Lecture Notes in Mathematics 1598

Jürgen Berndt Franco Tricerri
Lieven Vanhecke

Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces

Springer

Jürgen Berndt   Franco Tricerri
Lieven Vanhecke

Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces

Springer

Authors

Jürgen Berndt
Mathematisches Institut
Universität zu Köln
Weyertal 86-90
D-50931 Köln, Germany
E-mail: berndt@mi.uni-koeln.de

Franco Tricerri †
formerly:
Dipartimento di Matematica "U. Dini"
Università di Firenze

Lieven Vanhecke
Department of Mathematics
Katholieke Universiteit Leuven
Celestijnenlaan 200 B
B-3001 Leuven, Belgium
E-mail: fgaga03@cc1.kuleuven.ac.be

Mathematics Subject Classification (1991): 53C20, 53C25, 53C30, 53C40, 22E25

ISBN 3-540-59001-3 Springer-Verlag Berlin Heidelberg New York

CIP-Data applied for

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1995
Printed in Germany

Typesetting: Camera-ready TEX output provided by the authors
SPIN: 10130255 46/3142-543210 - Printed on acid-free paper

Editorial Policy

§ 1. Lecture Notes aim to report new developments - quickly, informally, and at a high level. The texts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes manuscripts from journal articles which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for Ph. D. theses to be accepted for the Lecture Notes series.

§ 2. Manuscripts or plans for Lecture Notes volumes should be submitted (preferably in duplicate) either to one of the series editors or to Springer- Verlag, Heidelberg . These proposals are then refereed. A final decision concerning publication can only be made on the basis of the complete manuscript, but a preliminary decision can often be based on partial information: a fairly detailed outline describing the planned contents of each chapter, and an indication of the estimated length, a bibliography, and one or two sample chapters - or a first draft of the manuscript. The editors will try to make the preliminary decision as definite as they can on the basis of the available information.

§ 3. Final manuscripts should preferably be in English. They should contain at least 100 pages of scientific text and should include

- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

Lecture Notes in Mathematics 1598

Editors:
A. Dold, Heidelberg
F. Takens, Groningen

此为试读,需要完整PDF请访问: www.ertongbook.com

Preface

The fundamental conjecture about harmonic manifolds has been a source of intensive research during the past decades. Curvature theory plays a fundamental role in this field and is intimately related to the study of the Jacobi operator and its role in the geometry of geodesic symmetries and reflections on a Riemannian manifold.

Our research about harmonic manifolds led in a natural way to the study of spaces with volume-preserving geodesic symmetries and several related classes of manifolds, in particular commutative spaces and Riemannian manifolds all of whose geodesics are orbits of one-parameter groups of isometries. It was also a part of our motivation for developing the theory of homogeneous structures. In this work, the classical and the generalized Heisenberg groups provided a rich collection of examples and counterexamples. It is also well-known that the latter ones take a nice and important place in the florishing research about nilpotent Lie groups and nilmanifolds.

Recently the picture has changed drastically on the one hand by the positive results of Z.I. Szabó and on the other hand by the discovery of the Damek-Ricci harmonic spaces which are the first counterexamples to the fundamental conjecture. These manifolds are Lie groups whose Lie algebras are solvable extensions of generalized Heisenberg algebras. The discovery of these spaces led to a renewed interest in the field, in particular because, just as in the case of the generalized Heisenberg groups, they were found during the work in harmonic analysis and not much attention was given to the detailed study of their geometry and the properties of their curvature as reflected in those of the Jacobi operator.

These notes present a more detailed treatment of this aspect for both classes of manifolds. We do this by relating our study to the several classes of Riemannian manifolds which we have introduced or studied recently in the field of the geometry of the Jacobi operator. It will be shown that they have a rich geometry and provide again answers, examples and counterexamples for several other conjectures and open problems. It is our hope that these notes will stimulate further fruitful research in this area.

During our work in this field, many friends, collaborators and colleagues have contributed by means of their lectures, discussions, joint work, encouragement and interest. They all made this result possible. We are very grateful for their help and for sharing with us their interest and love for mathematics and in particular for geometry. In particular, we thank O. Kowalski, F. Prüfer and F. Ricci.

We also take the opportunity to thank our respective universities, the Consiglio Nazionale delle Richerche (Italy) and the National Fund for Scientific Research (Belgium) for their continued financial support.

Finally we express our deep gratitude to our families for giving us the time needed to do what we enjoy so much.

Köln, Firenze, Leuven
May 1994

Jürgen Berndt, Franco Tricerri, Lieven Vanhecke

To our deep sorrow Franco Tricerri, his wife and his two children died in an airplane crash two weeks after completion of this manuscript. Our loss is immeasurable.

Jürgen Berndt and Lieven Vanhecke

Contents

cerning the geometry of generalized Heisenberg groups was that they are D'Atri spaces, that is, have volume-preserving geodesic symmetries (up to sign). On the other hand, these groups are not naturally reductive as a Riemannian homogeneous space unless the dimension m of the center is one or three. This answered negatively the question whether a D'Atri space is always locally isometric to a naturally reductive Riemannian homogeneous space or not. Moreover, nilmanifolds arising as compact quotients from generalized Heisenberg groups have attracted considerable attention in spectral geometry. For more details see [Gor2], where the author provides, by using suitable compact quotients of generalized Heisenberg groups, the first known examples of closed isospectral Riemannian manifolds which are not locally isometric to each other.

As will be shown in Chapter 3 of these notes, generalized Heisenberg groups also provide examples and counterexamples for other questions and conjectures. But up to now, a systematic study of the geometry of these groups, in particular the aspects relating to the Jacobi operator, was not available. One of the purposes of these notes is to provide a thorough treatment of these aspects based on the explicit research about the spectral theory of this operator and the explicit computation of the Jacobi vector fields vanishing at a point. This method of attack does not only give new geometrical properties but also yields new and more geometrical proofs of known results. Moreover, by doing this, we will relate our research to the different classes of Riemannian manifolds which have been introduced recently in the framework of the study of the geometry of the Jacobi operator. Chapter 2 contains a short survey about these classes including their definitions, known classifications, various characterizations and relations between them.

Our interest in the treatment of D'Atri spaces, as introduced in [Dat], [DaNi1], and [DaNi2], came from the research about harmonic spaces. The fundamental conjecture about harmonic spaces (also referred to as the conjecture of Lichnerowicz) stated that every Riemannian harmonic manifold is locally isometric to a two-point homogeneous space. It was shown that the condition of harmonicity is equivalent to two infinite series of conditions on the curvature tensor and its covariant derivatives, known as the even and odd Ledger conditions. The D'Atri property is equivalent to the set of odd Ledger conditions. Only during the past five years there was a breakthrough in this field on the one hand by the positive results by Z.I. Szabó (see 2.6) and on the other hand by the negative one by E. Damek and F. Ricci. More precisely, the last two authors showed in [DaRi1] that this conjecture is false by proving that there exist suitable extensions of arbitrary generalized Heisenberg groups which are harmonic. Any such extension is a simply connected solvable Lie group with a left-invariant Riemannian metric. Among these Lie groups are the complex hyperbolic spaces, the quaternionic hyperbolic spaces and the Cayley hyperbolic plane. Their horospheres provide realizations of the Heisenberg groups in the complex case and of suitable generalized Heisenberg groups with three- and seven-dimensional center, respectively, in the two other cases. In each of these particular cases the corresponding classical or generalized Heisenberg group is precisely the nilpotent part in the Iwasawa decomposition of the isometry group of the hyperbolic space. The above mentioned extension is then the solvable part in the Iwasawa decomposition and, as a group, is a semi-direct product of the nilpotent group and the real numbers. By imitating this construction of the hyperbolic spaces as solvable Lie groups one obtains from each generalized Heisenberg group a solvable Lie group with a left-invariant Riemannian metric. These particular extensions have been called Damek-Ricci spaces. Any of these spaces is a Hadamard manifold with the corresponding generalized Heisenberg group embedded as a horosphere, and is either one of the above hyperbolic spaces or is non-symmetric. In the latter case each one provides a counterexample to the fundamental conjecture about harmonic spaces. Moreover, as is mentioned in [Gor2], the study of the closed geodesic balls in Damek-Ricci spaces by Z.I. Szabó yielded the first examples of closed isospectral Riemannian manifolds with boundary which are not locally isometric to each other. As concerns the harmonic analysis on the Damek-Ricci spaces we again refer to [DaRi2].

But also here, a detailed study of the geometry of the Damek-Ricci spaces is appealing. Some aspects of it have already been considered by several authors (for more details see Chapter 4). Using again the Jacobi operator, in Chapter 4 we will consider those aspects which are related to some of the classes of manifolds considered in Chapter 2. This leads to several new geometrical characterizations of the symmetric Damek-Ricci spaces. It will also be proved that the Damek-Ricci spaces provide, as in the case for generalized Heisenberg groups, examples and counterexamples to open questions and conjectures. All this gives support for the belief that a further study of their geometry will lead to the discovery of other nice geometrical properties.

A more detailed description of the contents of each chapter will be given at the beginning of each of the respective chapters.

此为试读,需要完整PDF请访问: www.ertongbook.com

Chapter 2

Symmetric-like Riemannian manifolds

In this chapter we provide some basic material about various classes of Riemannian manifolds which may be regarded as generalizations of Riemannian (locally) symmetric spaces. Our list of such generalizations is not exhaustive. For example, we do not talk about the class of k-symmetric spaces [Kow1] which are natural generalizations of symmetric spaces too. Our selection contains only those spaces which are related to our research on generalized Heisenberg groups and their Damek-Ricci harmonic extensions. Concerning the material about the spaces presented here we have tried to be rather complete as regards known classifications and characterizations. The basic references given here will guide the reader to further results and details on these spaces. See also [Van2] for a selection.

All manifolds are supposed to be connected and of class C^∞. Our sign convention for the Riemannian curvature tensor R is given by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for all tangent vector fields X, Y, where ∇ denotes the Levi Civita connection.

2.1 Naturally reductive Riemannian homogeneous spaces

Let $M = G/H$ be a Riemannian homogeneous space endowed with a G-invariant Riemannian metric g. The Lie group G is supposed to be connected and to act effectively on M. A decomposition of the Lie algebra $\mathfrak{g}$ of G into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H, is said to be *reductive* if $Ad(H)\mathfrak{m} \subset \mathfrak{m}$. If H is connected, a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is reductive if and only if $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$. Note that in the present situation there always exists a reductive decomposition. For $X, Y \in \mathfrak{m}$ we denote by $[X,Y]_\mathfrak{m}$ the projection of $[X,Y]$ onto $\mathfrak{m}$. Each $X \in \mathfrak{g}$ generates a one-parameter subgroup of the group $I(M)$ of isometries of M via $p \mapsto (\exp tX) \cdot p$ and hence induces a Killing vector field X^* on M. If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition of $\mathfrak{g}$, the *natural torsion-free connection* $\bar{\nabla}$ with respect to this decomposition is defined by

$$(\bar{\nabla}_{X^*} Y^*)_o = \frac{1}{2}[X^*, Y^*]_o = -\frac{1}{2}[X, Y]_{\mathfrak{m}}$$

for all $X, Y \in \mathfrak{m}$, where $\pi(H) = o$ for $\pi : G \to G/H$. Finally, a *homogeneous structure* on M is a tensor field T of type (1,2) such that

$$\tilde{\nabla} g = \tilde{\nabla} R = \tilde{\nabla} T = 0$$

for $\tilde{\nabla} := \nabla - T$, where ∇ is the Levi Civita connection of (M, g) and R the corresponding Riemannian curvature tensor. Then we have the following characterizations (or definitions) of naturally reductive Riemannian homogeneous spaces (for (i) and (ii) see [KoNo,Chapter X,3]; for (iii) see [AmSi, Theorem 5.4] and [TrVa1, Theorem 6.2 and the subsequent remark]).

Proposition 1 [KoNo], [AmSi], [TrVa1] *Let (M, g) be a homogeneous Riemannian manifold. Then (M, g) is a naturally reductive Riemannian homogeneous space if and only if there exist a connected Lie subgroup G of $I(M)$ acting transitively and effectively on M and a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{g}$, where $\mathfrak{h}$ is the Lie algebra of the isotropy group H at some point in M, such that one of the following equivalent statements holds:*

(i) $g([X, Z]_{\mathfrak{m}}, Y) + g(X, [Z, Y]_{\mathfrak{m}}) = 0$ *for all* $X, Y, Z \in \mathfrak{m}$;

(ii) the Levi Civita connection of (M, g) and the natural torsion-free connection with respect to the decomposition are the same;

(iii) every geodesic in M is the orbit of a one-parameter subgroup of $I(M)$ generated by some $X \in \mathfrak{m}$.

An important observation is that a Riemannian homogeneous space $M = G/H$ might be naturally reductive although for any reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{g}$ none of the statements in the proposition holds. The point is that there might exist another appropriate subgroup $\tilde{G}$ of $I(M)$ such that $M = \tilde{G}/\tilde{H}$ and with respect to which a reductive decomposition satisfies the required conditions. Because of this ambiguity the following result has been proved worthwhile for verifying that certain Riemannian homogeneous spaces are naturally reductive without knowing their isometry group and its transitive subgroups explicitly (see [BeVa4], [BlVa], [GoGoVa1], [GoGoVa2], [Nag], [ToVa], [TrVa1], [TrVa2] for applications).

Proposition 2 [TrVa1] *Let (M, g) be a complete and simply connected Riemannian manifold. Then (M, g) is a naturally reductive Riemannian homogeneous space if and only if there exists a homogeneous structure T on M with $T_v v = 0$ for all tangent vectors v of M.*

Every Riemannian symmetric space is naturally reductive. As the classification of Riemannian symmetric spaces is known since the work of E. Cartan, we concentrate now on non-symmetric naturally reductive spaces. For dimension two the situation is clear since any Riemannian homogeneous space obviously has constant curvature and hence is a locally symmetric space. For non-symmetric naturally reductive Riemannian homogeneous spaces in dimensions three, four and five there are the following results (for dimension three see [TrVa1, Theorem 6.5] and in a more explicit way [Kow3]; for the geometric realizations see [BeVa4]).

Theorem 1 [TrVa1], [Kow3], [BeVa4] *Let (M, g) be a three-dimensional simply connected Riemannian manifold. Then (M, g) is a non-symmetric naturally reductive Riemannian homogeneous space if and only if it is one of the following spaces:*

(i) the Lie group $SU(2)$ with some special left-invariant Riemannian metric g. There is a two-parameter family of left-invariant Riemannian metrics on $SU(2)$ making it into a naturally reductive Riemannian homogeneous space. These metrics are precisely those obtained by considering $SU(2) \approx S^3$ as a geodesic sphere in some two-dimensional complex projective or hyperbolic space equipped with some Fubini-Study metric of constant holomorphic sectional curvature;

(ii) the Lie group $\widetilde{SL(2,\mathbb{R})}$ with some special left-invariant Riemannian metric g. There is a two-parameter family of left-invariant Riemannian metrics on $\widetilde{SL(2,\mathbb{R})}$ making it into a naturally reductive Riemannian homogeneous space. These metrics are precisely those obtained by taking the universal covering of any tube around a one-dimensional complex hyperbolic space embedded totally geodesically in a two-dimensional complex hyperbolic space equipped with some Fubini-Study metric of constant holomorphic sectional curvature. In explicit form, these spaces are given by $M = \mathbb{R}^3[t, x, y]$ with

$$ds^2 = \frac{1}{|a+b|}dt^2 + |a+b|e^{-2t}dx^2 + (dy + \sqrt{2b}e^{-t}dx)^2,$$

where $a, b \in \mathbb{R}$ with $b > 0$ and $a + b < 0$. Geometrically, a and b are the eigenvalues of the Ricci tensor of M, the first one with multiplicity two;

(iii) the three-dimensional Heisenberg group H_3 with any left-invariant Riemannian metric g. There is a one-parameter family of such metrics on H_3 and they are obtained by realizing H_3 as a horosphere in some two-dimensional complex hyperbolic space equipped with some Fubini-Study metric of constant holomorphic sectional curvature. Explicitly, $M = \mathbb{R}^3[x, y, z]$ with

$$ds^2 = \frac{1}{2b}(dx^2 + dz^2 + (dy - x dz)^2),$$

where $b \in \mathbb{R}_+$. Here, $-b$ and b are the eigenvalues of the Ricci tensor of M, the first one with multiplicity two.

Theorem 2 [KoVa1] *Let (M, g) be a four-dimensional simply connected Riemannian manifold. Then (M, g) is a non-symmetric naturally reductive Riemannian homogeneous space if and only if it is isometric to some Riemannian product*

$$SU(2) \times \mathbb{R}\ ,\ \ \widetilde{SL(2,\mathbb{R})} \times \mathbb{R}\ ,\ \ H_3 \times \mathbb{R}\ ,$$

where the first factor is equipped with a naturally reductive Riemannian metric according to the classification in dimension three.

Theorem 3 [KoVa5] *Every five-dimensional simply connected non-symmetric naturally reductive Riemannian homogeneous space is either a Riemannian product* $M_1 \times M_2$*, where* M_1 *is* $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ *or* H_3 *with some naturally reductive metric and* M_2 *is some standard space of constant curvature, or locally isometric to one of the following spaces:*

(i) $(SO(3) \times SO(3))/SO(2)_r$ *or* $(SO(3) \times SL(2,\mathbb{R}))/SO(2)_r$ *or* $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/SO(2)_r$*, where* $SO(2)_r$ *denotes the subgroup consisting of pairs of matrices of the form*

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cos rt & -\sin rt & 0 \\ \sin rt & \cos rt & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (t \in \mathbb{R})$$

and r *is a rational number. On each of these spaces there is a family of naturally reductive invariant Riemannian metrics depending on two real parameters. For each of the three types the whole family of locally non-isometric spaces depends on two real parameters and one rational parameter;*

(ii) $(H_3 \times SO(3))/SO(2)^{(r)}$ *or* $(H_3 \times SL(2,\mathbb{R}))/SO(2)^{(r)}$*, where* $SO(2)^{(r)}$ *denotes the subgroup consisting of all pairs of matrices of the form*

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cos rt & -\sin rt & 0 \\ \sin rt & \cos rt & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (t \in \mathbb{R})$$

and r *is a rational number. On each of these spaces there is a family of naturally reductive invariant Riemannian metrics depending on two real parameters. For each of the two types the whole family of locally non-isometric spaces depends on two real parameters and one rational parameter;*

(iii) the five-dimensional Heisenberg group H_5*. The naturally reductive left-invariant Riemannian metrics on* H_5 *form a two-parameter family. Explicitly, these spaces are* $M = \mathbb{R}^5[x,y,z,u,v]$ *with*

$$ds^2 = \frac{1}{\rho}(du^2 + dx^2) + \frac{1}{\lambda}(dv^2 + dy^2) + (udx + vdy - dz)^2$$

and $\lambda, \rho \in \mathbb{R}_+$*;*

(iv) $SU(3)/SU(2)$ *or* $SU(1,2)/SU(2)$*, and on each space there is a family of naturally reductive invariant Riemannian metrics depending on two real parameters.*

Geodesic spheres in two-point homogeneous spaces except CayP^2 and CayH^2 are naturally reductive Riemannian homogeneous spaces (see [Zil2] and [TrVa2]). Every simply connected η-umbilical hypersurface of a complex space form is naturally reductive [BeVa4]. This has been extended by Nagai [Nag] to the so-called hypersurfaces of type (A) in complex projective spaces and their corresponding ones in complex hyperbolic spaces. Every simply connected φ-symmetric space (that is, a Sasakian manifold with complete characteristic field such that the reflections with respect to the integral curves of that field are global isometries) is naturally reductive [BlVa]. Every simply connected Killing-transversally symmetric space (that is, a space equipped with a complete unit Killing vector field such that the reflections with respect to the flow lines of that field are global isometries) is naturally reductive (see [GoGoVa1] and [GoGoVa2]). Note that each φ-symmetric space is a Killing-transversally symmetric space.

For further results and references on naturally reductive Riemannian homogeneous spaces we refer to J.E. D'Atri and W. Ziller [DaZi], who also classified all naturally reductive compact simple Lie groups. For a treatment of the non-compact semisimple case, see C. Gordon [Gor1].

2.2 Riemannian g.o. spaces

A Riemannian manifold (M, g) is said to be a *Riemannian g.o. space* [KoVa7] if every geodesic in M is the orbit of a one-parameter subgroup of the group of isometries of M. Clearly, any such space is homogeneous. From Proposition 1(iii) in 2.1 we derive immediately

Proposition *Every naturally reductive Riemannian homogeneous space is a Riemannian g.o. space.*

O. Kowalski and the third author [KoVa7] have proved that the converse holds if the dimension is less than six.

Theorem 1 [KoVa7] *Every simply connected Riemannian g.o. space of dimension ≤ 5 is a naturally reductive Riemannian homogeneous space.*

Combining this with Theorems 1, 2 and 3 in 2.1 yields a classification of all simply connected Riemannian g.o. spaces of dimension less than six. For dimension six the converse does not hold. In fact, there is the following result:

Theorem 2 [KoVa7] *The following six-dimensional simply connected Riemannian g.o. spaces (and only those) are never naturally reductive:*

(i) (M, g) is a two-step nilpotent Lie group with two-dimensional center, provided with a left-invariant Riemannian metric such that the maximal connected isotropy group is isomorphic to $SU(2)$ or $U(2)$. All these Riemannian g.o. spaces depend on three real parameters;

(ii) $(\tilde{M}, g)$ is the universal covering space of a homogeneous Riemannian manifold of the form $M = SO(5)/U(2)$ or $M = SO(4,1)/U(2)$, where $SO(5)$ or $SO(4,1)$ is the identity component of the full isometry group, respectively. In each case all admissible Riemannian metrics depend on two real parameters.

Every geodesic sphere in a two-point homogeneous space except $\mathrm{Cay}P^2$ or $\mathrm{Cay}H^2$ is a Riemannian g.o. space since it is naturally reductive. For $\mathrm{Cay}P^2$ and $\mathrm{Cay}H^2$ it is still an open problem whether the geodesic spheres are g.o. spaces or not.

2.3 Weakly symmetric spaces

A Riemannian manifold M is said to be a *weakly symmetric space* [Sel] if there exist a subgroup G of the isometry group $I(M)$ of M acting transitively on M and an isometry f of M with $f^2 \in G$ and $fGf^{-1} = G$ such that for all $p, q \in M$ there exists a $g \in G$ with $g(p) = f(q)$ and $g(q) = f(p)$. It can easily be seen that any Riemannian symmetric space is weakly symmetric. There are the following geometrical characterizations of weakly symmetric spaces:

Proposition [BePrVa1], [BeVa5] *Let (M, g) be a Riemannian manifold. Then the following statements are equivalent:*

(i) M is a weakly symmetric space;

(ii) for any two points $p, q \in M$ there exists an isometry of M mapping p to q and q to p;

(iii) for every maximal geodesic γ in M and any point m of γ there exists an isometry of M which is an involution on γ with m as fixed point.

Note that Riemannian manifolds having property (iii) have been introduced by Szabó [Sza2] as *ray symmetric spaces*.

In dimensions three and four the simply connected weakly symmetric spaces are completely classified.

Theorem 1 [BeVa5] *A three- or four-dimensional simply connected Riemannian manifold is a weakly symmetric space if and only if it is a naturally reductive Riemannian homogeneous space (see Theorems* 1 *and* 2 *in* 2.1*).*

We also have the following further examples of non-symmetric weakly symmetric spaces.

Theorem 2 [BeVa5] *Each of the following hypersurfaces, endowed with the induced Riemannian metric of the ambient space, is a weakly symmetric space for $n \geq 2$:*

ambient space	hypersurface
$\mathbb{C}P^n$	*tube around* $\{p\}, \mathbb{C}P^1, \ldots,$ *or* $\mathbb{C}P^{n-1}$
$\mathbb{H}P^n$	*tube around* $\{p\}, \mathbb{H}P^1, \ldots,$ *or* $\mathbb{H}P^{n-1}$
$\mathrm{Cay}P^2$	*tube around* $\{p\}$ *or* $\mathrm{Cay}P^1$
$\mathbb{C}H^n$	*horosphere; tube around* $\{p\}, \mathbb{C}H^1, \ldots,$ *or* $\mathbb{C}H^{n-1}$
$\mathbb{H}H^n$	*horosphere; tube around* $\{p\}, \mathbb{H}H^1, \ldots,$ *or* $\mathbb{H}H^{n-1}$
$\mathrm{Cay}H^2$	*horosphere; tube around* $\{p\}$ *or* $\mathrm{Cay}H^1$.

此为试读，需要完整PDF请访问： www.ertongbook.com