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Vortex Methods

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PREFACE

One of the prominent features of incompressible fluid motion is the common occurrence of vortical structures concentrated in a small fraction of the flow field. Among the important vortical structures are vortex sheets, vortex tubes, and line vortices. In an effort to gain insight into the nature of fluid motion, and the behavior of solutions of the equations which are used to describe fluid motion, much research effort has gone into the analysis, both computational and analytical, of the evolution of these vortical structures. The papers which are presented in this collection, papers based on talks given at the U.C.L.A. Workshop on Vortex Methods, held during May 20–22, 1987, describe some of the research in this direction. One aim of the workshop was to bring together people carrying out theoretical and numerical investigations. Vortex methods, by which we mean numerical schemes in which the computational elements are pieces of vorticity, were particularly emphasized.

The first two papers in this collection are devoted to the study, by analytical and numerical tools, respectively, of vortex sheets. The next two articles address computations of three-dimensional flow and models of vortex stretching and turbulence. Articles five through eight concern convergence results for vortex methods. The last two articles discuss different techniques for achieving great improvements in the speed of vortex method calculations without suffering losses in accuracy.

We would like to thank the participants, as well as the Office of Naval Research, who provided funding, for making the meeting a success.

Chris Anderson

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Contents

LONG TIME EXISTENCE AND SINGULARITY FORMATION FOR VORTEX SHEETS

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ABSTRACT

The initial linear evolution of a nearly flat and uniform vortex sheet is given by the Kelvin-Helmholtz instability. Asymptotic analysis and numerical computations of the subsequent nonlinear evolution show several interesting features. At some finite time the vortex sheet develops a singularity in its shape; i.e. the curvature becomes infinite at a point. This is immediately followed by roll-up of the sheet into an infinite spiral. This paper presents two mathematical results on nonlinear vortex sheet evolution and singularity formation: First, for sufficiently small analytic perturbations of the flat sheet, existence of smooth solutions of the Birkhoff-Rott equation is proved almost up to the expected time of singularity formation. Second, we present a construction of exact solutions that develop singularities (infinite curvature) in finite time starting from analytic initial data. These results are derived within the framework of analytic function theory. The analysis of singular solutions is an independent construction of solutions first found by Duchon and Robert (1986,1988). All of these results are in the analytic function setting, since that is the only space in which the vortex sheet problem is known to be well-posed. We present a simple example to show ill-posedness of the 2D Euler equations in the energy norm.

For two-dimensional, inviscid, incompressible flow, vortex sheets are a phenomena of fundamental and technological importance. The perturbations of a nearly flat and uniform sheet will initially grow due to the Kelvin-Helmholtz instability until nonlinear interactions become important. Asymptotic expansions (Moore 1979, 1984) and numerical computations (Krasny 1986, 1987, Meiron, Baker and Orszag 1982) show several interesting features in the subsequent nonlinear evolution: At some finite critical time the vortex sheet develops a singularity in its shape; the curvature becomes infinite at a point. Although this is a weak singularity, it seems to be of fundamental importance since it is immediately followed by roll-up of the sheet into an infinite spiral. The spiral starts off at the critical time with center at the singularity point and with zero outer radius; then the radius grows as time evolves. However at any time after the critical time, the spiral has an infinite

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number of turns.

The first part of this paper focuses on the evolution of a vortex sheet up to and including the time of singularity formation. The reasons for such interest in the singularity are that it is a distinctive nonlinear phenomena, that its appearance signals the beginning of roll-up and that study of the singularity may clarify several other mathematical and physical issues, such as well-posedness of the vortex sheet problem. The second part of the paper presents an example of ill-posedness for the 2D Euler equations in the energy norm. This is presented as partial justification for the use of analyticity in the existence and singularity formation results: no weaker function space is known in which the vortex sheet equations are well-posed.

1. Existence and Singularity Formation for Vortex Sheets

The vortex sheet is described by a complex function $z(\gamma,t) = x(\gamma,t) + iy(\gamma,t)$ in which γ is the circulation variable and z is the complex position of the sheet. The evolution of the sheet is governed by the Birkhoff-Rott equation

$$\partial_t z^*(\gamma,t) = B[z](\gamma,t) \tag{1}$$

$$= (2\pi i)^{-1}\, PV \int_{-\infty}^{\infty} (z(\gamma,t) - z(\gamma',t))^{-1}\, d\gamma'$$

in which the integral is a Cauchy principal value integral and the last line defines the operator $B[z]$. Also $z^*(\gamma) = \overline{z(\overline{\gamma})}$, in which the bar denotes the usual complex conjugate. Our first result concerns existence of solutions of this equation and the approximation of those solutions.

EXISTENCE (Caflisch and Orellana 1986): Suppose that initially the vortex sheet has a small sinusoidal perturbation, so that $z(\gamma,t) = \gamma + i\varepsilon\sin\gamma$ in which ε is small. Then the vortex sheet equation (1) has a smooth solution for a time interval $0 \le t < 2\kappa|\log\varepsilon|$ in which $\kappa < 1$ and $\kappa \to 1$ as $\varepsilon \to 0$. Moreover the solution $z(\gamma,t)$ is close to the solution $z_M(\gamma,t)$ of Moore's approximate equations; i.e.

$$|z - z_M| \le constant\ \varepsilon^2. \tag{2}$$

This is an example of a more general theorem that applies for any small analytic, periodic perturbation of a flat vortex sheet. The time of existence is nearly optimal, since asymptotic analysis (Moore 1984) indicates that a singularity will form at the critical time $t = 2|\log\varepsilon| + O(\log|\log\varepsilon|)$.

Moore's approximate equations are

$$\partial_t g = h^{-2}\,\partial_y h \tag{3}$$

$$\partial_t h = \partial_y g \tag{4}$$

in which g and h are (roughly speaking) the phase and amplitude for the perturbation of $\partial_\gamma z$ and $y = i\gamma$. More precisely in which

$$\partial_\gamma s(\gamma,t) = -1 + (\, h(i\gamma,t)/2\,)^{1/2}\, e^{-ig(i\gamma,t)/2} \tag{6}$$

The proof of the Existence result is summarized as follows: Write z as $z = \gamma + s$ in which s is a small perturbation, and decompose s into its upper analytic (i.e. positive wavenumber) and lower analytic components as $s = s_+ + s_-$. If s is 2π-periodic and analytic in the strip $|Im\ \gamma| < \rho$, then

$$B[z] = (1/2)\{(1+(s_{+\gamma})^{-1} s_{+\gamma} - (1+(s_{-\gamma})^{-1} s_{-\gamma}\} \;+\; O(e^{-4\rho}). \tag{7}$$

Omitting the error term at the end of (7) results in precisely Moore's equations (3),(4). Existence for these equations can be shown using the a priori estimates of Lax (1964) for a system of two conservation laws. A solution for the full equation (1), with the error term in (7) considered as a perturbation, can be constructed iteratively, using a variant of the abstract Cauchy-Kowalewski Theorem (Nishida 1977, Asano 1988) .

The advantage of Moore's equations (3),(4) over the Birkhoff-Rott equation (1) is that the former are differential equations; they do not contain nonlocal operators, such as the integral in (1). Moreover they give an intuition about the mechanism for singularity formation on vortex sheets. Equations (5) and (6) are nonlinear hyperbolic equations with characteristics which extend in the y direction, that is the imaginary γ direction. Thus we may expect singularities to propagate in the imaginary γ direction until they hit the real γ line, at which time they will appear physically. Moore (1984) showed that the expected form of the singularity would be an envelope for the characteristics.

Although we are not yet able to mathematically analyze singularity formation with the detail described in the previous paragraph, we can construct exact solutions of the Birkoff-Rott equation for which singularities form at finite time starting from smooth initial data. This is the second main result:

SINGULARITY FORMATION (Caflisch and Orellana 1988): Let $\nu > 0$ and let ε be small. There are solutions $z(\gamma,t)$ of (1) that are analytic for $t < 0$ but have a singularity of order $1+\nu$ at $\gamma = 0$, $t = 0$; i.e. for $t = 0, \gamma \approx 0$,

$$z \approx \gamma + \varepsilon\, sgn(\gamma)\ |\gamma|^{\nu+1} \tag{8}$$

$$z_{\gamma\gamma} \approx \varepsilon\, sgn(\gamma)\ |\gamma|^{\nu-1} \tag{9}$$

In particular for $0 < \nu < 1$, the sheet has infinite curvature (second derivative) at $\gamma = 0$, $t = 0$.

The approximate vortex sheet strength $\sigma = |\partial z/\partial\gamma|^{-1}$ at the singularity time is plotted in figure 1. The cusp at $\gamma = 0$ corresponds to the singularity. Such singular solutions were first constructed by Duchon and Robert (1986,1988) using Fourier analysis and a fixed point theorem. Ours is an independent derivation which uses the abstract Cauchy-Kowalewski method.

The proof of the Singularity Formation result is summarized as follows: Decompose z as above into $z = \gamma + s_+ + s_-$. The linearized equations for s_+ and s_- are

$$\partial_t s_+ = -(1/2) s^*_{-\gamma}.$$

This is an elliptic system for which singularities travel at speeds $\pm i/2$. Thus singularities can appear on the real line by coming in from the complex plane. Likewise a singularity at $\gamma = 0$ initially will travel off into the complex plane, leaving the solution to be analytic on the real line. Such analytic solutions of the linearized equations (10) can be turned into analytic solutions of the full nonlinear equation (1), by using the abstract Cauchy-Kowalewski Theorem again. The resulting solution has a singularity at $\gamma = t = 0$ but is analytic in $|Im\gamma| < \kappa(|t|/2)$ for $t < 0$. The constant κ is smaller than 1, but nearly equal to 1 if the perturbation s is small.

By shift of the time dependence of the solution in the Singularity Formation result, one finds analytic initial data for which the solution of (1) develops a singularity at a finite time. Furthermore by rescaling this initial data, one can find initial data of arbitrarily small size in any of the usual energy (Sobolev) norms H^n, for which a singularity forms in an arbitrarily short time. However such data would be large in the analytic norm. A precise statement of this result is the following:

COROLLARY (Caflisch and Orellana 1988): Let ε and δ be arbitrarily small and let ν be positive. For any n, there is initial data $z = \gamma + s$ for (1) with $||s||_{H^n} < \varepsilon$, so that an infinite $(1 + \nu)$-th derivative develops in a time $t < \delta$.

These results lead to a clearer understanding of the well-posedness of the vortex sheet problem. The existence result shows that, within the class of analytic functions, equation (1) is well posed, in the sense that small initial data (for the perturbation of the flat uniform sheet) leads to existence for a long time. The singularity results show that singularities may form at some finite time. Moreover in any larger function space, such as the Sobolev spaces, initial data of arbitrarily small norm may lead to an infinite $(1 + \nu)$-th derivative in arbitrarily short time. In particular this shows the vortex sheet problem to be ill-posed in H^n for $n > 3/2$.

The restriction of analyticity (of z with respect to γ), is a mathematical requirement for well-posedness but may seem artificial physically. However we believe that restriction to analytic solutions is consistent with the infinite Reynolds number limit, at least within certain flow regimes.

These singularity results may be a useful analogue for more difficult problems of singularity development in fluid flows. An important outstanding mathematical problem is whether singularities form in finite time out of smooth initial data for the incompressible Euler equations in 3D. Although smooth initial data in 2D is known to lead to smooth solutions for all time, our result show that singular initial data (a vortex sheet) in 2D can become more singular (infinite curvature of the sheet) in finite time.

2. Ill-Posedness in the Energy Norm for the 2D Euler Equations

We present a simple example showing that the Euler equations are not well posed in the energy norm. This example shows that a bound on the total energy and entropy does not guarantee a bound on the modulus solution with velocity $u(x,t)$ for $x \in \mathbf{R}^2$, such that

$$\int_{\mathbf{R}^2} |u|^2 dx < 1 \tag{11}$$

$$\int_{\mathbf{R}^2} |\nabla \times u| dx < 1 \tag{12}$$

$$\int_{\mathbf{R}^2} |u(t) - u(0)|^2 dx > \frac{1}{2} \text{ for } t = \varepsilon \tag{13}$$

We call the integral in (13) the energy difference of the solution at two times.

The example will consist approximately of two circular patches each with constant vorticity and with equal but opposite circulation. This is only the simplest example; other examples are expected to show stronger instability.

This discussion is partly motivated by recent work of Majda and DiPerna (1987a,1987b,1988), who have successfully analyzed weak solutions and their limits, as well as various desingularizations (such as nonzero viscosity) using the energy norm and the constraint of finite total vorticity as in (11) and (12) (actually they only require bounded energy and total vorticity on bounded sets).

As a preliminary, consider a small isolated vortex patch of constant vorticity with radius a and total circulation Γ . The corresponding velocity field is

$$u = \begin{cases} (\Gamma/2\pi r)\hat{\Theta} & r \geq a \\ (\Gamma/2\pi a^2) r \hat{\Theta} & r \leq a \end{cases} \tag{14}$$

The velocity field near the patch contains energy E_{loc} , which we call the local energy and which is of size $\Gamma^2 |\log a|$ since

$$E_{loc} = \int_{|x|<1} |u|^2 dx \qquad = (2\pi)^{-1}\Gamma^2 |\log a| \tag{15}$$

for a small. The upper limit $|x| = 1$ in this integral is convenient but not necessary.

Now consider two small patches of radius a, separation distance l (which can be taken to be 1) and circulation Γ and $-\Gamma$. Because the total circulation is zero, the energy is small at large distances from the patches and most of the energy is in the local energies. To keep the total energy bounded as in (11) we require

$$\Gamma = |\log a|^{-1/2} \tag{16}$$

This also ensures small total vorticity as in (12). The vortices act on each other like point vortices if

$$a < < R \tag{17}$$

We also require that there is not significant shearing, so that the patches stay nearly circular (this is more than is necessary). This is true up to time t , if $t |\nabla u| < < 1$, i.e.

$$t\Gamma R^{-1} << 1 \tag{18}$$

Under the conditions the vortex patches will move like two point vortices, with speed

$$V = (2\pi)^{-1}\Gamma R^{-1} \tag{19}$$

Since almost all of the energy is in the local energies of the vortex patches, these will be a significant change for u in the energy norm as in (13) if

$$Vt >> a \tag{20}$$

It is easy to see that conditions (16)-(20) are compatible for arbitrarily small t. Therefore, for any small ε, there is a flow satisfying (11)-(13).

A modification of this argument yields two solutions that start off arbitrarily close in the energy norm, but in arbitrarily short time have an 0(1) difference in that norm. These examples show that a function space with the energy norm will not be a proper setting for analyzing certain properties, such as convergence of numerical methods, for time dependent flows.

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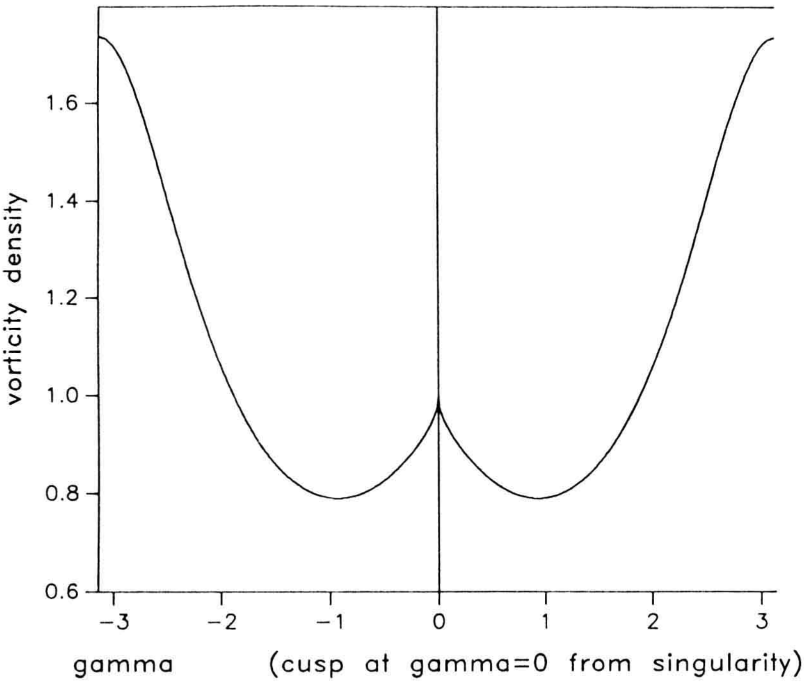
Figure 1. Approximate Vorticity Density

epsilon = .1, nu = .5



COMPUTATION OF VORTEX SHEET ROLL-UP

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1. Introduction

In this article we shall review some recent developments for computing vortex sheet roll-up. A vortex sheet is an asymptotic model of a free shear layer in which the transition region between the two fluid streams is approximated by a surface across which the tangential velocity component is discontinuous. A common theme in fluid dynamics is that the vortex sheet model can be useful in understanding the dynamics of coherent vortex structures observed in laminar and turbulent flows. If this goal is to be realized, reliable methods for computing vortex sheet evolution must be developed.

At present, numerical methods are available for studying the initial value problem in two space dimensions. For example, detailed analytical phenomena such as singularity formation in the shape of an evolving periodic vortex sheet can be studied with existing methods. The complex roll-up process and the interaction of several spiral vortices has also been investigated numerically. These calculations have been stimulated by recent theoretical results about vortex sheets and by progress in the convergence theory of general vortex methods.

First, results for the periodic vortex sheet will be reviewed. Then an application to some vortex sheet problems occurring in aerodynamics will be discussed in more detail. Finally, some open questions and directions for further research will be summarized.

2. The Vortex Sheet Evolution Equation

A vortex sheet in two dimensional ideal flow can be described by a curve in the complex plane, $z(\Gamma, t) = x(\Gamma, t) + iy(\Gamma, t)$, varying with time t. The Lagrangian parameter Γ measures the circulation contained between a base point and an arbitrary point along the vortex sheet [4]. The vorticity associated with a vortex sheet is in the form of a delta function with support on the curve. The vortex sheet strength $\sigma = |\partial z/\partial\Gamma|^{-1}$ is the jump in the tangential velocity component across the curve.

The vortex sheet evolution equation is,

$$\overline{\frac{\partial z}{\partial t}} = \int K(z - \tilde{z})\, d\tilde{\Gamma}. \qquad (1)$$

In this equation, $z = z(\Gamma, t)$, $\tilde{z} = z(\tilde{\Gamma}, t)$, $K(z) = 1/2\pi iz$ is the Cauchy kernel and the Cauchy principal value of the integral is taken. The bar over the time derivative on the left denotes the complex conjugate. The evolution equation is supplemented by an initial condition for the vortex sheet $z(\Gamma, 0)$.

This basic evolution equation takes other forms depending upon the particular geometry and initial conditions under consideration. The simplest class of problems concerns a vortex sheet which is periodic in the x-direction. In this case, the integrand used is $K(z) = \cot(\pi z)/2i$ and the circulation parameter Γ runs over a single period $[0, 1]$. The initial condition takes the form $z(\Gamma, 0) = \Gamma + p(\Gamma, 0)$. The function $p(\Gamma, t)$ is periodic in Γ and it describes the perturbation away from the equilibrium solution $z(\Gamma, t) = \Gamma$, corresponding to a flat vortex sheet of constant strength.

A straightforward method of discretization was introduced by Rosenhead [19] in 1931. Consider a finite number of point vortices per wavelength which approximately interpolate the vortex sheet at equidistant values of Γ. Thus the point vortices' positions are $z_j(t) \sim z(\Gamma_j, t)$ where $\Gamma_j = j\Delta\Gamma$ and $\Delta\Gamma = N^{-1}$. The point vortices evolve according to the following system of ordinary differential equations,

$$\overline{\frac{dz_j}{dt}} = N^{-1} \sum_{k \neq j} K(z_j - \tilde{z}_k). \tag{2}$$

By neglecting the singular term $k = j$, the sum appearing on the right side of equation (2) is formally an $O(N^{-1})$ approximation to the principal value integral in equation (1). The initial point vortex positions interpolate the exact initial vortex sheet $z_j(0) = z(\Gamma_j, 0)$. The viewpoint adopted here is that in order to study properties of the vortex sheet, one must determine whether solutions of the point vortex equations converge as the dimension $N \to \infty$. The next section reviews the theoretical results and numerical evidence relating to this issue.

3. Singularity Formation in a Periodic Vortex Sheet

The vortex sheet model does not include any physical mechanisms to stabilize the short wavelength modes. In fact, the linearized initial value problem for perturbations of a flat, constant strength vortex sheet is subject to the Kelvin-Helmholtz instability [4]. Just as in the classical example, i.e. the Cauchy problem for the Laplace equation, the linearized vortex sheet initial value problem is not well-posed in the sense of Hadamard. However, if the initial perturbation $p(\Gamma, 0)$ is an analytic function of Γ then the nonlinear vortex sheet problem has an analytic solution in some time interval [21,5]. Analyticity of the solution is equivalent to controlling the amplitude of the short wavelength modes.

These Cauchy-Kovaleski results for the periodic vortex sheet actually were preceded by an asymptotic analysis of the nonlinear problem by Moore [16]. For an initial perturbation consisting of a single Fourier mode of amplitude ϵ, Moore's analysis indicates that a singularity forms in the vortex sheet at time $t_c(\epsilon) \sim \log \epsilon^{-1}$. Meiron, Baker & Orszag [13] studied the vortex sheet's Taylor series in time around $t = 0$ and obtained a similar comclusion. At the critical time, the vortex sheet strength has a finite amplitude cusp and the curvature has an infinite jump discontinuity at isolated points. However, the sheet's slope remains bounded and its tangent vector is continuous.

Previous numerical studies of this problem using Rosenhead's point vortex approximation have experienced difficulty in converging when the number of point vortices was increased [4]. Explaining the source of this difficulty and providing a remedy for it have been longstanding issues [17,18,20].

In a numerical solution of the point vortex equations (2) one can examine the discrete Fourier transform $\widehat{p_k}$ of the computed perturbation quantities $p_j = z_j - \Gamma_j$. Using this discrete Fourier analysis to diagnose the solution, it was shown that computer roundoff error is responsible for the irregular point vortex motion that occurs at a smaller time as the number of points is increased [9]. This source of computational error can be controlled either by using higher precision arithmetic or by using a new filtering technique. The numerical evidence indicates that the point vortex approximation converges as $N \to \infty$ up to but not beyond the time of singularity formation in the vortex sheet. Good agreement is obtained with Moore's relation for the critical time's dependence upon the initial amplitude.

4. Roll-Up Past the Critical Time

When a singularity forms in the solution of a nonlinear evolution equation, it may still be possible to extend the solution beyond that time in a way which ensures that the extension has physical significance. A classical example is shock formation and the theory of weak solutions to nonlinear hyperbolic equations [12]. Even though the shock solution is a discontinuous function, it serves as a useful approximation to a viscous profile. One can ask whether a similar theory can be constructed for the vortex sheet evolution equation.

One approach to extending the vortex sheet solution past the critical time is motivated by Chorin's "vortex blob" method [6,1]. The singular kernel $K(z)$ appearing in the vortex sheet equation (1) is replaced by a smooth kernel $K_\delta(z)$ which depends upon an artificial smoothing parameter δ. For example, in free space one can choose $K_\delta(z) = K(z)\,|z|^2/(|z|^2 + \delta^2)$ as the desingularized kernel. For $\delta > 0$ the solution of this "δ-equation" is a curve which approximates the vortex sheet. The proposal is to view the vortex sheet as the limit of these desingularized solutions as the smoothing parameter δ tends to zero. To discretize the δ-equation, one simply uses the smooth kernel $K_\delta(z)$ in place of $K(z)$ in the system of ordinary differential equations (2). For $\delta > 0$ therefore the point vortex is replaced by a "vortex blob".

This approach has been applied to the periodic vortex sheet problem [10]. Linear stability analysis shows that the vortex sheet's short wavelength instability is diminished when $\delta > 0$. The resulting ordinary differential equations are numerically more tractable since the computer roundoff error difficulty is not as severe. Solutions of the δ-equation are obtained for a fixed value of δ, at a fixed time $t > t_c$ by integrating the vortex blob equations and converging to the limit $N \to \infty$. Detailed numerical convergence studies also indicate that the sequence of solutions of the δ-equation converges pointwise in Γ as $\delta \to 0$ and that the error has an asymptotic expansion in powers of δ. These results suggest that in this weak sense, as a limit of smooth curves, the vortex sheet rolls up into a double-branched spiral past the critical time.

There are other possible ways of desingularizing a periodic vortex sheet to obtain candidate weak solutions. Baker and Shelley [3] study the dynamics of a layer of constant vorticity in the limit of vanishing thickness. A key question is whether this sequence converges to the same object that solutions of the δ-equation converge to, particularly

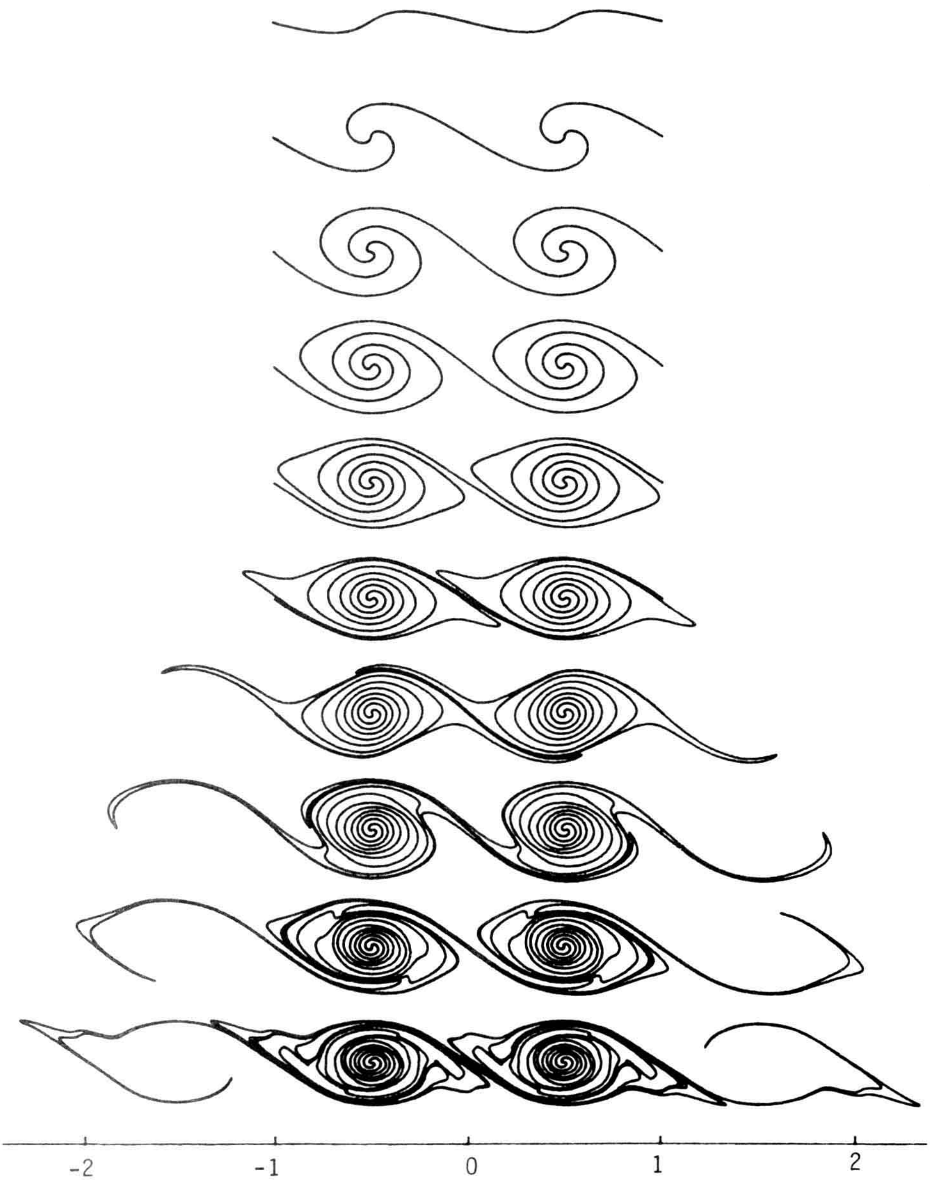

Figure 1. Periodic vortex sheet roll-up [10]. Two periods of the circulation parameter are plotted. The value of the smoothing parameter is $\delta = 0.5$.