a two sided unit and by setting $xj = j\bar{x}$ $(x \in C)$. The correspondence between these two points of view is given by $(a+ib) + j(c-id)$ $= a + ib + jc + kd$. If $q = x + jy$, then $\bar{q} = \bar{x} - jy$.

<u>Notation</u>. If $d = 1,2,$ or 4, then $F_d = \begin{cases} \text{Reals if } d = 1 \\ \text{Complexes if } d = 2 \\ \text{Quaternions if } d = 4 \end{cases}$

F will be used to denote F_1, F_2 or F_4. F^n denotes the n-dimensional vector space over F consisting of column vectors with entries in F.

<u>Definition</u>. If $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ are in F^n, the scalar product $\langle x,y \rangle$ is defined by $\langle x,y \rangle = \sum_{i=1}^{n} \bar{x}_i y_i$. The norm of x, $||x||$, is defined by $||x|| = \langle x,x \rangle$.

Direct calculation shows that $||x||$ is a real number and $||x|| = 0$ iff $x = 0$. If $||x|| = 1$, then x is called a unit vector in F^n. The space of unit vectors in F^n is the sphere, S^{nd-1}.

The scalar product satisfies

1. $\langle x, y_1+y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$
2. $\langle x_1+x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
4. $\langle x, y\lambda \rangle = \langle x, y \rangle \lambda, \ \lambda \in F$
5. $\langle x\lambda, y \rangle = \bar{\lambda} \langle x, y \rangle, \ \lambda \in F$ using $\overline{(xy)} = \bar{y} \cdot \bar{x}$, x and y in F.

Definition. $G(n)$ denotes the set of all linear transformations L of F^n into F^n which preserve scalar products. That is
$L \in G(n)$ iff $\langle Lx, Ly \rangle = \langle x, y \rangle$ for all x and y in F^n.

It is easily seen that

1. If L and T are in $G(n)$, then $L\,T$ is in $G(n)$.
2. If $L \in G(n)$, then L is invertible and L^{-1} is in $G(n)$.

Thus $G(n)$, with product given by the composition of linear transformations, is a group.

Definition. If $d = 1$, then $G(n)$, denoted by $O(n)$, is called the orthogonal group. If $d = 2$, then $G(n)$ is denoted by $U(n)$ and called the unitary group. If $d = 4$, then $G(n)$ is denoted by $Sp(n)$ and called the symplectic group.

Note: The map $G(1) \to$ (unit sphere in F) defined by $L \to L(1)$ is an isomorphism of groups.

So $O(1) = \{-1,1\} = Z_2$

$U(1) = \text{unit complexes} = S^1$

$Sp(1) = S^3 = \text{unit quaternions}$

Let u_i be the vector $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ place, in F^n, then

$\beta = \{u_1 \cdots u_n\}$ is an orthonormal basis for F^n. Given a linear transformation $L: F^n \to F^n$, with L in $G(n)$. Set $[L]_{\beta,\beta}$ equal to the $n \times n$ matrix with entries in F whose i^{th} column is the vector $L(u_i)$. The map $L \to [L]_{\beta,\beta}$ defines a group isomorphism between $G(n)$ and a certain subgroup of the group of all invertible $n \times n$ matrices with entries in F.

Proposition 1.1.1. $G(n)$ is isomorphic to the group of all $n \times n$ matrices, A, over F with the property that the columns of A form an orthonormal basis for F^n, which in turn equals the group of all $n \times n$ matrices over F, with the property that $\bar{A}^t A = I$.

Proof. Let $L \in G(n)$. L preserves scalar products and so $||L(x)|| = ||x||$ for all x in F^n. Thus L takes any orthonormal basis for F^n into an orthonormal basis for F^n. In particular, $\{Lu_1 \cdots Lu_n\}$ is an orthonormal basis for F^n. So if $L \in G(n)$, then the columns of $[L]_{\beta,\beta}$ form an orthonormal basis for F^n.

Conversely, let A be any $n \times n$ matrix with entries in F whose columns form an orthonormal basis for F^n. Since $Au_i = i^{th}$ column of A, we have that $\langle Au_i, Au_j \rangle = \delta_{ij}$ where

$$\delta_{ij} = \begin{Bmatrix} 0 & \text{if} & i \neq j \\ 1 & \text{if} & i = j \end{Bmatrix}.$$

Let x and y be in F^n, to show $\langle Ax, Ay \rangle = \langle x, y \rangle$, write $x = \sum_{i=1}^{n} u_i x_i$, $y = \sum_{i=1}^{n} u_i y_i$ with x_i and y_i in F. Then

$$\begin{aligned} \langle Ax, Ay \rangle &= \langle A\Sigma u_i x_i, A\Sigma u_j y_j \rangle \\ &= \langle \Sigma A u_i x_i, \Sigma A u_j y_j \rangle \\ &= \sum_i \sum_j \langle A u_i x_i, A u_j y_j \rangle \\ &= \sum_i \sum_j \bar{x}_i \langle A u_i, A u_j \rangle y_i \\ &= \sum_i \sum_j \bar{x}_i \delta_{ij} y_j = \sum_i \bar{x}_i y_i = \langle x, y \rangle. \end{aligned}$$

This, together with $[T \circ L]_{\beta,\beta} = [T]_{\beta,\beta} \circ [L]_{\beta,\beta}$, proves the first part of Proposition 1.1.1.

The second part follows from the observation that the i,j entry of the matrix $\bar{A}^t A$ is the scalar product $\langle Au_i, Au_j \rangle$. From now on $G(n)$ will be identified with the corresponding group of matrices.

Definition. If G is a group and a topological space, then G is called a topological group if

1. $G \to G$ defined by $x \to x^{-1}$ is continuous

and

2. $G \times G \to G$ defined by $(x,y) \to xy$ is continuous.

Examples of topological groups.

1. $G(n)$ topologized as a subspace of F^{n^2}.
2. R^1 and R^n (additively)
3. $R^1 - \{0\}$ (multiplicatively)
4. For $d = 1$ or 2 set $GL(n,F_d)$ equal to the group of all invertible $n \times n$ matrices with entries in F_d. Topologize $GL(n,F_d)$ as a subspace of $F_d^{n^2}$. For $d = 1$ or 2 set $SL(n,F_d)$ equal to the subgroup of $GL(n,F_d)$ consisting of matrices with determinant 1. Set $SO(n)$ equal to the subgroup of $O(n)$ consisting of matrices with determinant 1, and set $SU(n)$ equal to the subgroup of $U(n)$ consisting of matrices with determinant 1.
5. Let G be a group, then G with discrete topology is a topological group.
6. A product of topological groups is a topological group.
7. If H is a closed, normal subgroup of G, then G/H with the quotient space topology is a topological group. For example if I^n denotes the subgroup of R^n consisting of vectors with integer coordinates, then R^n/I^n can be

identified with the product $\underbrace{S' \times \cdots \times S'}_{n}$ of topological groups.

8. If X is a compact, Hausdorff space; set $h(X)$ equal to the group of homeomorphisms of X onto itself. Then $h(X)$ with the compact open topology is a topological group.

<u>Definition</u>. Let G be a topological group, and let H be a closed subgroup, then the space of left cosets, G/H, topologized as a quotient space of G is called a homogeneous space.

<u>Examples of homogeneous spaces</u>.

1. If $k < n$, define $G(k) \to G(n)$ by $A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. In this way $G(k)$ is a closed subgroup of $G(n)$. The homogeneous space $G(n)/G(k)$, denoted by $G(n,k)$, is the Stiefel manifold of ordered orthonormal $(n-k)$ frames in F^n. For example $G(n,n-1)$ is homeomorphic to the sphere S^{nd-1} for $n \geq 2$.

2. If $k < n$, define $G(k) \times G(n-k) \to G(n)$ by $(A,B) \to \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. In this way, $G(k) \times G(n-k)$ is a closed subgroup of $G(n)$. The homogeneous space $G(n)/G(k) \times G(n-k)$, denoted by $M(n,k)$ is the Grassmanian manifold of $(n-k)$ dimensional subspaces of F^n. Thus $M(n,k)$ is homeomorphic to $M(n,n-k)$ and $M(n,n-1)$ is the real, complex, Quaternionic projective space of real dimension $d(n-1)$.

To see that $G(n,k)$ is the space of ordered orthonormal $(n-k)$

frames in F^n, it suffices to prove that $AG(k) = BG(k)$ iff A and B have the same last $(n-k)$ columns. This is done by observing that if $C \varepsilon G(k)$, $A \varepsilon G(n)$; then AC and A have the same last $(n-k)$ columns, and that if A and B have the same last $(n-k)$ columns, then $A^{-1}B \varepsilon G(k)$.

To see that $M(n,k)$ is the space of $(n-k)$ dimensional subspaces of F^n, it suffices to prove that $A(G(k) \times G(n-k))$ equals $B(G(k) \times G(n-k))$ iff the last $(n-k)$ columns of A and the last $(n-k)$ columns of B are bases for the same $(n-k)$ dimensional subspaces of F^n. This is done by observing that if $C \varepsilon G(k) \times G(n-k)$, $A \varepsilon G(n)$; then the last $(n-k)$ columns of AC and the last $(n-k)$ columns of A span the same subspace of F^n, and that if the last $(n-k)$ columns of A and the last $(n-k)$ columns of B are bases for the same subspaces of F^n, then $A^{-1}B \varepsilon G(k) \times G(n-k)$. Note that since $A^{-1} = \bar{A}^t$, the $(i,j)^{th}$ entry of the matrix $A^{-1}B$ is the scalar product $\langle i^{th}$ column of A, j^{th} column of $B\rangle$.

2. Cell Structure of $O(n), U(n), Sp(n)$

Definition of ϕ. Let S^{nd-1} be the sphere of unit vectors in F^n. Define $\phi\colon S^{nd-1} \times S^{d-1} \to G(n)$ by requiring that $\phi(x,\lambda)y = y$ if $\langle x,y\rangle = 0$ and $\phi(x,\lambda)x = x\lambda$. Then

$$\phi(x,\lambda)y = x(\lambda-1)\langle x,y\rangle + y$$

$$[\phi(x,\lambda)]_{ij} = x_i(\lambda-1)\bar{x}_j + \delta_{ij} \quad \text{in matrix notation.}$$

If $m < n$, the usual inclusion of F^m in F^n induces a commutative diagram

$$\begin{array}{ccc} S^{md-1} \times S^{d-1} & \longrightarrow & S^{nd-1} \times S^{d-1} \\ \downarrow \phi & & \downarrow \phi \\ G(m) & \longrightarrow & G(n) \end{array}$$

<u>Definition of Q_n</u>. Let Q_n be the quotient space of $S^{nd-1} \times S^{d-1}$ induced by ϕ for $n \geq 1$, and let Q_o be a single point. Map Q_o into $G(n)$ by $Q_o \to I$.

<u>Note</u>: 1. The diagram $\begin{array}{ccc} Q_m & \longrightarrow & Q_n \\ \downarrow & & \downarrow \\ G(m) & \longrightarrow & G(n) \end{array}$ is commutative.

2. Q_n is the quotient space of $S^{nd-1} \times S^{d-1}$ by the relations $(x,\lambda) = (xv, v^{-1}\lambda v)$ and $(x,1) = (y,1)$.

A direct calculation shows that $\phi(x,\lambda) = \phi(xv, v^{-1}\lambda v)$ and $\phi(x,\lambda) = I$ iff $\lambda = 1$.

To see that these are all the relations, suppose $\phi(x,\lambda) = \phi(y,\gamma) \neq I$. Considering the fixed point sets of $\phi(x,\lambda)$ and $\phi(y,\gamma)$ shows that $y = xv$ for some v in S^{d-1}.

$\phi(x,\lambda)x = x\lambda$ so $\phi(x,\lambda)xv = x\lambda v$ and $\phi(xv,\gamma) = xv\gamma$. Thus $\phi(x,\lambda) = \phi(xv,\gamma)$ implies $v\gamma = \lambda v$ so $\gamma = v^{-1}\lambda v$.

3. Q_n is a compact Hausdorff space, $Q_n \to G(n)$ and

$Q_m \to Q_n$ for $m \leq n$, are embeddings

Q_n is compact since it is the image of $S^{nd-1} \times S^{d-1}$ which is compact. Since Q_n is compact and $Q_n \to G(n)$ is 1-1, the map $Q_n \to G(n)$ is an embedding. (Recall any continuous 1-1 map of a compact space into a Hausdorff space is an embedding.)

The next step is to show that Q_n is a C. W. complex.

Consider $E^{(n-1)d}$ as the ball of vectors x in $S^{nd-1} \subseteq F^n$ with x_n real and ≥ 0. (Note: $x_n = \sqrt{1 - \sum_{i=1}^{n-1} \bar{x}_i x_i}$.) Let $f_n \colon E^{(n-1)d} \to S^{nd-1}$ be the inclusion map. Let $g\colon (E^{d-1}, S^{d-2}) \to (S^{d-1}, 1)$ be the usual relative homeomorphism $(S^{-1} = \phi,\ E^0 = \{-1\})$. Set $h_n\colon E^{nd-1} \to Q_n (n \geq 1)$ equal to the composition

$$E^{nd-1} = E^{(n-1)d} \times E^{d-1} \xrightarrow{f_n \times g} S^{nd-1} \times S^{d-1} \xrightarrow{\phi} Q_n.$$

<u>Lemma 1.2.1</u>. The map h_n defines a relative homeomorphism $h_n\colon (E^{nd-1}, S^{nd-2}) \to (Q_n, Q_{n-1})$, if $n \geq 1$. Therefore Q_n is a C. W. Complex with a 0-cell Q_0 and with an (md-1)-cell for each m such that $1 \leq m \leq n$.

<u>Proof</u>. The image of $E^{nd-1} - S^{nd-2}$ in $S^{nd-1} \times S^{d-1}$ equals $\{(x,\lambda)\colon x_n$ is real and $> 0,\ \lambda \neq 1\}$, and $Q_n - Q_{n-1} = \{\phi(x,\lambda)\colon x_n \neq 0$ and $\lambda \neq 1\}$. So h_n maps $E^{nd-1} - S^{nd-2}$ into $Q_n - Q_{n-1}$ and is