Lecture Notes in Mathematics
1631

Kumiko Nishioka

Mahler Functions and Transcendence

Springer

Kumiko Nishioka

Mahler Functions and Transcendence

Springer

Author

Kumiko Nishioka
Mathematics
Keio University
Hiyoshi Campus
4-1-1 Hiyoshi Kohokuku
233 Yokohama, JAPAN

Cataloging-in-Publication data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Nishioka, Kumiko:
Mahler functions and transcendence / Kumiko Nishioka. -
Berlin ; Heidelberg ; New York ; Barcelona ; Budapest ; Hong
Kong ; London ; Milan ; Paris ; Santa Clara ; Singapore ;
Tokyo : Springer, 1996
(Lecture notes in mathematics ; 1631)
ISBN 3-540-61472-9
NE: GT

Mathematics Subject Classification (1991): 11J81

ISSN 0075-8434
ISBN 3-540-61472-9 Springer-Verlag Berlin Heidelberg New York

Typesetting: Camera-ready TEX output by the author
SPIN: 10479803 46/3142-543210 - Printed on acid-free paper

Preface

The present notes are based on the lectures the author gave at Keio University in 1989 and 1993. Recently the transcendence theory of Mahler functions has seen profound development and has found a diversity of applications. This volume is the first comprehensive treatise on the subject. The author hopes that it will be a source of further research.

A Mahler function, for example, of one variable is a function which satisfies a functional equation under the transformation $z \to z^d$, where d is an integer greater than unity. The study of transcendence and algebraic independence of the values of those functions were started by Mahler's three papers in 1929, 1930. After a gap of about fifty years, it was again investigated by Kubota, Loxton, van der Poorten and the author. Especially Masser's vanishing theorem in 1982 gave a complete solution to a problem of Mahler which is important for the study of the values of Mahler functions of several variables. Next the present author applied elimination-theoretic method by Nesterenko and Philippon to Mahler functions to obtain a general algebraic independence result and a zero-order estimate. Amou, Becker and Töpfer followed this approach. Very recently Barré-Sirieix, Diaz, Gramain and Philibert proved the transcendence of $J(q) = j(\log q/2\pi i)$ for algebraic q, where $j(w)$ is the modular invariant, which had been conjectured by Mahler.

Chapter 1 is concerned with transcendence of Mahler functions of one variable and their values. After some preliminaries algebraic functional equations are treated and an application to the Mandelbrot set by Becker and Bergweiler is given but the proof of the transcendence of $J(q)$ above is not included. Chapter 2 is mainly devoted to the proof of Masser's vanishing theorem. Here we present a proof given by the author, which is based on p-adic methods and simpler than Masser's original proof. Chapter 3 is on algebraic independence of Mahler functions and their values. Generalizations of Mahler's method by Kubota, Loxton, van der Poorten and the author are exposed. Chapter 4 contains Nesterenko's lemmas without proofs and their applications to Mahler functions. Only basic results are proved to clarify the idea. Chapter 5 is concerned with the connection between regular sequences and Mahler functions. Some examples are treated.

The author would like to express her heartfelt gratitude to Professor Iekata Shiokawa for his encouragement to publish this note; her husband

Professor Keiji Nishioka for giving her much valuable advice on the properties of functions; graduate students of Keio University, Mr. Taka-aki Tanaka, Mr. Yoshihisa Uchida and Ms. Hiromi Kimura for checking the text; Professor Yuji Ito for looking through the manuscript.

November 1995 Kumiko Nishioka

Contents

5 Regular sequences and Mahler functions

Chapter 1

Transcendence theory of Mahler functions of one variable

1.1 Introduction and preliminaries

We denote by **Q** and by **Z** the field of rational numbers and the ring of integers respectively. An *algebraic number* is one that satisfies some equation of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

with rational coefficients. A polynomial having leading coefficient 1 is called *monic*. If an algebraic number α satisfies some monic polynomial equation with rational integral coefficients, we say that α is an *algebraic integer*. A complex number that is not algebraic is called *transcendental*. The set of the algebraic numbers which is denoted by $\overline{\mathbf{Q}}$ is a field and the set of the algebraic integers which is denoted by **I** is a ring. We know that $\mathbf{I} \cap \mathbf{Q} = \mathbf{Z}$. In 1844, Liouville gave the first example of transcendental numbers, which was $\sum_{k=1}^{\infty} 10^{-k!}$. Here we shall prove this.

First we give some notations. Let α be an algebraic number. The algebraic number α satisfies a unique monic polynomial equation of least degree, which is called the *minimal polynomial* of α. We denote the degree of it by $\deg(\alpha)$. By $\overline{|\alpha|}$ and $\mathrm{den}(\alpha)$ we denote respectively the maximum of the absolute values of α and its conjugates and the least positive integer d such that $d\alpha \in \mathbf{I}$, that is

$$\overline{|\alpha|} = \max\{|\alpha^{\sigma}| \mid \sigma \in \mathrm{Aut}(\overline{\mathbf{Q}}/\mathbf{Q})\}$$

$$\mathrm{den}(\alpha) = \min\{d \in \mathbf{Z} \mid d > 0,\ d\alpha \in \mathbf{I}\}.$$

It is easily seen that

$$\overline{|\alpha+\beta|} \le \overline{|\alpha|} + \overline{|\beta|}, \quad \overline{|\alpha\beta|} \le \overline{|\alpha|}\,\overline{|\beta|},$$

and

$$d\alpha,\ d\beta \in \mathbf{I} \implies d(\alpha+\beta),\ d^2\alpha\beta \in \mathbf{I}.$$

The following is fundamental throughout this note.

Fundamental inequality (Liouville inequality). If α is a nonzero algebraic number with $\deg(\alpha) = n$, then

$$\log|\alpha| \ge -2n \max\{\log\overline{|\alpha|},\ \log \mathrm{den}(\alpha)\}.$$

Proof. Letting $d = \mathrm{den}(\alpha)$, we have $d\alpha \in \mathbf{I}$ and so

$$0 \ne N_{\mathbf{Q}(\alpha)/\mathbf{Q}}(d\alpha) \in \mathbf{I} \cap \mathbf{Q} = \mathbf{Z}.$$

Hence we have

$$1 \le |N_{\mathbf{Q}(\alpha)/\mathbf{Q}}(d\alpha)| \le d^n|\alpha|\,\overline{|\alpha|}^{n-1}.$$

Therefore

$$\begin{aligned} \log|\alpha| &\ge (1-n)\log\overline{|\alpha|} - n\log d \\ &\ge -2n\max\{\log\overline{|\alpha|},\ \log d\}. \end{aligned}$$

Theorem 1.1.1. If α is an algebraic number with $0 < |\alpha| < 1$, then $\sum_{k=1}^{\infty} \alpha^{k!}$ is transcendental.

Proof. Let $f(z) = \sum_{k=1}^{\infty} z^{k!}$ and suppose that $f(\alpha)$ is an algebraic number. Put $\gamma_m = f(\alpha) - \sum_{k=1}^{m-1} \alpha^{k!}$. Then we have $\gamma_m \in \mathbf{Q}(\alpha, f(\alpha))$ and

$$\gamma_m = \sum_{k=m}^{\infty} \alpha^{k!} = \alpha^{m!} + o\left(|\alpha|^{m!}\right).$$

Therefore, $\gamma_m \ne 0$ and $|\gamma_m| < c_1|\alpha|^{m!}$ if m is sufficiently large (in what follows, $c_1, c_2, \ldots$ denote positive constants depending on α but independent of m). On the other hand,

$$\overline{|\gamma_m|} \le \overline{|f(\alpha)|} + m\max\{1,\ \overline{|\alpha|}\}^{(m-1)!} \le c_2^{(m-1)!}$$

and

$$\operatorname{den}(\gamma_m) \le \operatorname{den}(f(\alpha))\operatorname{den}(\alpha)^{(m-1)!} \le c_3^{(m-1)!}.$$

By the Fundamental inequality, we obtain

$$\begin{aligned}\log c_1 + m! \log|\alpha| &\ge \log|\gamma_m| \\ &\ge -2[\mathbf{Q}(\alpha, f(\alpha)) : \mathbf{Q}] \max\{\log \overline{|\gamma_m|}, \log \operatorname{den}(\gamma_m)\} \\ &\ge -2[\mathbf{Q}(\alpha, f(\alpha)) : \mathbf{Q}](m-1)! c_4.\end{aligned}$$

Dividing both sides by $m!$ and letting m tend to infinity, we get $\log|\alpha| \ge 0$. This contradicts the assumption $|\alpha| < 1$.

It is known that e^α is transcendental for nonzero algebraic α. Especially e and π are transcendental numbers. Hilbert's seventh problem in 1900 asked whether α^β is transcendental for any algebraic number $\alpha \ne 0,\ 1$ and any algebraic irrational β and it was solved by Gelfond and Schneider independently in 1934. See Baker [1] for such topics. Here we show a simple case of Mahler's transcendence theory.

Theorem 1.1.2. Let d be an integer greater than 1. If α is an algebraic number with $0 < |\alpha| < 1$, then $\sum_{k=0}^{\infty} \alpha^{d^k}$ is transcendental.

Proof. Let $f(z) = \sum_{k=0}^{\infty} z^{d^k}$. Then $f(z)$ satisfies the functional equation

$$f(z^d) = f(z) - z.$$

First we prove that $f(z)$ is not algebraic over $\mathbf{C}(z)$. Assume that $f(z)$ satisfies the following irreducible equation

$$f(z)^n + a_{n-1}(z) f(z)^{n-1} + \cdots + a_0(z) = 0, \tag{1.1.1}$$

where $a_i(z) \in \mathbf{C}(z)$ $(i = 0, \ldots, n-1)$. Substituting z^d for z, we have

$$f(z^d)^n + a_{n-1}(z^d) f(z^d)^{n-1} + \cdots + a_0(z^d) = 0.$$

Since $f(z^d) = f(z) - z$, we get

$$f(z)^n + (-nz + a_{n-1}(z^d)) f(z)^{n-1} + \cdots = 0. \tag{1.1.2}$$

Since the left hand sides of (1.1.1) and (1.1.2) must coinside as polynomials of $f(z)$, we obtain

$$a_{n-1}(z) = -nz + a_{n-1}(z^d).$$

Letting $a_{n-1}(z) = a(z)/b(z)$, where $a(z)$, $b(z)$ are coprime polynomials, we get

$$a(z)b(z^d) = -nzb(z)b(z^d) + a(z^d)b(z). \tag{1.1.3}$$

Hence $b(z^d)$ must divide $b(z)$, because $a(z^d)$, $b(z^d)$ are coprime. Therefore $\deg b(z) = 0$ and so we may assume $b(z) = 1$. By (1.1.3),

$$a(z) = -nz + a(z^d).$$

Comparing the degrees of both sides, we have $a(z) \in \mathbf{C}$ and so $-nz = 0$, a contradiction.

Suppose that $f(\alpha)$ is an algebraic number. Let K be an algebraic number field, i.e. a finite extension of $\mathbf{Q}$, which contains α and $f(\alpha)$, and p a positive integer. We assert that there are $p+1$ polynomials $P_0, \ldots, P_p \in \mathbf{Z}[z]$ with degrees at most p such that the auxiliary function

$$E_p(z) = \sum_{j=0}^{p} P_j(z) f(z)^j = \sum_{h=0}^{\infty} b_h z^h$$

is not identically zero and all the coefficients b_h, with $h \leq p^2$, vanish. To see this, we put $P_j(z) = \sum_{\ell=0}^{p} x_{j\ell} z^\ell$. Then b_h is a linear form of $x_{j\ell}$ $(0 \leq j,\ \ell \leq p)$ over $\mathbf{Q}$. Since $(p+1)^2 > p^2 + 1$, the system of equations $b_0 = b_1 = \cdots = b_{p^2} = 0$ has a nontrivial solution $x_{j\ell}$ in $\mathbf{Z}$. Since $f(z)$ is not algebraic over $\mathbf{C}(z)$, $E_p(z)$ is not identically zero. Let H be the least integer such that $b_H \neq 0$. Then $H > p^2$. In what follows, $c_1, c_2, \ldots$ denote positive constants independent of p, k and $c_1(p), c_2(p), \ldots$ denote positive constants depending on p but independent of k. Since

$$\lim_{z \to 0} E_p(z) z^{-H} = b_H,$$

we have for any $k \geq c_1(p)$,

$$0 \neq |E_p(\alpha^{d^k})| \leq c_2(p)|\alpha|^{d^k H} \leq c_2(p)|\alpha|^{d^k p^2}. \tag{1.1.4}$$

On the other hand

$$\begin{aligned} E_p(\alpha^{d^k}) &= \sum_{j=0}^{p} P_j(\alpha^{d^k}) f(\alpha^{d^k})^j \\ &= \sum_{j=0}^{p} P_j(\alpha^{d^k}) \left(f(\alpha) - \alpha - \alpha^d - \cdots - \alpha^{d^{k-1}}\right)^j. \end{aligned}$$

Hence we obtain

$$\begin{aligned} E_p(\alpha^{d^k}) &\in K, \\ \overline{\left|E_p(\alpha^{d^k})\right|} &\le c_3(p)(p+1)^2 c_4^{d^k p}(k+1)^p c_5^{d^{k-1}p} \le c_6(p) c_7^{d^k p}, \end{aligned}$$

and

$$\mathrm{den}(\alpha)^{2d^k p}\mathrm{den}(f(\alpha))^p E_p(\alpha^{d^k}) \in \mathbf{I}.$$

Therefore

$$\max\{\log \overline{\left|E_p(\alpha^{d^k})\right|},\ \log \mathrm{den}(E_p(\alpha^{d^k}))\} \le \log c_6(p) + c_8 d^k p. \qquad (1.1.5)$$

By (1.1.4), (1.1.5) and the fundamental inequality, we have

$$\log c_2(p) + d^k p^2 \log|\alpha| \ge -2[K:\mathbf{Q}]\left(\log c_6(p) + c_8 d^k p\right), \qquad (1.1.6)$$

for $k \ge c_1(p)$. Dividing both sides of (1.1.6) by d^k and letting k tend to infinity, we get

$$p^2 \log|\alpha| \ge -2[K:\mathbf{Q}]c_8 p.$$

This is a contradiction for every large p and we complete the proof.

1.2 Mahler's theorem

We generalize Theorem 1.1.2. For an algebraic number field K, we put $\mathbf{I}_K = K \cap \mathbf{I}$ and $K[[z]]$ denotes the ring of formal power series in variable z with coefficients in K. Suppose that $f(z) \in K[[z]]$ has convergence radius $R > 0$ and satisfies the following functional equation for an integer d greater than one,

$$f(z^d) = \frac{\sum_{i=0}^m a_i(z) f(z)^i}{\sum_{i=0}^m b_i(z) f(z)^i}, \qquad m < d,\ a_i(z), b_i(z) \in \mathbf{I}_K[z]. \qquad (1.2.1)$$

By $\Delta(z)$ we denote the resultant of $\sum_{i=0}^m a_i(z)u^i$ and $\sum_{i=0}^m b_i(z)u^i$ as polyomials in u. If one of them is a constant $c(z)$ as a polynomial in u, then $\Delta(z) = c(z)$.

Theorem 1.2 (Mahler [1]). Assume that $f(z)$ is not algebraic over $K(z)$. If α is an algebraic number with $0 < |\alpha| < \min\{1, R\}$ and $\Delta(\alpha^{d^k}) \ne 0$ $(k \ge 0)$, then $f(\alpha)$ is transcendental.

Remark. $f(z)$ is algebraic over $K(z)$ if and only if $f(z)$ is algebraic over $\mathbf{C}(z)$. For we assume $f(z)$ satisfies

$$a_n(z)f(z)^n+a_{n-1}(z)f(z)^{n-1}+\cdots+a_0(z)=0, \qquad a_i(z)\in\mathbf{C}[z],\ a_n(z)\neq 0.$$

Let $\{b_1,\ldots,b_m\}$ be a maximal subset of the set of all the coefficients of $a_0(z),\ldots,a_n(z)$ which is linearly independent over K. Then

$$a_i(z)=\sum_{j=1}^{m}a_{ij}(z)b_j, \qquad a_{ij}(z)\in K[z],$$

and

$$\sum_{j=1}^{m}\left(a_{nj}(z)f(z)^n+\cdots+a_{0j}(z)\right)b_j=0.$$

Comparing the coefficients of z, we have

$$a_{nj}(z)f(z)^n+\cdots+a_{0j}(z)=0, \qquad j=1,\ldots,m.$$

Since at least one of $a_{nj}(z)$ $(j=1,\ldots,m)$ is not zero, $f(z)$ is algebraic over $K(z)$. The converse is trivial.

Proof. Suppose that $f(\alpha)$ is algebraic. We may assume α, $f(\alpha)\in K$. Let p be a positive integer. For a reason similar to the one in the proof of Theorem 1.1.2, there are $p+1$ polynomials $P_0,\ldots,P_p\in\mathbf{I}_K[z]$ with degrees at most p such that the auxiliary function

$$E_p(z)=\sum_{j=0}^{p}P_j(z)f(z)^j=\sum_{h=0}^{\infty}b_hz^h$$

is not identically zero and all the coefficients b_h, with $h\le p^2$, vanish. Since $f(z)$ is not algebraic over $K(z)$, $E_p(z)$ is not identically zero. Let H be the least integer such that $b_H\neq 0$. Then $H>p^2$. Since

$$\lim_{z\to 0}E_p(z)z^{-H}=b_H,$$

we have for any $k\ge c_1(p)$,

$$0\neq|E_p(\alpha^{d^k})|\le c_2(p)|\alpha|^{d^kH}\le c_2(p)|\alpha|^{d^kp^2}. \qquad (1.2.2)$$

There are polynomials $S(z,u),T(z,u)\in\mathbf{I}_K[z,u]$ such that

$$\Delta(z)=S(z,u)\sum_{i=0}^{m}a_i(z)u^i+T(z,u)\sum_{i=0}^{m}b_i(z)u^i.$$

Hence

$$\Delta(\alpha) = S(\alpha, f(\alpha)) \sum_{i=0}^{m} a_i(\alpha) f(\alpha)^i + T(\alpha, f(\alpha)) \sum_{i=0}^{m} b_i(\alpha) f(\alpha)^i.$$

Suppose that $\sum_{i=0}^{m} b_i(\alpha) f(\alpha)^i = 0$. Since

$$\left(\sum_{i=0}^{m} b_i(\alpha) f(\alpha)^i \right) f(\alpha^d) = \sum_{i=0}^{m} a_i(\alpha) f(\alpha)^i,$$

we get $\sum_{i=0}^{m} a_i(\alpha) f(\alpha)^i = 0$ and so $\Delta(\alpha) = 0$. This contradicts the assumption. Therefore $\sum_{i=0}^{m} b_i(\alpha) f(\alpha)^i \neq 0$ and $f(\alpha^d) \in K$. Proceeding in this way, we see that $f(\alpha^{d^k}) \in K$ and therefore $E_p(\alpha^{d^k}) \in K$ $(k \geq 0)$. Define Y_k $(k \geq 1)$ inductively as follows,

$$\begin{aligned} Y_1 &= \sum_{i=0}^{m} b_i(\alpha) f(\alpha)^i, \\ Y_{k+1} &= Y_k^m \sum_{i=0}^{m} b_i(\alpha^{d^k}) f(\alpha^{d^k})^i, \qquad k \geq 1. \end{aligned}$$

Then $Y_k \in K$ and $Y_k \neq 0$ $(k \geq 1)$. We estimate $\overline{\left| Y_k^p E_p(\alpha^{d^k}) \right|}$ and $\operatorname{den}(Y_k^p E_p(\alpha^{d^k}))$. Let $\deg_z a_i(z)$, $\deg_z b_i(z) \leq \ell$, $\overline{|\alpha|}$, $\overline{|f(\alpha)|} \leq c_3$ $(c_3 > 1)$ and D a positive integer such that $D\alpha, Df(\alpha) \in \mathbf{I}$. Then we have

$$\begin{aligned} \overline{|Y_1|} &= \overline{\left| \sum_{i=0}^{m} b_i(\alpha) f(\alpha)^i \right|} \leq \sum_{i=0}^{m} \overline{|b_i(\alpha)|}\, \overline{|f(\alpha)|}^i \leq c_4 c_3^{\ell} c_3^{m}, \\ \overline{|Y_1 f(\alpha^d)|} &= \overline{\left| \sum_{i=0}^{m} a_i(\alpha) f(\alpha)^i \right|} \leq \sum_{i=0}^{m} \overline{|a_i(\alpha)|}\, \overline{|f(\alpha)|}^i \leq c_4 c_3^{\ell} c_3^{m} \end{aligned}$$

and

$$D^{\ell+m} Y_1, \ D^{\ell+m} Y_1 f(\alpha^d) \in \mathbf{I}.$$

Since $Y_2 = Y_1^m \sum_{i=0}^{m} b_i(\alpha^d) f(\alpha^d)^i$ and $Y_2 f(\alpha^{d^2}) = Y_1^m \sum_{i=0}^{m} a_i(\alpha^d) f(\alpha^d)^i$, we have

$$\overline{|Y_2|}, \ \overline{|Y_2 f(\alpha^{d^2})|} \leq \left(c_4 c_3^{d\ell} \right) \left(c_4 c_3^{\ell+m} \right)^m$$

and

$$D^{d\ell} \left(D^{\ell+m} \right)^m Y_2, \ D^{d\ell} \left(D^{\ell+m} \right)^m Y_2 f(\alpha^{d^2}) \in \mathbf{I}.$$

Proceeding in this way, we obtain

$$\overline{|Y_k|},\ \overline{\left|Y_k f(\alpha^{d^k})\right|} \leq c_4^{1+m+\cdots+m^{k-1}} (c_3^\ell)^{d^{k-1}+d^{k-2}m+\cdots+m^{k-1}} c_3^{m^k}$$

and

$$\begin{aligned} (D^\ell)^{d^{k-1}+d^{k-2}m+\cdots+m^{k-1}} D^{m^k} Y_k &\in \mathbf{I}, \\ (D^\ell)^{d^{k-1}+d^{k-2}m+\cdots+m^{k-1}} D^{m^k} Y_k f(\alpha^{d^k}) &\in \mathbf{I}. \end{aligned}$$

By the assumption $m < d$, we have

$$d^{k-1} + d^{k-2}m + \cdots + m^{k-1} = d^{k-1}\left(1 + \frac{m}{d} + \cdots + \left(\frac{m}{d}\right)^{k-1}\right) \leq c_5 d^{k-1},$$

where we take a positive integer as c_5. Hence

$$\overline{|Y_k|},\ \overline{\left|Y_k f(\alpha^{d^k})\right|} \leq c_4^{c_5 d^{k-1}} \left(c_3^\ell\right)^{c_5 d^{k-1}} c_3^{d^k} \leq c_6^{d^k}$$

and

$$D_0^{d^k} Y_k,\ D_0^{d^k} Y_k f(\alpha^{d^k}) \in \mathbf{I}, \qquad D_0 = D^{\ell c_5 + 1}.$$

Since

$$Y_k^p E_p(\alpha^{d^k}) = \sum_{j=0}^{p} P_j(\alpha^{d^k}) Y_k^{p-j} \left(Y_k f(\alpha^{d^k})\right)^j,$$

we obtain

$$\overline{\left|Y_k^p E_p(\alpha^{d^k})\right|} \leq c_7(p) c_3^{d^k p} c_6^{d^k p}, \quad D_0^{2d^k p} Y_k^p E_p(\alpha^{d^k}) \in \mathbf{I}. \tag{1.2.3}$$

By (1.2.2), (1.2.3) and the fundamental inequality,

$$\begin{aligned} d^k p \log c_6 + \log c_2(p) + d^k p^2 \log|\alpha| &\geq \log\left|Y_k^p E_p(\alpha^{d^k})\right| \\ &\geq -2[K : \mathbf{Q}]\left(\log c_7(p) + d^k p \log c_3 c_6 + 2d^k p \log D_0\right), \end{aligned}$$

for $k > c_1(p)$. Dividing both sides above by d^k and letting k tend to infinity, we have

$$p \log c_6 + p^2 \log|\alpha| \geq -2[K : \mathbf{Q}]\left(p \log c_3 c_6 + 2p \log D_0\right).$$

Dividing both sides above by p^2 and letting p tend to infinity, we have $\log|\alpha| \geq 0$, a contradiction.

1.3 Transcendence of functions

To apply Theorem 1.2 to Mahler functions, we need the transcendence of formal power series. Let C be an algebraically closed field of characteristic 0 and $C[[z]]$ the formal power series ring over C.

Theorem 1.3 (Keiji Nishioka [2]). Suppose that $f(z) \in C[[z]]$ satisfies one of the following for an integer $d > 1$.
(i) $f(z^d) = \varphi(z, f(z))$,
(ii) $f(z) = \varphi(z, f(z^d))$,
where $\varphi(z, u)$ is a rational function in z, u over C. If $f(z)$ is algebraic over $C(z)$, then $f(z) \in C(z)$.

Proof. We need the notion from the theory of algebraic function fields of one variable (cf. Cohn [1]). Let M be the quotient field of $C[[z]]$ and τ the endomorphism from M into itself defined by

$$\tau z = z^d, \quad \tau a = a \ (a \in C).$$

Suppose that $f(z)$ satisfies (i) and is algebraic over $C(z)$. Letting $y = f(z)$, we get an algebraic function field $R = C(z, y)$ of one variable and τ is an endomorphism from R into itself. For a place P of R, there is a unique place $P\tau^{-1}$ of τR such that

$$v'_{P\tau^{-1}}(\tau u) = v_P(u), \quad u \in R,$$

where v and v' are associated normalized valuations with P and $P\tau^{-1}$ respectively. If t is a uniformizer of P, then τt is a uniformizer of $P\tau^{-1}$. Let Q be a place of $C(z)$, P an extension of Q in R and e_{PQ} the ramification index of the extension. Then $Q\tau^{-1}$ is a place of $C(z^d)$ and $P\tau^{-1}$ is an extension of $Q\tau^{-1}$ in τR with ramification index e_{PQ}. Let Q_0 and Q_∞ be the places of $C(z)$ with unformizers z and z^{-1} respectively. If Q is a place distinct from Q_0, Q_∞, then $Q\tau^{-1}$ is not ramified at the extension $C(z)/C(z^d)$. Let $P_1, \ldots, P_r$ be all the places of R that are ramified at $R/C(z)$ and not the extensions of Q_0, Q_∞. Let Q_i be the restriction of P_i to $C(z)$. Then $e_{P_iQ_i} > 1$. Suppose that P is an extension of $P_i\tau^{-1}$ in R and Q is the restriction of P to $C(z)$. Since Q is an extension of $Q_i\tau^{-1}$,

$$\begin{aligned} e_{PQ} &= e_{PQ}e_{QQ_i\tau^{-1}} = e_{PQ_i\tau^{-1}} = e_{PP_i\tau^{-1}}e_{P_i\tau^{-1}Q_i\tau^{-1}} \\ &\geq e_{P_i\tau^{-1}Q_i\tau^{-1}} = e_{P_iQ_i} > 1. \end{aligned}$$

Hence P is one of $P_1, \ldots, P_r$. Since $P_1\tau^{-1}, \ldots, P_r\tau^{-1}$ are distinct, each $P_i\tau^{-1}$ has a unique extension P_j and so $e_{P_j P_i \tau^{-1}} = [R : \tau R] = d$. By the equalities above,

$$e_{P_j Q_j} = d\, e_{P_i Q_i}.$$

This leads to a contradiction by choosing i for which $e_{P_i Q_i}$ is maximum. Therefore $r = 0$. By Riemann's formula (Theorem 6.6 in Cohn [1]) for the extension $R/C(z^d)$,

$$2(g-1) = -2dn + \sum_{i=1}^{h}(e_{P_{0i}} - 1) + \sum_{i=1}^{k}(e_{P_{\infty i}} - 1),$$

where g is the genus of R, $n = [R : C(z)]$ and the sums range over all the extensions of Q_0, Q_∞ respectively. For only Q_0, Q_∞ are ramified at the extension $R/C(z)$ or $C(z)/C(z^d)$. Since

$$\sum_{i=1}^{h}(e_{P_{0i}} - 1) = \sum_{i=1}^{h}(d\, e_{P_{0i}Q_0} - 1) = dn - h,$$

and

$$\sum_{i=1}^{k}(e_{P_{\infty i}} - 1) = \sum_{i=1}^{k}(d\, e_{P_{\infty i}Q_\infty} - 1) = dn - k,$$

we have

$$-2 \le 2(g-1) = -2dn + dn - h + dn - k = -h - k \le -2.$$

Therefore $g = 0$, $h = k = 1$. Hence $Q_0 = P_0^n$, $Q_\infty = P_\infty^n$ and so $(z) = Q_0 Q_\infty^{-1} = P_0^n P_\infty^{-n}$. Since $g = 0$, there is an element t in R such that $(t) = P_0 P_\infty$ and $R = C(t)$. Then $zt^{-n} \in C$ and $R = C(z^{1/n})$. Because $f(z) \in C[[z]] \cap C(z^{1/n})$, $f(z)$ must belong to $C(z)$.

Next suppose that $f(z)$ satisfies (ii) and is algebraic over $C(z)$. Then $R = C(z, f(z))$ is a function field of one variable and

$$R \subset C(z)\tau R.$$

Hence we have

$$[R : C(z)] = [\tau R : C(z^d)] \ge [C(z)\tau R : C(z)] \ge [R : C(z)].$$

Therefore $R = C(z)\tau R \supset \tau R$ and so $f(z)$ satisfies (i). This completes the proof.