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Dynamical Systems

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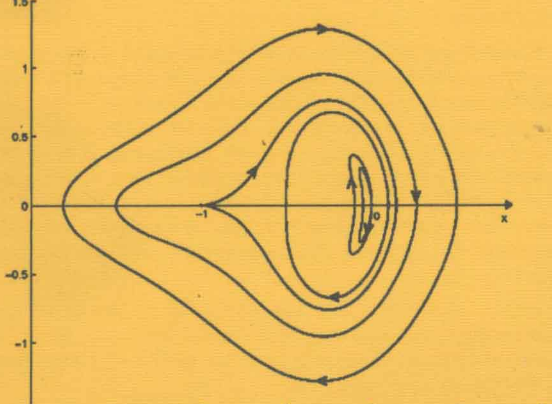

FIG. 11

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Preface

In June of 2000, a group of 39 mathematicians and graduate students met at the beautiful Hotel San Michele in Cetraro, Italy. The purpose of this CIME summer school was to study and investigate the latest results in certain areas of dynamical systems. The summer school was also dedicated to celebrating the career of Professor Roberto Conti of the Istituto G. Sansone of the Università di Firenze. Professor Conti, now in his 70's, has been a major force in developing young researchers working in dynamical systems. In addition, his own research has been of major importance in the field, in fact a series of lectures at this summer school was devoted to his recent joint work with Professor Marcello Galleotti.

This summer school necessarily focussed on a small number of the many important recent developments in dynamical systems. In each case the lecturers presented these developments in full historical context.

Professors Conti and Galleotti gave a series of lectures on the classical, extremely difficult, problem of classifying the orbits of systems of the form

$$x'(t) = X(x, y), \quad y'(t) = Y(x, y),$$

with X and Y polynomials. Such systems arise in applications when the complicated functions that represent the driving forces in a dynamical system are approximated by Taylor Polynomials. Given that X and Y are of a specific degree, a fundamental open problem in dynamical systems theory is to classify the possible phase space portraits (orbits). The possible orbits have been shown to be of amazing complexity. Professors Galleotti and Conti reported on the progress made for the case when X and Y are polynomials of degree 3, with some discussion of general results. The problem for degree 3 exhibits all the difficulties associated with higher degree polynomials. They showed how an elegant series of transformations and insights can reduce the system to one of a small number of canonical forms for which, with some assumptions, one can classify the possible orbits. Their work constitutes fundamental progress, and their lectures brought young researchers completely up to date in this challenging area.

The lectures of Professor Russell Johnson of the Università di Firenze dealt with the application of dynamical system methods to the study of periodic and quasi-periodic orbits for non-autonomous systems $x'(t) = f(t, x)$, $x(t)$ belonging to a finite or infinite dimensional vector space. Under very minimal hypotheses, the solutions of an *autonomous* differential equation define a local flow or semiflow. If the differential equation is *non-autonomous*, then the flow property does not hold, and so one has fewer tools available to study the solutions of such equations. However, if one considers a translation-invariant *family* F of non-autonomous differential equations, generated from a given non-autonomous equation, then (under mild hypotheses) the solutions of the equations do define a local flow or semiflow (in an extended space). If one takes account of the fact that a deterministic differential equation disturbed by a stationary real-noise process generates such a translation-invariant family, then one has ample motivation to develop a theory of such families.

Professor Johnson's lectures described how several tools from ergodic theory, including Lyapounov exponents and the Osedelec theorem, can be applied when the family F supports an invariant measure. If F carries a compact metric topology, then methods of topological dynamics such as rotation numbers and almost automorphy can be used to study various problems. In addition, the basic concept of exponential dichotomy - though initially defined for a single non-autonomous linear equation - can be adapted to study compact metric families F.
He showed how these tools can be used to study problems involving nonlinear parabolic PDEs with almost periodic time dependence, the qualitative theory of random ODEs, the theory of the random Schroedinger operator, and random bifurcation theory. In fact non-autonomous differential equations (especially those which depend periodically on time) can be studied using methods related to those mentioned above but without reference to any family F. Professor Johnson discussed how "dynamical" methods of this type can be used to investigate the ground-state problem for the scalar curvature equation, and the asymptotic behavior of solutions of the Navier-Stokes equations.

The lectures of Professors Shui-Nee Chow (Singapore National University and Georgia Tech) and John Mallet-Paret (Brown University) dealt with their recent work on the dynamics of *lattice differential equations* (LDE's), a field in which they are among the pioneers. These are systems of ordinary differential equations (with time as the independent variable) with a discrete structure in space. Such systems occur in electrical circuit theory, materials science, the theory of chemical reactions, image processing, and biology. The literature only goes back to about 1987, with the full mathematical development starting in the 1990's. Their lectures presented their comprehensive program to study

such systems from the point of view of, and using the tools of, dynamical systems.
Lattice differential equations are systems of ordinary differential equations, generally of infinite order, in which the system coordinates are parameterized by an underlying spatial lattice $\Lambda \subseteq \mathbb{R}^D$. One example is the discrete Allen-Cahn equation, obtained by coupling local nonlinear scalar dynamics with a nearest-neighbor spatial interaction. The one-dimensional system takes the form

$$\dot{u}_i = \alpha(u_{i+1} - 2u_i + u_{i-1}) - f(u_i), \qquad i \in \mathbb{Z}, \tag{1}$$

where typically f is a bistable nonlinearity, such as $f(u) = -u + u^3$, or $f(u) = (u^2 - 1)(u - a)$, or even a monostable function such as $f(u) = u + u^3$. The coupling coefficient α is real, and of either sign; in any case, the initial value problem for this equation is well posed in $l^\infty = l^\infty(\mathbb{Z})$. The PDE limit corresponds only to the limit $\alpha \to \infty$, but their interest is for general α. Analogous systems in other lattices, for example

$$\dot{u}_{i,j} = \alpha\Delta u_{i,j} - f(u_{i,j}), \qquad \Delta u_{i,j} = \sum_{|a-i|+|b-j|=1} [u_{a,b} - 4u_{i,j}], \qquad (i,j) \in \mathbb{Z}^2, \tag{2}$$

in the two-dimensional lattice $\mathbb{Z}^2$, can be considered. Here $\Delta = \Delta^+$ is a discrete Laplacian modeled on a + shaped stencil, although one could also model such an operator on other stencils, for example $\Delta^\times$ modeled on a $\times$ shaped stencil. One might also allow nonsymmetric couplings, and also nonlinear couplings, between lattice points. A general feature of higher-dimensional lattice systems is that the anisotropy and discrete symmetry of the lattice plays a role in the analysis, in contrast to the case of a PDE such as $u_t = \Delta u - f(u)$ in $\mathbb{R}^2$ which is rotationally and translation invariant.
Numerical simulations show that LDE's can exhibit a rich variety of dynamic phenomena, including pattern formation (e.g., spontaneous appearance of stripes, checks, and other figures, from randomly chosen initial conditions), traveling waves, and spatial chaos (spatially disordered patterns which are temporally stable). Their lectures covered:
1. Traveling Waves in Lattices .
2. The Fredholm Alternative Theorem for Nonlocal Equations.
3. Propagation Failure and the Effects of Anisotropy.
4. Stability and Perturbation of Traveling Waves.
As mentioned above, the theoretical discussions were complemented by presentations of extensive numerical simulations demonstrating the complex behavior of solutions to these systems.

Professor Roger Nussbaum of Rutgers University presented a series of lectures on the applications of fixed point theorems to dynamical systems theory. His lectures focussed on recent applications of the fixed point index (a generalization of the Leray-Schauder degree) to singular perturbation problems with delay. His lectures dealt with two topics of recent interest in the theory

of nonlinear differential-delay equations:
(a) asymptotic behaviour of solutions of differential-delay equations with state-dependent time lags as a certain singular parameter approaches 0 and
(b) existence and nonexistence of "super-high frequency solutions" of certain nonlinear, discontinuous, differential-delay equations.

The model equation for topic (a) was

$$\epsilon x'(t) = f(x(t), x(t-r)),\ r := r(x(t)), \tag{3}$$

where $\epsilon > 0$ and f and r are given functions. A simple-looking but non-trivial example of Equation (3) is given by

$$\epsilon x'(t) = -x(t) - kx(t-r),\ r := 1 + cx(t), \tag{4}$$

where $\epsilon > 0,\ k > 1$, and $c > 0$. His lectures focussed on how (under appropriate assumptions on f and r) one can prove for (3) and more general equations the existence of "slowly oscillating periodic solutions". A basic asymptotic problem then is this: Suppose that $\epsilon_j \to 0^+$, that x_j is a slowly oscillating periodic solution of (3) for $\epsilon = \epsilon_j$, and that $G_j \subset \mathbb{R}^2$ denotes the graph of x_j. If $(j_k : k \geq 1)$ is an appropriate subsequence, what can be said about the limiting shape Ω of the graphs G_{j_k} as $k \to \infty$? What can be said about the limiting shape of G_j as $j \to \infty$? The topology of convergence here is that of the Hausdorff metric on compact subsets of $\mathbb{R}^2$. He described the theory that has been developed to answer such questions, and in many cases one can explicitly determine a unique, nontrivial set Ω and prove that $G_j \to \Omega$ as $j \to \infty$. Questions involving the limiting profile Ω lead directly to some corresponding questions involving analogues of Krein-Rutman theory for nonlinear, noncompact maps of a closed cone in a Banach space into itself. He discussed such nonlinear Krein-Rutman theorems and mentioned other contexts in which they arise.

As a model equation for topic (b) Professor Nussbaum used

$$x'(t) = -sgn(x(t-1)) + f(x(t)). \tag{5}$$

Here $sgn(w) = 1$ if $w > 0$, $sgn(w) = -1$ if $w < 0$, and $sgn(0) = 0$; f is a Lipschitzian function with $|f(x)| < 1$ for all x. If $\theta : [-1, 0] \to \mathbb{R}$ is a given continuous function, there exists a unique, absolutely continuous function $x : [-1, \infty] \to \mathbb{R}$ such that $x|_{[-1,0]} = \theta$ and x satisfies Equation (5) for almost all $t \geq 0$. Assume either that θ is not identically zero or that $f(0) \neq 0$. It has been proved by Akian-Bliman and Nussbaum-Shustin that there exists $T = T_\theta < \infty$ such that for any interval $(t-1, t]$ with $t \geq T$ the function $x|_{(t-1,t]}$ has at most finitely many zeros. In this situation, we say that (5) has no super-high frequency solutions. For generalizations of Equation (5), questions about the existence of super-high frequency solutions remain open.

The approach of Nussbaum-Shustin leads to a class of operators which take the cone of nonnegative vectors K in $l_1(\mathbb{Z})$ into itself and which are nonexpansive with respect to the l_1 norm. He discussed this class of operators in some detail, and indicated the connection to Equation (5) and why these operators are of interest in their own right.

Virtually all participants attended all of the scheduled lectures and participated actively in the evening seminars which were held each day. The scientific atmosphere was invigorating and led to several new potential collaborations. We would like to express our appreciation to CIME for sponsoring this exciting and worthwhile summer school. We also wish to express our appreciation to the Hotel San Michele for being such an excellent host. Finally, we thank our speakers, who made the conference an outstanding experience for all participants.

Jack Macki and Pietro Zecca
University of Alberta and Università di Firenze

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Contents

Lattice Dynamical Systems

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Introduction

The following notes are based on my lectures given at CIME Session on "Dynamical Systems" from June 19 to June 26, 2000 in Cetraro (Cosenza). In Section 1, we study spatially discrete nonlinear diffusion equations and discuss various phenomenon; in Section 2, we introduce the idea of spatial chaos by studying Nagumo equation; in Section 3, 4 and 5, we present a general theory for pattern formation and spatial chaos in lattice dynamics systems; finally in Section 6 and 7, we study the special case of synchronisation phenomena of lattice systems.

I would like to express my thanks to Professors P. Zecca and J. Macki and the host of the conference for inviting me to participate in this conference and making the publication of this note possible. I would also like to express my gratitude to Ms G. M. Wu, without whose help these notes would not have been completed. Mimmie, thank you for your patience and hard work.

1 Spatially Discrete Nonlinear Diffusion Equations

1.1 Introduction

In this section we consider spatially discrete nonlinear diffusion equations that occur as models for binary alloys. These equations are spatially discrete and are not space discretized partial differential equations. In fact, for some of the parameter values that we consider there may not exist a well-posed PDE even in a weak sense. The differential equations we consider are analogous in form to the Cahn-Hilliard equation (see [27]) and the Cahn-Allen equation (see [16]). The Cahn-Hilliard equation with Neumann boundary condition is given by

$$\begin{aligned} u_t &= \Delta(f(u) - \varepsilon\Delta u), & x \in \Omega \\ n \cdot \nabla u &= n \cdot \nabla \Delta u = 0, & x \in \partial\Omega \end{aligned} \tag{1}$$

for $\varepsilon > 0$ where $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$ is a bounded domain and f is a "cubic" nonlinearity, typically $f(u) = u^3 - u$. The Cahn-Allen equation with Neumann boundary conditions is given by

$$\begin{aligned} u_t &= -f(u) + \varepsilon\Delta u, & x \in \Omega \\ n \cdot \nabla u &= 0, & x \in \partial\Omega \ . \end{aligned} \tag{2}$$

The Cahn-Hilliard equation models the evolution of a binary alloy after it has been quenched to a constant temperature. The Cahn-Allen equation models the motion of the interface between two phases of a binary alloy.

Spatially discrete equations have long been considered in the material sciences (see [37], [48]). The model considered by Hilbert in [48] is a one-dimensional model and allows for order-disorder and spinodal decomposition. There is no restriction on the amplitude of the composition in Hilbert's model. Our models are for subsets of one- and two- lattices. We have observed spinodal decomposition, order-disorder, twinning and the coexistence of up to three distinct phases. We will present high amplitude equilibrium solutions for certain model equations.

We will show that spatially discrete diffusion equations on a finite subset of a lattice can be analyzed in terms of a variational calculus and that the systems we consider possess a gradient structure and hence have a global attractor. Our initial task is to set up a discrete variational calculus for discrete nonlinear diffusion equations. Using the fact that these are gradient systems we are able to prove that there exists a compact, connected invariant set for a large range of parameter values that includes the case in which there is no continuum limit PDE. We present a numerical method that we have implemented on a massively parallel SIMD machine. Using this efficient algorithm, we are able to exhibit the pattern formation that occurs for a wide range of parameter values.

1.2 Spatially Discrete Models

In this section we consider a general class of spatially discrete nonlinear diffusion equations on integer valued subsets of lattices in one- and two- dimensions. We then consider specific equations that occur as models for the evolution of binary alloys. We define what we mean by a subset of a lattice and its boundary with respect to discrete Laplacian type operators. In this way we are able to develop a variational calculus for spatially discrete diffusion equations using a discrete Green's formula that is easy to derive by considering one-dimensional summation by parts formulas. A general form for equations is given that, although spatially discrete, are of reaction diffusion and Cahn-Hilliard type. With a boundary-sum boundary condition, the spatially discrete Cahn-Hilliard equations conserve mass. For rectangular boundaries it is shown that these boundary-sum conditions are satisfied with the discrete analog of periodic and Neumann boundary conditions.

Given a $\mathbb{Z}^N$ module for $N = 1, 2, 3$ we wish to construct a finite subset, L, of this lattice. For $N = 1, 2, 3$ we have that $\eta \in L$ is of the form $\eta = (i), \eta = (i, j), \eta = (i, j, k)$, respectively, where i, j, k are integers. First we construct a subgroup of $\mathbb{Z}^N$ using a set $\{\eta_k\}_{k \in D}$ of translations (see [22]) where D is an indexing set and $\eta_k \in \mathbb{Z}^N$ act as generators. An example of such a subgroup of $\mathbb{Z}^3$ would be the fact-centered cell where the η_k are all permutations of (1). Here the translations in the subgroup are those $\eta = (i, j, k)$ for which $i + j + k$ is even and the size of face-centered unit cell is double that of the unit cell for $\mathbb{Z}^3$. This allows us to retain integer components for centered cells. We may also wish to consider the lattice complexes that are formed by the complement of a subgroup or the union of some of the cosets of the subgroup with respect to $\mathbb{Z}^N$. We specify L to be the points in a finite subset of this lattice or lattice complex, i.e., a collection of N-tuples with integer components, a finite subset of $\mathbb{Z}^N$.

We now define the boundary and interior of L with respect to the set of translations $\{\eta_k\}_{k \in D}$. A point $\eta \in L$ is in the interior of L if $\eta \pm \eta_k \in L$ for all the translations η_k. A point $\eta \in \partial L^-_{\eta_k}$, the negative boundary with respect to the translation η_k, if $\eta \in L$ and $\eta - \eta_k \notin L$. Similarly, a point $\eta \in \partial L^+_{\eta_k}$, the positive boundary with respect to η_k, if $\eta \in L$ and $\eta + \eta_k \notin L$. We define the boundary of L to be those $\eta \in L$ such that either $\eta \in \partial L^-_{\eta_k}$ or $\eta \in \partial L^+_{\eta_k}$ for some translation η_k.

For a one-dimensional discrete Laplacian, we have the following summation by parts formula.

Theorem 1.1 (Summation by parts). *Given a positive integer M and sequences $\{v(i)\}_0^{M+1}$ and $\{w(i)\}_0^{M+1}$, we have*

$$\sum_{i=1}^{M} \{\Delta v(i) \cdot w(i) + \nabla v(i) \cdot \nabla w(i)\} = -\nabla v(0) \cdot w(1) + \nabla v(M) \cdot w(M+1), \quad (3)$$

where $\Delta v(i) = v(i+1) - 2v(i) + v(i-1)$ and $\nabla v(i) = v(i+1) - v(i)$.

We define higher dimensional discrete Laplacians as the sum of one-dimensional discrete Laplacians using the set of translations $\{\eta_k\}_{k \in D}$ as follows

$$\Delta_i u(\eta) = \sum_{k \in D_i} \{u(\eta - \eta_k) - 2u(\eta) + u(\eta + \eta_k)\}, \quad (4)$$

where $\{\eta_k\}_{k \in D_i}$ is the set of crystallographically equivalent translations. For point groups with little symmetry, each D_i may contain only one element, while for point groups with a large degree of symmetry, the corresponding D_i will contain several elements.

Using the summation of parts formula (3) and the construction (4) of higher dimensional discrete Laplacian type operators Δ_i, we have the following discrete Green's formula

Theorem 1.2 (Green's Formula). .

$$\begin{aligned}
&\sum_{\eta\in L}\Delta_i v(\eta)\cdot w(\eta)+\sum_{\eta\in L}\nabla_i v(\eta)\cdot\nabla_i w(\eta) \qquad (5)\\
&=\sum_{k\in D_i}\Big\{\sum_{\eta\in L}(v(\eta-\eta_k)-2v(\eta)+v(\eta+\eta_k))\cdot w(\eta)\\
&\qquad +(v(\eta+\eta_k)-v(\eta))\cdot(w(\eta+\eta_k)-w(\eta))\Big\}\\
&=\sum_{k\in D_i}\Big\{\sum_{\eta\in\partial L^-_{\eta_k}}-(v(\eta)-v(\eta-\eta_k))\cdot w(\eta)\\
&\qquad +\sum_{\eta\in\partial L^+_{\eta_k}}(v((\eta+\eta_k)-v(\eta))\cdot w(\eta+\eta_k)\Big\}\ .
\end{aligned}$$

Given L, we consider differential equations that take on a value at each point $\eta\in L$. The differential equations are similar in form to the Cahn-Hilliard equation and the corresponding Cahn-Allen equation. For $\eta\in L$, what we will call the Spatially Discrete Cahn-Hilliard equation (SDCH) has the form

$$\dot u(\eta,t)=\Delta_B(f(u(\eta,t))-\Delta_A u(\eta,t)), \tag{6}$$

where $\dot u=du/dt, f(u(\eta,t))=\log((l+u(\eta,t))/(l-u(\eta,t)))+\sigma u(\eta,t)$ and $\Delta_A=\sum_{i\in I_A}\alpha_i\Delta_i, \Delta_B=\sum_{i\in I_B}\beta_i\Delta_i$ where I_A and I_B are indexing sets. Note that for $\sigma<-2$, $f(x)$ has three unique real roots. In what follows, we will assume that the σ_i and β_i are real constants, but $\sigma_i\equiv\sigma_i(\eta)$ or $\beta_i\equiv\beta_i(\eta)$, the weights as functions of position, may also be useful in some applications. For $\eta\in L$, the Spatially Discrete Cahn-Allen equation (SDCA) has the form

$$\dot u(\eta,t)=\Delta_A u(\eta,t)-f(u(\eta,t))\ . \tag{7}$$

For both equations, we consider the following boundary conditions

$$\begin{aligned}
&\sum_{i\in I_A}\alpha_i\sum_{k\in D_i}\Big\{-\sum_{\eta\in\partial L^-_{\eta_k}}(u(\eta,t)-u(\eta-\eta_k,t))\\
&\qquad+\sum_{\eta\in\partial L^+_{\eta_k}}(u(\eta+\eta_k,t)-(u(\eta,t))\Big\}=0\ . \qquad (8)
\end{aligned}$$

Additionally, for the SDCH equation, we employ the boundary condition