COMPLEX VARIABLES FOR SCIENTISTS AND ENGINEERS

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John D. Paliouras

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Complex Variables

FOR

Scientists AND Engineers

John D. Paliouras
Professor of Mathematics
Rochester Institute of Technology

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PREFACE

A FIRST course on complex variables taught to students in the sciences and engineering is invariably faced with the difficult task of meeting two basic objectives: (1) It must create a sound foundation based on the understanding of fundamental concepts and the development of manipulative skills, and (2) it must reach far enough so that the student who completes such a course will be prepared to tackle relatively advanced applications of the subject in subsequent courses that utilize complex variables. This book has been written with these two objectives in mind. Its main goal is to provide a development leading, *over a minimal and yet sound path*, to the fringes of the promised land of applications of complex variables or to a second course in the theory of analytic functions. The arrangement of the topics allows a variety of choices depending on the objectives of the course. The standard topics that are necessary, regardless of the goals of a particular course, are in the mainstream of the development, while peripheral topics are available but not stressed.

The level of the development is quite elementary, and its main theme is the *calculus of complex functions*. The only prerequisite for the study of this book is a standard course in elementary calculus. The topological aspects of the subject are developed only to the extent necessary to give the reader an intuitive understanding of these matters. However, the material contained in the exercises, in the appendices, and in Part III of the book provides ample opportunity for in-depth treatment of most of the concepts if desired. Theorems are discussed informally and, whenever possible, are illustrated via examples, but their proofs are given in the appendices at the end of each chapter. Numerous examples illustrate new concepts soon after they are introduced as well as theorems that lend themselves readily to problem solving. Most of the examples are discussed in detail although, occasionally, some less elementary steps are included which are intended to prompt the inquisitive and conscientious student to seek and provide justifications, thus affording himself the opportunity to review the underlying fundamental notions. Similar practices are followed in the proofs of theorems. Exercises are usually divided into three categories in order to accommodate problems that range from the routine type to the more formidable ones. Constant reference is made to concepts from elementary calculus that are analogous to the concepts under discussion. Thus the student is constantly reminded of the similarities between real and complex analysis. At the same time, cases in which such similarities cease to exist are pointed out.

Earlier versions of this book were used, in the form of notes, in a course given to science and engineering students at the Rochester Institute of Technology. During that period I was fortunate to have received the constructive criticism of many of my colleagues and students, as a result of which many improvements were effected; I am grateful to all of them. I am especially indebted to Professors A. Erskine, L. Fuller, C. Haines, and T. Upson, who read and corrected various parts of the final form of the manuscript. My deep appreciation goes to my wife for her infinite patience and silent encouragement.

J. D. P.

Rochester, N.Y.

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PART I
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Section 1
Complex Numbers and Their Algebra

It is assumed that the reader is familiar with the system of real numbers and their elementary algebraic properties. Our work in this book will take us to a larger system of numbers, which have been given the unfortunate name of "imaginary" or "complex numbers." A historical account of the discovery of such numbers and of their development into prominence in the world of mathematics is outside the scope of this book. We only remark that the need for imaginary numbers arose from the need to find square roots of negative numbers.

The system of complex numbers can be formally introduced by use of the concept of an "ordered pair" of real numbers (a, b). The set of all such pairs with appropriate operations defined on them can be defined to constitute the system of complex numbers. The reader who is interested in this approach is referred to Appendix 1(A). Here, with due apologies to the formalists, we shall proceed to define the complex numbers in the more conventional, if somewhat incomplete, manner.

The set of **complex numbers** is defined to be the totality of all quantities of the form

$$a + ib \qquad \text{or} \qquad a + bi,$$

where a and b are real numbers and $i^2 = -1$. To the reader who may wonder what is so incomplete about this approach, we point out that nothing is said as to the meaning of the implied multiplication in the terms "ib" or "bi."

If $z = a + ib$ is any complex number, then a is called the **real part** of z and b is called the **imaginary part** of z; we sometimes denote them

$$R(z) \qquad \text{and} \qquad I(z),$$

respectively. We point out that both $R(z)$ *and* $I(z)$ *are real numbers*. If $R(z) = 0$ and $I(z) \neq 0$, then z is called **pure imaginary**; e.g., $z = 3i$ is such a number. In particular, if $R(z) = 0$ and $I(z) = 1$, then we write $z = i$ and we call this number the **imaginary unit.** If $I(z) = 0$, then z reduces to the real number $R(z)$; in that sense, one can think of any real number x as being a complex number of the form $z = x + 0i$.

For the remainder of this section

$$z_n = x_n + iy_n, \qquad n = 1, 2, 3,$$

are three arbitrary complex numbers.

Equality of complex numbers is defined quite naturally. Thus

$$z_1 = z_2 \qquad \text{provided that} \qquad x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

The **sum** of two complex numbers is defined by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

and their **product** by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).*$$

The **zero** (additive identity) of the system of complex numbers is the number

$$0 + 0i,$$

which we simply write 0, and the **unity** (multiplicative identity) is the number

$$1 + 0i,$$

which we simply write 1. It is very easy to show that for any $z = x + iy$,

$$z + (0 + 0i) = z \qquad \text{and} \qquad z(1 + 0i) = z.$$

Again, if z is any complex number, there is one and only one complex number, which we will denote by $-z$, such that

$$z + (-z) = 0;$$

* The **nonnegative integral powers** of a complex number z are defined as in the case of real numbers. Thus,

$$z^1 = z,\ z^2 = zz,\ z^3 = z^2 z, \ldots, z^{n+1} = z^n z,$$

and if $z \neq 0$, then $z^0 = 1$.

$-z$ will be called the **negative** (additive inverse) of z and it turns out that

$$\text{if } z = x + yi, \qquad \text{then} \qquad -z = -x - yi.$$

For any nonzero complex number $z = x + iy$ there is one and only one complex number, which we will denote by z^{-1} or $1/z$, such that

$$zz^{-1} = 1;$$

z^{-1} is called the **reciprocal** (multiplicative inverse) of z and a direct calculation yields

$$z^{-1} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i.$$

In order to facilitate further algebraic manipulations, we now define the **difference** of two numbers by

$$z_1 - z_2 = z_1 + (-z_2),$$

which yields

$$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i.$$

We also define the **quotient** of two numbers by

$$\frac{z_1}{z_2} = z_1 z_2^{-1}, \qquad \text{for } z_2 \neq 0,$$

which yields

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}i.$$

In addition to the operations defined above, we have a "new" operation, called **conjugation,** defined on the complex numbers as follows: if $z = x + iy$, then the **conjugate** of z, denoted $\bar{z}$, is defined by

$$\bar{z} = x - yi.$$

Unlike the four "binary operations" defined above, conjugation is a "unary operation": i.e., it acts on one number at a time and has the effect of negating its imaginary part.

Algebraic Properties of Complex Numbers

The operations defined above obey the following laws.

1. Commutative laws:

$$z_1 + z_2 = z_2 + z_1; \qquad z_1 z_2 = z_2 z_1.$$

2. Associative laws:

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3; \qquad z_1(z_2z_3) = (z_1z_2)z_3.$$

3. Distributive law:

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3.$$

4. Distributivity of conjugation:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2; \qquad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2; \qquad \overline{z_1z_2} = \bar{z}_1\bar{z}_2;$$
$$\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2.$$

5. $\bar{\bar{z}} = z$.
6. $z\bar{z} = [R(z)]^2 + [I(z)]^2$.

Some of these properties are proved in the examples that follow; the remaining ones are left for the exercises.

NOTE: The reader may have noted already that the product of two complex numbers is found by effecting an ordinary multiplication of two binomials in which use is made of the reduction formula $i^2 = -1$.

On the other hand, one finds the quotient z_1/z_2 by first multiplying numerator and denominator by $\bar{z}_2$ and then simplifying the resulting expression. Thus,

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2}$$

and, in particular, for the reciprocal of z we have

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}.$$

EXAMPLE 1

If $z = 5 - 5i$ and $w = -3 + 4i$, find $z + w$, $z - w$, zw, z/w, $\bar{z}$, and $\bar{w}$.

Using the definitions of the algebraic operations, we find

$$z + w = (5 - 5i) + (-3 + 4i) = (5 - 3) + (-5 + 4)i = 2 - i.$$

$$z - w = (5 - 5i) - (-3 + 4i) = (5 + 3) + (-5 - 4)i = 8 - 9i.$$

$$zw = (5 - 5i)(-3 + 4i) = (-15 + 20) + (15 + 20)i = 5 + 35i.$$

$$\frac{z}{w} = \frac{5 - 5i}{-3 + 4i} = \frac{(5 - 5i)(-3 - 4i)}{(-3 + 4i)(-3 - 4i)} = -\frac{7}{5} - \frac{1}{5}i.$$

$$\bar{z} = 5 + 5i.$$

$$\bar{w} = -3 - 4i.$$

EXAMPLE 2

Prove the commutative law for addition: $z_1 + z_2 = z_2 + z_1$.

We effect this proof by employing the corresponding law for real numbers, which states that, for any two real numbers a and b, $a + b = b + a$. Thus, we have

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) && \text{definition of sum} \\ &= (x_2 + x_1) + i(y_2 + y_1) && \text{commutativity of real numbers} \\ &= (x_2 + iy_2) + (x_1 + iy_1) && \text{definition of sum} \\ &= z_2 + z_1. \end{aligned}$$

EXAMPLE 3

Prove that conjugation distributes over multiplication: $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

On the one hand, we have

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i} && \text{definition of product} \\ &= (x_1x_2 - y_1y_2) - (x_1y_2 + x_2y_1)i && \text{definition of conjugate} \\ &= (x_1x_2 - y_1y_2) + (-x_1y_2 - x_2y_1)i && \text{definition of negative.} \end{aligned}$$

On the other hand,

$$\begin{aligned} \bar{z}_1\bar{z}_2 &= \overline{(x_1 + y_1 i)}\,\overline{(x_2 + y_2 i)} \\ &= (x_1 - y_1 i)(x_2 - y_2 i) && \text{definition of conjugate} \\ &= (x_1x_2 - y_1y_2) + (-x_1y_2 - x_2y_1)i && \text{definition of product.} \end{aligned}$$

Clearly, the two sides are equal and the proof is complete.

EXAMPLE 4

Prove property 6, namely, that $z\bar{z} = [R(z)]^2 + [I(z)]^2$.

Let $z = x + iy$; then $\bar{z} = x - iy$. Therefore,

$$\begin{aligned} z\bar{z} &= (x + iy)(x - iy) \\ &= x^2 + y^2 \\ &= [R(z)]^2 + [I(z)]^2. \end{aligned}$$

This property simply says that, for any number z, the product $z\bar{z}$ is always a nonnegative real number, since it is the sum of squares of real numbers.

It should be apparent to the reader that most of the familiar algebraic properties of the real numbers are shared by the complex numbers. There is, however, a particular property of the real numbers, namely, the property of order, which does not carry over to the complex case. By this we mean that given two arbitrary complex numbers z and w, no reasonable meaning can be attached to the expression

$$z < w;$$

discussion and proof of this fact are left as an exercise for the reader. See Review Exercise 19 at the end of the chapter.

EXERCISE 1

A

In Exercises 1.1–1.10 perform the indicated operations, reducing the answer to the form $A + Bi$.

1.1. $(5 - 2i) + (2 + 3i)$.
1.2. $(2 - i) - (6 - 3i)$.
1.3. $(2 + 3i)(-2 - 3i)$.
1.4. $-i(5 + i)$.
1.5. $i\bar{i}$.
1.6. $(a + bi)(a - bi)$.
1.7. $6i/(6 - 5i)$.
1.8. $(a + bi)/(a - bi)$.
1.9. $1/(3 + 2i)$.
1.10. $i^2, i^3, i^4, i^5, \ldots, i^{10}$.
1.11. From the results of the preceding exercise, formulate a rule for all the positive integral powers of i and then for the negative ones.
1.12. Show that if $z = -1 - i$, then $z^2 + 2z + 2 = 0$.
1.13. Show that the imaginary unit has the property that $-i = i^{-1} = \bar{i}$.
1.14. If $z = a + bi$, express z^2 and z^3 in the form $A + Bi$.
1.15. Reduce each of the following to the form $A + Bi$:

(a) $\dfrac{1 + i}{1 - i}$. (b) $\dfrac{i}{1 - i} + \dfrac{1 - i}{i}$.

(c) $\dfrac{1}{i} - \dfrac{3i}{1 - i}$. (d) $i^{123} - 4i^9 - 4i$.

B

1.16. (a) For which complex numbers, if any, is $z^{-1} = z$ true?
(b) Similarly for the equation $\bar{z} = -z$.
(c) Similarly for the equation $\bar{z} = z^{-1}$.
1.17. Prove that for any number z,

$$R(z) = \frac{1}{2}(z + \bar{z}) \qquad \text{and} \qquad I(z) = \frac{1}{2i}(z - \bar{z}).$$