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Mathematical Analysis

MATHEMATICAL ANALYSIS

A Modern Approach to Advanced Calculus

by

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ADDISON-WESLEY PUBLISHING COMPANY, INC.

READING, MASSACHUSETTS, U.S.A.

PREFACE

A glance at the Table of Contents will reveal that most of the topics which usually fall under the heading "Advanced Calculus" are treated in this book. The author's aim has been to provide a development of the subject matter which is honest, rigorous, up-to-date, and, at the same time, not too pedantic. Most of the "hard" theorems which are either omitted or treated rather skimpily in many texts on advanced calculus are proved here with great care. Some of them are ordinarily considered too difficult for a standard course in advanced calculus but too elementary for a course in real or complex function theory. With the inclusion of these theorems, the book helps to fill the gap between elementary calculus and advanced courses in analysis. More important than this, it introduces the reader to some of the abstract thinking that pervades modern mathematics.

Some of the features to be found in the book include:

A chapter on abstract set theory which contains a precise formulation of the function concept.

Point set topology in n-dimensional Euclidean space.

A satisfactory treatment of differentials.

An elementary discussion of connectedness.

An extensive treatment of the Riemann-Stieltjes integral, including complex integration and a discussion of the winding number.

A development of Jordan content and outer Lebesgue measure.

A proof of Green's theorem for plane regions bounded by arbitrary rectifiable Jordan curves.

A careful treatment of surfaces and surface integrals.

A thorough discussion of the interchange of limit processes.

A chapter on Fourier series and Fourier integrals, including the Fourier integral theorem, the convolution theorem for Fourier transforms, and the complex inversion formula for Laplace transforms.

An introduction to the theory of functions of a complex variable.

There is ample material here for a year's course, but many parts can easily be omitted without disturbing the continuity of the presentation. Nearly 500 exercises are included; many of these are designed to illustrate the general theory or to show where it can break down.

For various reasons, one of which is simply lack of space, there is a minimum of emphasis on applications and physical motivation in this book. It is a fairly easy matter for a lecturer to give a leisurely heuristic discussion that motivates a difficult concept, but in many instances the very same discussion may appear somewhat ridiculous when set down in print. Furthermore, the approach to motivation is largely a matter of personal taste and hence is best provided in the form of classroom lectures as the individual instructor sees fit to introduce it.

A clear understanding of the basic concepts of calculus is indispensable for anyone who wishes to learn about more advanced concepts in analysis. Therefore it is hoped that this book, although designed primarily for mathematicians, may also be of interest to students of the physical and engineering sciences.

While preparing the manuscript the author has had the best assistance one could wish for. He is particularly grateful to Professors Paul R. Halmos of the University of Chicago and M. E. Munroe of the University of Illinois. Their thorough criticism of the preliminary version of the manuscript had a profound effect on the final form of the book. Special thanks are also due to Dr. Basil Gordon of the California Institute of Technology, who read the final draft and pointed out a number of errors. Through the generosity of the California Institute, the expert services of Miss Rosemarie Stampfel were made available for typing of the manuscript. The author also owes much to the students of Caltech, who provided the original incentive for this work.

Finally, thanks are due to the Addison-Wesley Publishing Company; first, for their sincere interest in producing a book of this kind, and second, for the excellent help and cooperation which they have constantly proffered.

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January, 1957

CONTENTS

CHAPTER 1

THE REAL AND COMPLEX NUMBER SYSTEMS

1–1 Introduction. The real number system is one of the fundamental concepts of mathematics. A thorough and exhaustive study of mathematical analysis would have to include a careful definition of what is meant by a real number, a discussion of how real numbers are constructed (starting, for example, with the integers), and a derivation of the principal properties of real numbers. Although these elements form a very interesting part of the foundations of mathematics, they will not be treated in detail here. As a matter of fact, in most phases of analysis it is only the *properties* of real numbers that concern us, rather than the methods used to construct the real number system. Therefore we shall simply list a set of axioms from which all the properties of real numbers can be derived. For discussions of the methods used to construct real numbers the reader should consult the references at the end of this chapter.

We shall assume that the reader is familiar with most of the properties of real numbers discussed in the next few pages, as well as some of their elementary consequences. We shall also assume that the reader has some knowledge of the elementary functions of calculus, such as the trigonometric, exponential, and logarithmic functions. Although infinite series will be treated in detail in a later chapter, a few basic facts about series (with which the reader is probably already familiar from his study of elementary calculus) will be used in the early part of this chapter.

1–2 Arithmetical properties of real numbers. Given any two real numbers x and y, we can form their sum $x + y$ and their product xy, and these satisfy the following axioms:

AXIOM 1. $x + y = y + x,\ xy = yx$ (*commutative laws*).

AXIOM 2. $x + (y + z) = (x + y) + z,$
$x(yz) = (xy)z$ (*associative laws*).

AXIOM 3. $x(y + z) = xy + xz$ (*distributive law*).

AXIOM 4. Given any two real numbers x and y, there exists a real number z such that $x + z = y$. This z is denoted by $y - x$; the number $x - x$ is denoted by 0. (It can be proved that 0 is independent of x.) We write $-x$ for $0 - x$.

AXIOM 5. There exists at least one real number $x \neq 0$. If x and y are two real numbers with $x \neq 0$, then there exists a real number z such that $xz = y$. This z is denoted by y/x; the number x/x is denoted by 1 and can be shown to be independent of x. We write x^{-1} for $1/x$ if $x \neq 0$.

From these axioms we can derive all the usual laws of arithmetic; for example, $-(-x) = x$, $(x^{-1})^{-1} = x$, $-(x - y) = y - x$, $x - y = x + (-y)$, etc. (For a more detailed explanation see Ref. 1–4.)

1–3 Order properties of real numbers. We also have a relation $<$ which establishes an ordering among the real numbers and which satisfies the following axioms:

AXIOM 6. Exactly one of the relations $x = y$, $x < y$, $x > y$ holds.

NOTE. $x > y$ means the same thing as $y < x$.

AXIOM 7. If $x < y$, then for every z we have $x + z < y + z$.

AXIOM 8. If $x > 0$ and $y > 0$, then $xy > 0$.

AXIOM 9. If $x > y$ and $y > z$, then $x > z$.

From these axioms we can derive the usual rules for operating with inequalities. For example, if we have $x < y$, then $xz < yz$ if $z > 0$, whereas $xz > yz$ if $z < 0$. Also, if $x > y > 0$ and $z > w > 0$, then $xz > yw$. (For a complete discussion of these rules see Ref. 1–1.) A tenth axiom is given in Section 1–9.

NOTE. The symbolism $x \leq y$ is used as an abbreviation for the statement: "$x < y$ or $x = y$." Thus we have $2 \leq 3$ since $2 < 3$, and $2 \leq 2$ since $2 = 2$. The symbol $\geq$ is similarly used.

1–4 Geometrical representation of real numbers. The real numbers can be represented geometrically as points on a line (the *real axis*). A point is selected to represent 0 and another point to represent 1, and these points determine the scale. Then each point on the real axis corresponds to one and only one real number and, conversely, each real number is represented by a single point.

1–5 Decimal representation of real numbers. Every real number x has a decimal expansion of the form

$$x = \pm N \cdot a_1 a_2 \cdots a_n \cdots,$$

where N is either 0 or a positive integer and each a_i is one of the digits from 0 to 9. The notation $N \cdot a_1a_2 \cdots a_n \cdots$ is really an abbreviation for the infinite series

$$N + a_1(10)^{-1} + a_2(10)^{-2} + \cdots + a_n(10)^{-n} + \cdots.$$

When N is 0 and all the a_i are equal (say, to a), then the infinite series becomes a geometric series with ratio 1/10 and sum

$$a\left(\frac{1/10}{1 - 1/10}\right) = \frac{a}{9}.$$

If $a = 9$, this series has the sum 1, which means that the number 1 has the *two* decimal expansions

$$1 = 1.000... = 0.999....$$

More generally, if a number has a decimal expansion which ends in zeros, for example, $1/8 = 0.125000...$, then this number can also be written as a decimal expansion which ends in nines by decreasing the last nonzero digit by one unit. Thus we have $1/8 = 0.124999....$ Except for situations like this, decimal expansions are unique.

1–6 Rational numbers. Real numbers are classified into two main types, rational and irrational. Rational numbers are those real numbers which are the quotient of two integers; for example, 1/2, 7/5, −5. Irrational numbers are those real numbers which are not rational; for example, π, $\sqrt{2}$, e. Rational and irrational numbers can also be distinguished by their decimal representations. A rational number has a decimal expansion which eventually repeats itself: $1/2 = 0.5\underline{0}$, $2/3 = 0.\underline{6}$, $1/7 = 0.\underline{142858}$, $13/22 = 0.59\underline{09}$, $1/8 = 0.124\underline{9}$. That portion of the decimal which is underscored is understood to be repeated indefinitely in each case. (The reader should have no difficulty in showing that such repetitions must always occur when a rational number is written in its decimal expansion.) Conversely, every repeating decimal represents a rational number. This fact is not so obvious, but it can easily be proved by observing that a decimal which repeats forms a geometric series of the form

$$a + b(1 + r + r^2 + \cdots + r^n + \cdots),$$

where a, b, and r are rational, r being some power of 1/10. Such a series has the sum $a + b/(1 - r)$, which is rational. As a case in point, take the repeating decimal $x = 0.3\underline{14}$. We can write this out as a series as follows:

$$\begin{aligned} x &= \frac{3}{10} + 14(10^{-3}) + 14(10^{-5}) + 14(10^{-7}) + \cdots \\ &= \frac{3}{10} + 14(10^{-3})[1 + (10^{-2}) + (10^{-4}) + \cdots] \\ &= \frac{3}{10} + 14(10^{-3})\left[\frac{100}{99}\right] = \frac{311}{990}. \end{aligned}$$

Given two rational numbers, say a and b, their average $(a + b)/2$ is also rational. Hence between any two rational numbers there lies another rational number. It then follows that between any two rational numbers there must be *infinitely many* rational numbers [Why?], which implies that if we are given a certain rational number x, we cannot speak of the "next largest" rational number.

1-7 Some irrational numbers. Ordinarily it is not too easy to show that some particular number is irrational. There is no simple proof, for example, of the irrationality of e^{π}. However, the irrationality of certain numbers such as $\sqrt{2}$ and $\sqrt{3}$ is not too difficult to establish and, in fact, we can easily prove the following:

1-1 THEOREM. *If n is an integer which is not a perfect square, then $\sqrt{n}$ is irrational.*

Proof. Suppose first that n contains no square factor >1. We assume that $\sqrt{n}$ is rational and obtain a contradiction. Let $\sqrt{n} = a/b$, where a and b are integers having no factor in common. Then $nb^2 = a^2$ and, since the left side of this equation is a multiple of n, so too is a^2. However, if a^2 is a multiple of n, a itself must be a multiple of n, since n has no square factors >1. (This is easily seen by examining the factorization of a into its prime factors.) This means that $a = cn$, where c is some integer. Then the equation $nb^2 = a^2$ becomes $nb^2 = c^2n^2$, or $b^2 = nc^2$. The same argument shows that b must also be a multiple of n. Thus a and b are both multiples of n, which contradicts the fact that they have no factor in common. This completes the proof if n has no square factor >1.

If n has a square factor, we can write $n = m^2k$, where $k > 1$ and k has no square factor >1. Then $\sqrt{n} = m\sqrt{k}$; and if $\sqrt{n}$ were rational, the number $\sqrt{k}$ would also be rational, contradicting that which was just proved.

A different type of argument is needed to prove that the number e is irrational.

1-2 THEOREM. *If* $e^x = 1 + x + x^2/2! + x^3/3! + \cdots + x^n/n! + \cdots$, *then the number e is irrational.*

Proof. We shall prove that e^{-1} is irrational. The series for e^{-1} is an alternating series with terms which decrease steadily in absolute value. In such an alternating series the error made by stopping at the nth term has the algebraic sign of the first neglected term and is less in absolute value than the first neglected term. Hence, if $s_n = \sum_{k=0}^{n} (-1)^k/k!$, we have the inequality

$$0 < e^{-1} - s_{2k-1} < \frac{1}{(2k)!},$$

from which we obtain

$$0 < (2k-1)!\,(e^{-1} - s_{2k-1}) < \frac{1}{2k} \leq \frac{1}{2}$$

for any integer $k \geq 1$. Now $(2k-1)!\,s_{2k-1}$ is always an integer. If e^{-1} were rational, then we could choose k so large that $(2k-1)!\,e^{-1}$ would also be an integer. But the last inequality says that the difference of these two integers would be a number between 0 and $\frac{1}{2}$, which is impossible. Thus e^{-1} cannot be rational, and hence e cannot be rational.

The ancient Greeks were aware of the existence of irrational numbers as early as 500 B.C. However, a satisfactory theory of such numbers was not developed until late in the nineteenth century, at which time three different theories were introduced by Cantor, Dedekind, and Weierstrass. For an account of the theories of Dedekind and Cantor and their equivalence, see Chapter I of E. W. HOBSON, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series.* Vol. 1, 3rd ed. Cambridge: University Press, 1927.

1-3 Some fundamental inequalities. Calculations with inequalities arise quite frequently in analysis. They are of particular importance in dealing with the notion of absolute value. If x is any real number, then we define the absolute value of x, denoted by $|x|$, as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

A fundamental inequality concerning absolute values is given in the following:

1-3 THEOREM. *If* $a \geq 0$, *then we have the inequality* $|x| \leq a$ *if, and only if,* $-a \leq x \leq a$.

Proof. From the definition of $|x|$, we have the inequality $-|x| \leq x \leq |x|$, since $x = |x|$ or $x = -|x|$. If we assume that $|x| \leq a$, then we can write $-a \leq -|x| \leq x \leq |x| \leq a$ and thus half of the theorem is proved. Conversely, let us assume $-a \leq x \leq a$. Then if $x > 0$, we have $|x| = x \leq a$, whereas if $x < 0$, we have $|x| = -x \leq a$. In either case we have $|x| \leq a$ and the theorem is proved.

As an important consequence we have

1-4 THEOREM. $|x + y| \leq |x| + |y|$.

Proof. We have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Addition gives us $-(|x| + |y|) \leq x + y \leq |x| + |y|$, and from Theorem 1-3 we conclude that $|x + y| \leq |x| + |y|$.

Since $|-x| = |x|$, Theorem 1-4 also yields the inequality

$$|x - y| \leq |x| + |y|.$$

Replacing x by $y - x$, this can also be written in the form

$$|x - y| \geq |x| - |y|,$$

and by induction, we can prove the generalizations

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

and

$$|x_1 + x_2 + \cdots + x_n| \geq |x_1| - |x_2| - \cdots - |x_n|.$$

We shall now derive a very useful result known as the *Cauchy-Schwarz inequality.*

1-5 THEOREM. (*Cauchy-Schwarz inequality*). *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are arbitrary real numbers, we have*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

Proof. A sum of squares can never be negative. Hence we have

$$\sum_{k=1}^{n} (a_k x + b_k)^2 \geq 0$$

for every real x. This inequality can be written in the form

$$Ax^2 + 2Bx + C \geq 0,$$

where

$$A = \sum_{k=1}^{n} a_k^2, \qquad B = \sum_{k=1}^{n} a_k b_k, \qquad C = \sum_{k=1}^{n} b_k^2.$$

If $A > 0$, put $x = -B/A$ to obtain $B^2 - AC \leq 0$, which is the desired inequality. If $A = 0$, the proof is trivial.

NOTE. For an alternative proof of this theorem, see Exercise 1–15.

1–9 Infimum and supremum. Let S denote a collection of real numbers. The notation $x \in S$ means that the real number x is in the collection S, and we write $x \notin S$ to indicate that x is not in S.

The notation $\{x \mid x \text{ satisfies } P\}$ will be used to designate the collection of all real numbers x which satisfy the property P.

1–6 DEFINITION. *Let A be a set of real numbers. If there is a real number x such that $a \in A$ implies $a \leq x$, then x is called an upper bound for the set A and we say that A is bounded above.* [Lower bound is similarly defined.]

1–7 DEFINITION. *Let A be a set of real numbers bounded above. Suppose there is a real number x satisfying the following two conditions:*

(i) *x is an upper bound for A, and*

(ii) *if y is any upper bound for A, then $x \leq y$.*

Such a number x is called a least upper bound, or a supremum, of the set A. [The abbreviation lub is used for least upper bound and the abbreviation sup is used for supremum. The concept of greatest lower bound (glb), or infimum (inf), is similarly defined if A is bounded below.*]

It is easy to see that the sup and inf of a set are uniquely determined, whenever they exist (see Exercise 1–10).

Our final axiom for the real number system involves the notion of supremum:

AXIOM 10. If A is a nonempty set of real numbers which is bounded above, then A has a supremum.

* We shall use the term "supremum" rather than "least upper bound" and the term "infimum" rather than "greatest lower bound."