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Preface

Random Schrödinger operators are models for the quantum mechanical description of disordered media. The main aim of the analysis of such models is the understanding of the (charge) transport properties of the material. It turns out that many of the properties of interest of random Schrödinger operators are related to a quantity called *integrated density of states*. It measures or 'counts' the number of electron energy levels of the Hamiltonian per unit volume. An alternative name for the integrated density of states is *spectral distribution function* since it is the distribution function of a spectral measure associated to the random family of operators. Many features of this quantity have an intuitive physical interpretation, others play a prominent role in proofs of key theorems. Moreover, the spectral distribution function is an object of study in other fields of mathematics, like differential geometry, group and von Neumann algebras, and homological algebra.

What are the properties of the integrated density of states which have been studied in the literature? It would be hard to give an exhaustive answer, but there are several classes of questions that have drawn the attention of many authors.

The first class is concerned with the definition and construction of the integrated density of states. Can it be expressed as a limit of a sequence of distribution functions associated to the spectra of 'simpler' operators? These operators are usually restrictions of the original Schrödinger operator to some finite volume set. There are various ways how to choose the approximation sequence of operators. Thus another question comes up naturally: Does this choice influence the final outcome, or does one obtain the same distribution function, independently of the approximation procedure? Furthermore, is there a closed formula for the integrated density of states? If there are various such formulas, are some of them better suited for certain applications than others?

Another circle of ideas concerns the continuity properties of the integrated density of states and its set of points of increase: Can one characterise the location and size of the discontinuities? What is the structure of the sets of constancy of the integrated density of states? Since it is a spectral measure

distribution, these questions are intimately related to the spectrum of the random operator. Can one prove quantitative regularity properties of the integrated density of states: Is it log-Hölder, Hölder, or Lipschitz continuous? Does it exhibit even stronger regularity properties like differentiability or analyticity? Is it possible to give upper and lower bounds on the derivative?

Finally, one is interested in the behaviour of the integrated density of states as the energy variable approaches a spectral boundary. The most studied case is the infimum of the spectrum, although the behaviour at very high energies, or at internal spectral edges is of interest, too. Can one give a characteristic law of the asymptotics of the integrated density of states at the boundaries? Is it polynomial, is it exponential? Can one specify or estimate the characteristic exponents?

Of course, the answers to the above questions often depend on the type of random operator one is considering. Thus one my also ask: How do the parameters entering the model influence the abovementioned features? Is there a universal behaviour or some phase transition phenomenon?

We used above the term *random Schrödinger operator* although this terminology refers to just one instance among many types of equivariant Hamiltonians in various geometric settings for which the integrated density of states may be defined. A substantial body of papers is devoted to operators acting on combinatorial graphs, the simplest being the integer lattice $\mathbb{Z}^d$. Others consider operators acting on $L^2(\mathbb{R}^d)$, which includes Schrödinger operators. For models on $\mathbb{R}^d$, one may require a $\mathbb{R}^d$ or a $\mathbb{Z}^d$-equivariance condition. In the latter case, there is still a discrete structure present in the random operator, albeit it acts on continuous space. All settings mentioned so far concern the spaces $\mathbb{Z}^d$ or $\mathbb{R}^d$, and thus Euclidean geometry. Going beyond these, there are interesting related models on covering manifolds, finitely generated groups, as well as combinatorial and metric graphs with a quasi-transitive structure.

In the remainder of the preface we describe briefly the potential audience of the book, the recommended prerequisites, the approach taken to present the material, the selection of topics and the structure of the text.

The aim of the text is to give researchers interested in the subject of random Schrödinger operators an overview of known results and methods. Specialists may find it useful as a guide to further reading. The subject matter of the book draws on various mathematical disciplines. For that reason it was not possible to include all the background material, but the reader can find detailed descriptions of the relevant facts using the references to textbooks and monographs. Thus, the text should be accessible to graduate students who have a working knowledge of selfadjoint operators and quadratic forms, possibly from a course on linear operators in Hilbert space or an advanced functional analysis class. For students without this background any one of the following books is recommended as a reading companion: [18, 19] by Akhiezer and Glazman, [47] by Birman and Solomyak, [110] by Davies, [494] or [495] by Weidmann, or [497] by Werner, the last two references being in German. The reader will find the relevant material also in the treatises [140, 141, 142]

by Dunford and Schwartz, [239] by Kato, or [407, 408, 409, 410] by Reed and Simon. For the reader who wants to know more about the physical background of the models discussed here we recommend the monographs [53, 145, 340, 312, 143] where properties of disordered systems are discussed from the point of view of theoretical physics.

As already mentioned, the integrated density of states can be defined in various geometric settings and for operators with various equivariance types. If one aims at discussing all such models in a text, one could treat them subsequently one by one, or order the text according to various results and properties and discuss each time all models. One could also first develop a general approach which covers all models, and prove theorems on an abstract level. Although all these seem viable options, here we choose a different and more modest way: We consider just one model in detail and refer in remarks to sources in the literature where the proof for other variants may be found. More precisely, we concentrate here on operators on a continuous configuration space with a discrete group structure. The most important example is the alloy-type model, a $\mathbb{Z}^d$-ergodic operator acting on $L^2(\mathbb{R}^d)$, but operators on Riemannian manifolds with non-abelian group actions are also considered.

While most of the relevant aspects of the spectral theory of random Schrödinger operators figure in the text, the presentation is centred around the integrated density of states. A broader view is taken in the monographs [81] by Carmona and Lacroix and [389] by Pastur and Figotin which describe the state of the art at the beginning of the 1990s. There are several other text of a survey nature on the subject from the second half of the 1980s, including the introductory article [247] by Kirsch, a section on random Jacobi matrices in [102] by Cycon, Froese, Kirsch and Simon, and the Lifshitz memorial issue [335]. In recent years there have been two more monographs treating related topics. The theory of Anderson localisation for random Schrödinger operators is exposed in detail in [458] by Stollmann. Many features of the spectral distribution function in the context of geometry, group theory and K-theory are discussed in Lück's book [346].

Let us mention a few recent overview articles which discuss certain aspects of the theory covered only marginally in the present book. A survey of localisation results for one-dimensional random models is provided in [459] by Stolz. A detailed account of $\mathbb{R}^d$-ergodic random Schrödinger operators, which model amorphous media, is given by Leschke, Müller and Warzel in [331]. The present text emphasises the construction of the integrated density of states and its continuity properties, while its asymptotic behaviour at the infimum of the spectrum is discussed only in remarks. An overview of the results devoted to the last mentioned topic can be found in the recent [259] by Kirsch and Metzger. There also spectral properties of random surface models are discussed. We mentioned above that it is possible to introduce the integrated density of states in a general, abstract framework applicable to various types of equivariant operators and geometric settings. Such an approach is taken

for instance in [328], which covers e.g. Hamiltonians on Euclidean space and lattices, on covering manifolds, Delone sets, and on percolation graphs.

Of course, there are many other excellent sources in the literature not mentioned here in the preface. We hope to have them adequately quoted in the main text.

Here is a sketch of the contents and the structure of the book: The introduction explains how one arrives at random Schrödinger operators starting from the quantum mechanical theory of disordered solids. There we also fix frequently used notation, recall basic facts about selfadjoint operators, and define our models. Finally, the introduction explains the relation between spectral and transport properties of Schrödinger operators as well as the notion of spectral fluctuation boundaries.

The second chapter presents two alternative proofs of the approximation of the integrated density of states by its finite volume analoga. One of the approaches is general enough to be applicable to random Schrödinger and Laplace-Beltrami operators on manifolds.

The third chapter explains the relevance of Wegner estimates and regularity properties of the integrated density of states for other aspects of the theory of random Schrödinger operators. A prominent example would be the use of a Wegner bound in the multiscale proof of localisation.

The last two chapters present each a proof of Wegner's estimate for the alloy type or continuum Anderson model on $L^2(\mathbb{R}^d)$. The reason to present two different methods is that each of them has its own advantages when applied to models exhibiting various non-trivial features. We consider alloy type models with long-range or negative correlations, as well as singular and non-monotone dependence on the coupling constants. Several remarks are devoted to similar results for operators on graphs and manifolds mentioned above. Finally, an appendix is devoted to some facts from the theory of the spectral shift function which are used in the main text. More details can be found in the table of contents.

The present text is a revised version of the thesis [486] prepared for the habilitation at the Department of Mathematics of the Technische Universität Chemnitz, which in turn is based on [483]. W. König, P. Stollmann, and S. Teufel have kindly accepted the request of the Department to act as referees of the thesis and I would like to thank them at this occasion. The material presented here draws to a large extent on joint work with colleagues: I have greatly profited from discussions with T. Antunović, D. Borisov, M. Gruber, M. Helm, D. Hundertmark, R. Killip, W. Kirsch, S. Kondej, V. Kostrykin, D. Lenz, P. Müller, S. Nakamura, N. Peyerimhoff, O. Post and P. Stollmann and enjoyed working with them. This work has been made possible through the financial support of the Deutsche Forschungsgemeinschaft. I thank the staff of Springer in charge of the LNM series for their flexibility and efficiency in the course of the preparation of the manuscript.

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Random Operators

1.1 Physical Background

Random Schrödinger operators are used as models of disordered solids within the framework of quantum mechanics.

A macroscopic solid consists of an order of magnitude of 10^{23} of nuclei and electrons. The resulting Hamiltonian taking into account all interactions is highly complicated. To arrive at a Schrödinger operator which can be studied in some detail one neglects the electron-electron interaction and treats the nuclei in the infinite mass approximation. Thus one arrives at an one-electron Schrödinger operator with an external potential due to the electric forces between the electron and the nuclei, which are assumed to be fixed in space.

In the case that the nuclei are arranged periodically on a lattice, the potential energy of the electron is a periodic function of the space variable.

On the other hand, the electron could be moving in an amorphous medium, in which case there is no large group of symmetries of the Hamiltonian. However, from the physical point of view it is reasonable to assume that the local structure of the medium will be translation invariant *on average*. This means that we consider the potential which the electron experiences as a particular realisation of a random process and assume stationarity with respect to some group of translations. Moreover, physical intuition suggests to assume that the local properties of the medium in two regions far apart (on the microscopic scale) are approximately independent from each other. Therefore the stochastic process describing the potential should have a correlation function which decays to zero, or — more generally — should be ergodic.

There are interesting models which lie between the two extreme cases of lattice-periodic and amorphous media. They still have an underlying lattice structure which is, however, modified by disorder. Probably the best studied Hamiltonian with this properties is the alloy type model. We leave its precise definition for the next section and introduce here a special case on the intuitive level. Consider first the potential

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} u_k(\omega, x)$$

Each k corresponds to a nucleus sitting on a lattice point. The function $u_k(\omega, \cdot)$ describes the atomic or nuclear potential at the site k and depends on the random parameter ω which models the different realisations of the configuration of the nuclei. If there is only one type of atom present, which has a spherically symmetric potential, all the $u_k(\omega, \cdot)$ are the same, and V_ω is periodic. Now assume that there are two kinds a and b of atoms present, which have spherically symmetric atomic potentials of the same shape, but which differ in their nuclear charge numbers q_a and q_b.

In this case the potential looks like

$$V_\omega(x) := \sum_{k \text{ occupied by } a} q_a\, u(x-k) + \sum_{k \text{ occupied by } b} q_b\, u(x-k)$$

If the two sorts of atoms are arranged on the lattice in a regular pattern, this again gives rise to a periodic potential.

However, there are physically interesting situations (e.g. binary alloys) where the type of atom sitting on site k is random, for example obeying the law

$$\mathbb{P}\{k \text{ is occupied by atom } a\} = P, \quad \mathbb{P}\{k \text{ is occupied by atom } b\} = 1 - P$$

with some $P \in]0, 1[$. Here $\mathbb{P}\{\dots\}$ denotes the probability of an event. If we furthermore assume that the above probabilities are independent at each site and the parameter P is the same for all k, we arrive at the continuum *Bernoulli-Anderson potential*

$$V_\omega(x) = \sum_k q_k(\omega)\, u(x-k)$$

Here $q_k(\omega) \in \{q_a, q_b\}, k \in \mathbb{Z}^d$ denotes a collection of independent, identically distributed Bernoulli random variables and u is some atomic potential.

This model is a prototype which has motivated much research in the physics and mathematics literature, a part of which we will review in the present work.

1.2 Model and Notation

We introduce here, respectively recall, basic notions on L^p spaces, selfadjoint operators in general and Schrödinger operators in particular, and specify the model Hamiltonian which will be studied throughout the text. We suppose that the reader is acquainted with the theory of selfadjoint operators in Hilbert space. In the Preface one can find a list of monographs which provide the necessary background of this theory.

Let us start with some mathematical notation. The symbols $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0$ denote the set of reals, the set of integers, the set of natural numbers, and the set of non-negative integers, respectively. For a set $A \subset B$ we denote by $A^c := B \setminus A$ its complement. A measurable subset of $\mathbb{R}^d$ will be often denoted by Λ, and if there is a sequence of such sets its members will be denoted $\Lambda_1, \ldots, \Lambda_l, \ldots$. The symbol $|\Lambda|$ is used for the Lebesgue measure of Λ. We write $|x|$ for the norm of $x \in \mathbb{R}^d$, while the norm of a vector f in a function space is denoted by $\|f\|$.

The Hilbert space of (equivalence classes of) measurable functions on Λ which are square integrable with respect to Lebesgue measure is denoted by $L^2(\Lambda)$. Similarly, $L^p(\Lambda)$ with $p > 0$ stands for the Banach space of measurable functions f such that $|f|^p$ is integrable, while $L^\infty(\Lambda)$ is the set of measurable functions which are essentially bounded with respect to Lebesgue measure. The space of sequences $\{a_n\}_{n\in\mathbb{N}}$ such that $|a_n|^p$ is summable is denoted by $\ell^p(\mathbb{N})$. Note that the case $p \in]0, 1[$ is included in our notation. In our context we will often choose the exponent p dependent on the dimension of the configuration space. In the following we denote by $p(d)$ any number in $[1, \infty[$ which satisfies

$$p(d) \begin{cases} \geq 2 & \text{if } d \leq 3, \\ > d/2 & \text{if } d \geq 4 \end{cases} \tag{1.1}$$

For $\Lambda \subset \mathbb{R}^d$ open, the symbols $C(\Lambda), C^\infty(\Lambda)$ stand for the continuous, respectively smooth, functions on Λ. The subscript $_c$ in $C_c(\Lambda), C_c^\infty(\Lambda), L_c^p(\Lambda)$ means that we consider only those functions which have compact support in Λ. In the sequel we will often consider potentials from the class of functions which are *uniformly locally* in L^p. More precisely, f is in the set of uniformly locally L^p-functions, denoted by $L^p_{\text{unif,loc}}(\mathbb{R}^d)$, if and only if there is a constant C such that for each $y \in \mathbb{R}^d$

$$\int_{|x-y|<1} |f(x)|^p \, dx \leq C$$

The infimum over all such constants C equals by definition $\|f\|^p_{p,\,\text{unif,loc}}$. Finally, we introduce Sobolev spaces $W^{k,2}$ of order k. For $\Lambda \subset \mathbb{R}^d$ open, a function $f \in L^2(\Lambda)$ is in $W^{k,2}(\Lambda)$ if all its partial derivatives up to order k exist in the sense of distributions and are elements of $L^2(\Lambda)$. Obviously, $C_c^\infty(\Lambda)$ is a subset of $W^{k,2}(\Lambda)$. Its closure (with respect to the canonical norm of $W^{k,2}(\Lambda)$) is denoted by $W_0^{k,2}(\Lambda)$.

Let Δ denote the Laplacian on $\mathbb{R}^d$. If we choose its *operator domain* $\mathcal{D}(\Delta)$ to be the Sobolev space $W^{2,2}(\mathbb{R}^d)$, it becomes a selfadjoint operator. The restriction of Δ to an open true subset $\Lambda \subset \mathbb{R}^d$ becomes selfadjoint only if we specify appropriate boundary conditions (b.c.). Dirichlet b.c. are defined in Remark 2.2.3. For the definition of Neumann and periodic b.c. see for instance [408].

Let A, B be two densely defined symmetric operators on a Hilbert space $\mathcal{H}$, whose norm we denote by $\|\cdot\|$. The associated scalar product we denoted by $<.,.>$. We say that B is *(relatively) A-bounded* if the domains obey the inclusion $\mathcal{D}(A) \subset \mathcal{D}(B)$ and there are finite constants a and c_a such that for all $f \in \mathcal{D}(A)$

$$\|Bf\| \leq a\|Af\| + c_a\|f\| \tag{1.2}$$

The infimum over all a such that the estimate holds with some c_a is called *relative bound* (of B with respect to A). If B is A-bounded with relative bound zero, we call it *infinitesimally A-bounded.* Let A be selfadjoint, and B symmetric and relatively A-bounded with relative bound smaller than one. Then the operator sum $A + B$ on the domain $\mathcal{D}(A)$ is selfadjoint by the Kato-Rellich Theorem, see e.g. Sect. X.2 in [407]. We will apply this result to the sum of the negative Laplacian and a potential. A multiplication operator by a function $V \in L^p_{\text{unif,loc}}(\mathbb{R}^d)$ is infinitesimally Δ-bounded if $p = p(d)$, cf. Theorem XIII.96 in [408]. Moreover, the constant c_a in (1.2) depends only on $\|V\|_{p,\,\text{unif,loc}}$. Thus the sum $H := -\Delta + V$ is selfadjoint on $W^{2,2}(\mathbb{R}^d)$. In a similar way it is possible to introduce the notion of *relative form-boundedness.* Here we consider A which is selfadjoint and bounded below. Denote by $\mathcal{D}(Q_A)$ its quadratic form domain. A symmetric operator B is said to be (relatively) *A-form bounded* if the quadratic form domains obey the inclusion $\mathcal{D}(Q_A) \subset \mathcal{D}(Q_B)$ and there are finite constants a, C_a such that

$$|\langle\phi, B\phi\rangle| \leq a\langle\phi, A\phi\rangle + C_a\langle\phi, \phi\rangle \tag{1.3}$$

The *relative A-form bound* of B is the infimum of all a which satisfy (1.3). See Sect. VI.1.7 in [239] for more details.

Before we introduce random operators we want to fix notation concerning some terminology in probability theory. The triple $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ stands for a probability space with associated σ-algebra and probability measure, while $\mathbb{E}\{\dots\}$ denotes the expectation value with respect to $\mathbb{P}$. A collection $T_j : \Omega \to \Omega, j \in \mathcal{J}$ of measure preserving transformations is called *ergodic* if all measurable sets in Ω which are invariant under the action of all $T_j, j \in \mathcal{J}$ have measure zero or one.

Definition 1.2.1. *Let $p = p(d)$ be as in (1.1), $u \in L^p_c(\mathbb{R}^d)$ and $q_k : \Omega \to \mathbb{R}$, $k \in \mathbb{Z}^d$ a sequence of bounded, independent, identically distributed random variables, called* coupling constants. *Then the family of multiplication operators given by the stochastic process*

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} q_k(\omega)\, u(x-k) \tag{1.4}$$

is called alloy type potential. *The function u is called* single site potential. *Let $H_0 := -\Delta + V_{\text{per}}$ be a periodic Schrödinger operator with $V_{\text{per}} \in L^p_{\text{unif,loc}}(\mathbb{R}^d)$. The family of operators*

$$H_\omega := H_0 + V_\omega, \quad \omega \in \Omega \tag{1.5}$$

is called alloy type model.

The distribution measure of the random variable q_0 *will be called* single site distribution *and denoted by* μ. *If not stated otherwise, in the sequel we assume that* μ *is absolutely continuous with respect to the Lebesgue measure and has a bounded density. The density function is denoted by* f.

Due to our assumptions on the boundedness of the coupling constants, for each $a > 0$ there is a constant c_a such that for all ω and all $\psi \in \mathcal{D}(\Delta)$

$$\|V_\omega \psi\| \leq a\|\Delta\psi\| + c_a\|\psi\|, \quad \|V_{\text{per}}\psi\| \leq a\|\Delta\psi\| + c_a\|\psi\|$$

In particular H_0 and all H_ω are selfadjoint on the operator domain of Δ. It will be of importance to us that the constant c_a may be chosen independently of the random parameter ω.

Remark 1.2.2. (a) In several sections we study Hamiltonians as in Definition 1.2.1, but where some of the hypotheses on the single site potential or the coupling constants are relaxed. More precisely, we will consider single site potentials with non-compact support and coupling constants which are unbounded, correlated, or do not have a bounded density.

(b) If the coupling constants are not bounded, one has to impose some moment condition to make sure that the alloy type model still makes sense. The main difference (to the bounded case) is that for ω in a set $\Omega' \subset \Omega$ of full measure the operator H_ω will be (essentially) selfadjoint, however this may fail to hold for ω in the complement $\Omega \setminus \Omega'$. See for example [255, 256, 258] for more details.

(c) There is a group of measure preserving transformations $T_k, k \in \mathbb{Z}^d$ on $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ such that (1.4) obeys

$$V_\omega(x - k) = V_{T_k\omega}(x)$$

In other words, the stochastic process $V\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is *stationary* with respect to translations by vectors in $\mathbb{Z}^d$. Moreover, the group $T_k, k \in \mathbb{Z}^d$ acts ergodically on Ω, therefore we call V an $\mathbb{Z}^d$*-ergodic potential.*

To see that the above statements are true we pass over to the canonical probability space $\Omega = \times_{k\in\mathbb{Z}^d} \mathbb{R}$, equipped with the product measure $\mathbb{P} := \otimes_{k\in\mathbb{Z}^d} \mu$. Now the stochastic process $\{\pi_k\}_{k\in\mathbb{Z}^d}$, defined by $\pi_k(\omega) = \omega_k$ for all $k \in \mathbb{Z}^d$, has the same finite dimensional distributions as $\{q_k\}_k$. It is easily seen that the transformations $(T_k(\omega))_j := \omega_{j-k}$ are measure preserving and that the group $(T_k)_{k\in\mathbb{Z}^d}$ acts ergodically on Ω. See Sect. 3 in [247] or Sect. I.1 in [389] for more details.

Using the stochastic process $\{\pi_k\}_k$ the alloy type potential can be written as

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} \omega_k\, u(x - k) \tag{1.6}$$

We will use notations (1.4) and (1.6) without distinction in the sequel.

Abstracting the properties of stationarity and ergodicity we define general random potentials and operators with a $\mathbb{Z}^d$-ergodic structure.

Definition 1.2.3. *Let $V\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a stochastic process such that for almost all $\omega \in \Omega$ the realisation of the potential obeys $V_\omega \in L^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$, $p = p(d)$ and additionally $\mathbb{E}\{\|V_\omega \chi_\Lambda\|_p^p\} < \infty$, where Λ is a unit cube. Let $T_k, k \in \mathbb{Z}^d$ be a group of measure preserving transformations acting ergodically on $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ such that*

$$V_\omega(x-k) = V_{T_k\omega}(x)$$

Then we call $\{V_\omega\}_\omega$ a ($\mathbb{Z}^d$-ergodic) random potential and $\{H_\omega\}_\omega$ with $H_\omega = -\Delta + V_\omega$ a ($\mathbb{Z}^d$-ergodic) random operator.

The restriction of H_ω to an open subset Λ will be denoted by H_ω^Λ if we impose Dirichlet boundary conditions and by $H_\omega^{\Lambda,N}$ in the case of Neumann b.c. While we will be mainly concerned with $\mathbb{Z}^d$-ergodic operators we will give some comments as asides on their counterparts which are ergodic with respect to the group $\mathbb{R}^d$. The overview [331] is devoted to such models that model amorphous media. Insight in the research on almost-periodic operators can be obtained for instance in the papers [431, 432, 32, 40], the literature quoted on page 9, the monographs [102, 389], and the references therein.

Remark 1.2.4. All $\mathbb{Z}^d$-ergodic potentials can be represented in a form which resembles alloy type potentials. In fact, for such $V\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ there exists a sequence $f_k, k \in \mathbb{Z}^d$ of random variables on Ω taking values in the separable Banach space $L^p(\mathbb{R}^d)$ such that V can be written as

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^d} f_k(\omega, x-k). \tag{1.7}$$

This representation is of interest because it ensures that after passing to an equivalent probability space and stochastic process one may assume that the sigma algebra on Ω is countably generated. See [245] and Remark 2.8 in [328] for more information.

1.3 Transport Properties and Spectral Types

The main interest in the study of random operators is to understand the transport properties of the materials they model. In the particular case of the quantum mechanical Hamiltonian of an electron in a disordered solid the electric conductance properties are the principal object of interest.

The Hamiltonian governs the equation of motion, i.e. the time dependent Schrödinger equation

$$\frac{\partial\psi(t)}{\partial t} = -iH_\omega\psi(t) \tag{1.8}$$

The time evolution of the vector $\psi(t)$ in Hilbert space describes the movement of the electron. Since we chose the space representation in the Schrödinger picture, we can think of $\psi(t)$ as a wave packet which evolves in time. The square of its absolute value $|\psi(t,\cdot)|^2 \in L^1(\mathbb{R}^d)$ is a probability density. More precisely, $\int_A |\psi(t,x)|^2 dx$ is the probability to find the electron in the set $A \subset \mathbb{R}^d$ at time t. We will be only concerned with Hamiltonians H_ω which are time independent. In this case the solution to the equation (1.8) is given by $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi(0)$ where $\psi(0)$ denotes the initial condition at time 0.

For a given initial state $\psi(0)$ supported in a compact set $A \subset \mathbb{R}^d$ one would like to know whether for large times the function $\psi(t)$ stays (essentially) supported near A, or moves away to infinity. In the first case one speaks of a *bound state*, since it remains localised near its original support for all times. The other extreme case would be that $\psi(t)$ leaves any compact region in $\mathbb{R}^d$ (and never comes back) as time goes to infinity. Such a state is called a *scattering* or *extended state*. By the RAGE theorem it is possible to relate the dynamical properties of states just described to the spectral properties of the Hamiltonian. Roughly speaking, bound states correspond to pure point spectrum and scattering states to (absolutely) continuous spectrum.

For a more precise formulation we assume that the Schrödinger operator H satisfies the following local compactness property: If χ_B is the characteristic function of an arbitrary ball $B = B_R(x) \subset \mathbb{R}^d$ and $P(I)$ the spectral projection of H associated to a bounded interval $I \subset \mathbb{R}$, then the operator $\chi_R P(I)$ is compact. Under this condition a vector ψ is in the subspace associated to the continuous spectrum of H if and only if for arbitrary $B_R(x)$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} dt \int_{B_R(x)} |\psi(t,x)|^2 dx = 0 \tag{1.9}$$

Here we used the notation $\psi(t,x) = (\mathrm{e}^{-\mathrm{i}tH}\psi)(x)$. A vector ψ satisfying (1.9) is called a scattering state in time mean. Under the same local compactness assumption, a vector ψ is in the subspace associated to the pure point spectrum of H if and only if for any $\varepsilon > 0$ these exists a radius $R = R(\psi,\varepsilon)$ such that

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^d \setminus B_R(0)} |\psi(t,x)|^2 \leq \varepsilon \tag{1.10}$$

For a broader discussion consult for instance [409, 102, 458, 496].

The above relation to the dynamical properties of states motivates the systematic study of spectral features of the Schrödinger operators appearing in the time evolution equation (1.8). If a random Schrödinger operator exhibits almost surely only pure point spectrum in an energy region one speaks of *Anderson* or *spectral localisation*. The name goes back to Anderson's seminal work [20]. This property has been established for a variety of random models. In most of those cases one can additionally prove that the corresponding eigenfunctions decay exponentially in configuration space, a phenomenon

called *exponential (spectral) localisation.* The situation is different for random potentials with long range correlations, where sometimes only power-law decay of the eigenfunctions has been established [263, 170, 500].

If an energy interval contains almost surely only pure point spectrum, we call it *localisation interval.* An eigenfunction of H_ω which decays exponentially is called an *exponentially localised eigenstate.* The region or point in space where the localised state has its highest amplitude will be called *localisation centre* (we will not need a mathematically precise definition of this notion).

However, it turns out that the spectrum provides only a rough view on the dynamical properties of the quantum mechanical system. A more detailed understanding can be obtained by studying the time evolution of the moments of the position operators. This led to a formulation of several criteria of *dynamical localisation.* One possible characterisation of this phenomenon, namely *strong dynamical localisation in Hilbert-Schmidt* topology means that for all $q > 0$

$$\mathbb{E}\left\{\sup_{\|f\|_\infty \leq 1} \left\| |X|^{q/2} f(H_\omega) P_\omega(I) \chi_K \right\|_{HS}^2 \right\} < \infty \tag{1.11}$$

Here $P_\omega(I)$ denotes the spectral projection onto the energy interval I associated to the operator H_ω, $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm, $K \subset \mathbb{R}^d$ is any compact set, and $|X|$ denotes the operator of multiplication with the function $|x|$. For the interpretation of (1.11) as non-spreading of wave-packets one chooses $f(y) = \mathrm{e}^{-\mathrm{i}ty}$. Dynamical localisation (1.11) implies in particular that the random Hamiltonian H_ω exhibits spectral localisation in I. In [414] it was first pointed out that, in general, it is important to distinguish between dynamical and spectral localisation. The derivation of various forms of dynamical localisation using the multiscale analysis can be found in [183, 107, 186, 187, 188, 190], while a proof of the same fact based on the fractional moment method is the content of the papers [6, 60]. Actually, dynamical localisation has first been proven for discrete Schrödinger operators using the latter method in [9, 4, 7, 11]. In [414, 111, 235] examples are discussed where spectral localisation occurs, but certain dynamical criteria for localisation are not satisfied.

For the operators discussed in the present volume these distinctions are not crucial. In the case of the alloy type model, to which we devote most attention, spectral and dynamical localisation coincide, cf. [107, 190]. In the sequel we mean by localisation that the considered operator exhibits in a certain energy interval only pure point spectrum, and that the corresponding eigenfunctions decay sufficiently fast.

Since we are dealing not just with a single Hamiltonian, but with a whole family of them, we have to say something on how the spectral properties depend on the parameter ω describing the randomness: many properties of the spectrum of an operator pertaining to the family $\{H_\omega\}_\omega$ hold almost surely, i.e. for ω in a set such that its complement has measure zero in Ω. This is at