Fixed Points

ALGORITHMS and APPLICATIONS

Edited by

Stepan Karamardian

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Preface

Since the appearance of Brouwer fixed point theorem in 1912 and its subsequent generalizations, fixed point theorems provided powerful tools in demonstrating the existence of solutions to a large variety of problems in applied mathematics. However, from the computational standpoint, their usefulness was limited. Up to 1967 all computational methods used for computing an approximate fixed point for a given map were based on iterative procedures that required additional restrictions on the map to guarantee convergence.

In 1967 H. Scarf developed a finite algorithm for approximating a fixed point to a continuous map from a simplex into itself. Scarf's algorithm is based on first subdividing the simplex into finite subsets called primitive sets and then utilizing Lemke's complementarity pivoting procedure. This algorithm provided the first constructive proof to Brouwer's fixed point theorem.

Scarf's work stimulated considerable interest. During the following few years several important refinements and extensions to his algorithm were developed. Among those are the works of H. Kuhn, B. C. Eaves, and O. Merrill.

These algorithms were applied to a number of test and real problems, with reasonable degrees of efficiency. During this period there was also considerable interest in a related unifying model, the so-called "complementarity problem," the problem of finding a nonnegative vector whose image under a given map is also nonnegative, and such that the two vectors are orthogonal. In fact, the fixed point problem over the nonnegative orthant of a finite-dimensional Euclidean space is equivalent to a complementarity problem.

In early 1974 several researchers expressed an interest in holding a conference to bring together those who were active in the fields of fixed point algorithms, the complementarity problem, and those who were involved in their applications to economics and other problems.

The first International Conference on Computing Fixed Points with Applications was held in the Department of Mathematical Sciences at Clemson University, Clemson, South Carolina, June 26-28, 1974.

The Conference was sponsored by the Office of Naval Research and the Office of the Army Research Center. The participants included mathematicians and economists from several European countries, Japan, and the United States.

Nine one hour invited addresses and twelve one half hour contributed papers were presented during the Conference. Each presentation was followed by a half hour discussion period. All papers were refereed, edited, and finally approved by the authors before their publication in this volume.

Professor Herbert Scarf who attended the Conference and participated actively in the discussions was kind enough to write a very illuminating introduction to the proceedings.

The organizing committee consisted of J. Kenelly, C. J. Ancoin, C. B. Garcia, and S. Karamardian. Mrs. M. Hinton was most helpful in her role as secretary of the conference. Ms. Leslie Cobb typed the manuscripts, Ms. Gini Nordyke drew the technical figures; and Ms. Lynn Mayeda did the final proofreading. We extend our warmest appreciation to these dedicated persons and organizations.

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Introduction

Herbert E. Scarf

I would like to take this opportunity to give a personal view of some of the major developments in the field of fixed point computations during the last half dozen years. As the present volume clearly indicates, the field has progressed substantially during this brief period. My selection of topics will necessarily omit many contributions of continuing importance, which a more leisurely discussion would include.

It may be difficult to recapture the view which many of us had some ten years ago -- that the problems of mathematical programming fall naturally into two quite distinct categories in terms of whether or not they can be solved numerically. On the one hand, there were powerful techniques for solving the linear and non-linear programming problems arising when the production side of the economy is studied in isolation. But on the other hand as soon as a number of independent agents were introduced into the problem -- either as players in an n-person game or consumers in a model of economic equilibrium -- we were forced to be content with theorems advising us of the existence of a solution but with no indication whatsoever of a constructive method for its determination.

By the late 1960's -- the date which I would like to take as the point of departure for the present discussion -- this difficulty had, of course, been overcome. A variety of novel computational techniques had been developed for the approximation of the fixed points of a continuous mapping or correspondence. A substantial number of numerical examples had been successfully attempted and it had become clear that the efficiency of these algorithms was sufficiently high so as to justify their application to realistic problems of moderate size. Moreover, these algorithms were sufficiently flexible, so that a given problem might be solved by one of several methods -- selected so as to exploit the features of the particular problem in question. A survey of the state of the art at this moment of time, together with a number of specific applications may be found in the monograph, The Computation of Economic Equilibria, written in collaboration with Terje Hansen.

In order to introduce the discussion of recent developments, it is useful to remark on one major drawback of the early techniques: they typically required that the computational procedures be initiated on the boundary of the underlying simplex on which the mapping was defined -- in some instances at a vertex of the simplex -- and that a rough measure of accuracy be assigned in advance. If, after the completion of the computation, the accuracy were judged to be insufficient, the only available recourse was to perform the entire computation again with a finer grid size. The results of the earlier computation, which provided a rough indication of the location of the answer, were completely discarded on the grounds that they could not serve as the starting point for a subsequent attempt at higher precision.

The ability to obtain high precision -- say seven or eight significant digits -- with fixed point methods may be of considerable importance even if the underlying data of the problem lack a similar degree of accuracy. In order to estimate the consequences of a particular change in economic policy -- for example, an increase in tariffs -- a general equilibrium model will be solved numerically both before and after the imposition of that policy. But then in order to assert that the policy will lead to a 5% increase in the price of a certain commodity or a 10% decrease in its level of production, these prices and production levels must be calculated with sufficiently high accuracy to reflect the difference in their values.

There are, of course, well established numerical techniques, such as Newton's method, which converge locally to the solution of a system of non-linear equations. In the early programs, I frequently tacked on, after the fixed point approximation, a relatively crude variant of Newton's method which was written specifically for the problem at issue. The results were sufficiently satisfactory so that I know of at least one user who discards fixed point methods completely and is content with guessing an answer which is subsequently refined by Newton's method.

The major drawback of Newton's method -- aside from its apparent lack of harmony with fixed point techniques -- is that a specific program is required for any basic variation in approach to the problem being analyzed. A general equilibrium model, to take one example, may be solved in three of four quite distinct ways depending on its special structure; ideally one would like a method for obtaining high accuracy which is independent of the specific technique selected.

One of the major advances of recent years -- introduced by Curtis Eaves -- may be viewed as a technique which permits a continued improvement in accuracy without recourse to Newton's method, though its ramifications are considerably greater than this somewaht technical justification would suggest. As distinct from earlier authors, Eaves does not work with a simplicial subdivision of the simplex on which the mapping is defined, but rather with a subdivision of the cylinder formed by taking the product of this simplex with a finite interval. The mapping whose fixed point we wish to determine is placed on one end of the cylinder. A trivial mapping of the simplex into itself, whose fixed point is unique and can be placed in an arbitrary location, appears on the opposite end. The two mappings are then joined by a piecewise linear homotopy throughout the cylinder. Given this setting, the earlier methods of simplicial pivoting can be extended so as to construct an algorithm which begins at the pre-assigned location on the end bearing the trivial mapping, and which terminates on the opposite end with an approximate fixed point of the mapping in question.

At the cost of one additional dimension, Eaves' methods permit us to initiate the computation at an arbitrary point and to continue without starting the procedure again, until the desired degree of accuracy is reached. In his remarkable thesis, Orin Merrill describes his independent discovery of a similar technique. Merrill takes the opposite ends of the cylinder described above to be close together, so that each simplex in the decomposition of the cylinder touches both ends. His method (given the illuminating name of the "sandwich" method by later writers) therefore moves from the initial guess to the opposite end of the cylinder very rapidly and as a consequence must be content, in each iteration, with a modest refinement of the previous guess. This approximation, however, may be taken as an initial guess for a subsequent round of the algorithm; the basic cycle is then repeated until a satisfactory accuracy is obtained.

Several variations of these two methods have been programmed and the substantial computational experience obtained by Merrill, Kuhn, Wilmuth and others indicate that they perform remarkably well in contrast to the older techniques which had previously been available. I have not yet seen an explicit comparison to the naive approach in which a rough guess obtained by a fixed point technique is refined by Newton's method, but I feel sure the algorithms of Eaves and Merrill will remain one of the major approaches to fixed point computations in the years to come.

During the last decade all of the fixed point methods with which I had been familiar were based upon a decomposition of the simplex into combinatorial objects such as subsimplices or primitive sets. These objects permit a replacement operation which allows a discrete movement from an initial guess to a final approximation. The difficulties involved in solving a system of non-linear equations and inequalities by directly tracing out a path leading to the solution were replaced by a combinatorial approach to Brouwer's theorem with a constructive flavor.

In the 1950's, however, it had become apparent to mathematicians that many of the intricate arguments of combinatorial topology could be replaced by constructions which were simpler and more intuitive, if adequate differentiability assumptions were placed on the underlying manifolds and functions specifying the particular problem. For example, in a paper written in 1963 Morris Hirsch gives an elegant proof of Brouwer's theorem involving simplicial subdivisions, which has many points of similarity to our computational procedures. In a final paragraph he remarks that an alternative proof can be based on the concept of "regularity" of a differentiable map.

For many of us one of the great surprises of the conference at Clemson was the paper by Kellogg, Li and Yorke which presented the first computational method for finding a fixed point of a continuous mapping making use of the considerations of differential topology instead of our customary combinatorial techniques. In this paper the authors show how Hirsch's argument can be used to define paths leading from virtually any pre-assigned boundary point of the simplex to a fixed point of the mapping. Stephen Smale has also communicated to me recently the results of a similar study analyzing, in detail, the systems of differential equations which arise in this fashion. Both Smale and the three authors mentioned above make the important observation that the path which is being calculated by their methods -- near the fixed point -- is virtually identical with that which would be followed were Newton's method being used.

In order to explain why I find this observation important, let me begin by saying that the differentiable methods consist essentially of tracing the solutions of a system of non-linear equations, say $F(x) = c$, which typically involve one less equation than unknown. Under suitable assumptions this set of solutions forms a one-dimensional differentiable manifold which, by the proper selection of the constant c, will have a component leading from the pre-assigned boundary point to a desired fixed point. But as Eaves and I have shown in a recent paper, virtually all of the simplicial methods which have dominated the horizon during the last decade can be put in an identical form with the function F being <u>piecewise linear</u> rather than differentiable. In other words, there is essentially no distinction between the differentiable methods and those we have been in the habit of using, at least in their general outlines. (There are, of course, technical distinctions: differential equations may have great advantages over difference equations, since the step size for the former need not be assigned in advance -- on the other hand a discrete system may more easily accommodate mappings which are not smooth).

The great similarity between the combinatorial and differentiable algorithms suggests to me the possibility that a discrete method -- such as Eaves' -- in which the grid size is continually decreasing, may very well be behaving like Newton's method near the solution. If this argument could be made precise, it might provide an explanation for the unexpectedly small number of iterations typically required by fixed point methods, and their virtually linear behavior in the neighborhood of the solution. It might also suggest a similarity between the sophisticated methods based on homotopy arguments and the naive approaches in which a fixed point approximation is followed by a conventional Newton's method. The paper by Fisher, Gould and Tolle, presented here, may represent a step in the direction of understanding these phenomena.

A number of authors, including Shapley, Lemke, Kuhn, and Scarf and Eaves have recently been concerned with the application of the topological concepts of index theory to fixed point methods. It is possible to associate with each solution of a particular problem -- say a completely labelled simplex -- an index which is either +1 or -1 . The index is defined fully in terms of the data specifying the solution and is independent of the path which has been used in