NEW CONSTRUCTIONS OF UNCTIONS HOLOMORPHIC IN THE UNIT BALL OF C^n

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Walter Rudin

Madison, Wisconsin
July 1985

Introduction

It was tempting to choose something flippant like

On Beyond Inner Functions!

as a title for these lectures because that would have indicated their background and their content much more precisely, at least to those who have heard of the so-called inner function problem.

Let me begin by explaining what this was all about. A holomorphic function f that maps the open unit disk U in $\mathbb{C}$ into U is called *inner* if its radial limits

$$f^*(e^{i\theta}) = \lim_{r \nearrow 1} f(re^{i\theta})$$

satisfy $|f^*(e^{i\theta})| = 1$ almost everywhere on the unit circle T.

Examples of inner functions in U are, first, the Blaschke products

$$b(z) = cz^k \prod_1^\infty \frac{|\alpha_i|}{\alpha_i} \cdot \frac{\alpha_i - z}{1 - \overline{\alpha}_i z},$$

where $|c| = 1$, k is a nonnegative integer, $\{\alpha_i\}$ is a sequence in $U \backslash \{0\}$ (possibly finite or even empty) that satisfies the convergence-ensuring "Blaschke condition"

$$\sum_1^\infty (1 - |\alpha_i|) < \infty,$$

and, second, the zero-free (usually called "singular") inner functions

$$g(z) = \exp\left\{ - \int_T \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(e^{i\theta}) \right\},$$

one for every positive Borel measure μ on T (including 0) that is singular with respect to Lebesgue measure.

And that's all there is:

Every inner function in U is a product bg.

We thus have a *formula* that gives all inner functions in U.

There are many reasons why these functions are important; here are four:

(i) Every f in any of the classical H^p-spaces, and even in the Nevanlinna class, is a Blaschke product times a zero-free function in the same space.

(ii) Inner functions, and especially Blaschke products, are a fertile source for counterexamples.

(iii) Beurling's theorem: The proper closed invariant subspaces of the unilateral shift operator on a separable Hilbert space are in natural one-to-one correspondence with the inner functions in U (if we identify any two whose ratio is constant).

(iv) The Chang-Marshall theorem: The closed algebras between H^∞ and $L^\infty(T)$ are completely determined by the complex conjugates of the inner functions that they contain.

An excellent reference for all this is Garnett [**1**], the most recent major book on these topics, where many older references may also be found.

Let us now see what happens when the domain U is replaced by the open unit ball B of $\mathbf{C}^n$, $n > 1$. Again, a holomorphic $f: B \to U$ is called *inner* if its radial limits $f^*(\varsigma) = \lim_{r \nearrow 1} f(r\varsigma)$ satisfy $|f^*(\varsigma)| = 1$ for almost every ς in the sphere S that bounds B. (The term "almost every" refers now to Lebesgue measure on S, i.e., to the unique rotation-invariant positive Borel measure σ on S whose total mass is 1.)

When the function theory of B began to be investigated, one wanted of course to describe the inner functions in B and to find out whether their role is as important in B as it is in U.

As a first question—just to illustrate the problem—are there any analogues of finite Blaschke products in B? In other words, are there any inner functions in the so-called *ball algebra* $A(B)$, i.e., in the set of all functions continuous on the closure $\overline{B}$ of B and holomorphic in B?

The answer is no. In fact the following is true:

If $n > 1$ and $f \in A(B)$, then $f(S) = f(\overline{B})$.

This must be obvious to anyone who has even the slightest acquaintance with analytic varieties. But there is also a completely elementary proof which Frank Forelli showed me many years ago and which I find very appealing:

Assume 0 is not in $f(S)$, pick z_0 in B, let L be a complex line in $\mathbf{C}^n$ through z_0, put $D = B \cap L$, and let $C = S \cap L$ be the (oriented) circle that bounds the analytic disc D. Since $n > 1$, S is simply connected, hence C can be shrunk to a point within S, and this shrinks $f \circ C$ to a point within $\mathbf{C} \setminus \{0\}$. The winding number of $f \circ C$, around the origin of $\mathbf{C}$, is therefore 0, so that f (being holomorphic in D) has no zero in D. In particular, $f(z_0) \neq 0$. Q.E.D.

A very similar idea can be used to prove that *no* inner function in B extends continuously to even *one* boundary point, and that its oscillations must in fact be really wild near every boundary point (Theorem 1.2).

On the other hand, it is very easy to prove that the boundary values f^* of any inner function f in B must map S to T in an extremely "evenly distributed" manner (Theorem 1.3).

In view of all this, the question arose whether there were any inner functions in B at all when $n > 1$. (I remember asking this in 1966, but others may, of course, have asked it earlier.) Most people who thought about it came to believe that the answer was no. Section 19.1 of my book *Function Theory in the Unit Ball of* $\mathbb{C}^n$—called UB$\mathbb{C}^n$ in these notes—contains a more elaborate list of related conjectures.

Every one of these turned out to be false:

In the fall of 1981, A. B. Aleksandrov proved that inner functions did exist in B for all n. (He also rewrote the above-mentioned section for the recently published Russian edition of UB$\mathbb{C}^n$.) A few weeks later, Erik Løw, unaware of Aleksandrov's result, proved the same thing by pushing an earlier construction of Hakim and Sibony a bit further than they had done.

Thus, inner functions exist in B. However, it seems very unlikely, because of their inherent pathologies, that they will ever be as explicitly known as they are in U, or that they will ever be as important in B as they are in U. For example, they can certainly not be used in any decent factorization theory (see §10.5), and they fail to have some of the good approximation properties that they have in U (Chapter 13).

But the *techniques* that were developed to solve the inner function problem have already been used very successfully to prove a variety of existence theorems for holomorphic functions f with $|f^*|$ or $\operatorname{Re} f^*$ prescribed on S (almost everywhere or in some other approximate sense) and that can also be made to satisfy various additional interpolation data and growth conditions in B. The present notes are an exposition of some results obtained in this way.

The Hakim-Sibony-Løw proof uses a "bare hands" attack: The idea is to start with a small function and to push its absolute value up toward 1 on the boundary with sufficiently good control to achieve the desired result. The ingenious inductive procedure that is used is quite complicated, but, as Løw [**2**] showed, it works also on arbitrary strictly pseudoconvex domains.

Aleksandrov [**1**] used an approximation theorem in L^p, for $0 < p < 1$ (see Chapter 9, where, for simplicity, I restrict myself to $p = 1/2$), to get his "L^1-modification theorem", which has all sorts of easy consequences (Chapter 10). There are some technical similarities between Aleksandrov's original proof and the Hakim-Sibony paper [**1**], but later he found a much simpler proof, based on the Ryll-Wojtasczcyk polynomials (Chapter 2). These are homogeneous polynomials W_k of degree k that satisfy $\sup_{0<k<\infty} \|W_k\|_\infty / \|W_k\|_2 < \infty$.

Note that the inner functions are (up to multiplication by constants) precisely those functions $f \in H^2(B)$ that have $\|f\|_\infty = \|f\|_2$. Finding such polynomials was thus a step in the right direction towards finding inner functions. However, they were actually constructed in response to the following question by Steve Wainger: Is the $H^2(B)$-closure of the unit ball of $H^\infty(B)$ compact in the norm-topology of $H^2(B)$ when $n > 1$? (Answer: No.)

In these lectures I follow Aleksandrov's second approach for several reasons.

First, it is the easiest one that I know, and it leads to the spectacularly simple L^2-proof given in Chapter 4.

Second, it automatically allows one to solve many existence problems of the type considered here by means of holomorphic functions whose power series have large prescribed gaps (the "E-functions" of Chapter 3). This may not be too exciting, but it should please anyone who, like me, has spent a lot of time and energy on gap series. See, for example, §4.2 and Chapter 6.

Third, it allows one to replace Lebesgue measure by arbitrary positive Borel measures in many of the results with hardly any extra work. This pays off, for instance, in Chapter 5. Also (as Aleksandrov [**2**] has pointed out) it can be used to extend theorems proved in B to domains in B whose Šilov boundary lies in S; this idea is not pursued here, however.

The results up to and including Chapter 8 deal mainly with prescribing $|f^*|$ on S. Problems that involve $\operatorname{Re} f^*$ (Chapters 10, 15, 16) seem to be more delicate. They depend, at least in the present arrangement, on the $L^{1/2}$-approximation theorem of Chapter 9.

In Chapter 17 the mere existence of inner functions is shown to lead to other interesting functions in B by "pulling back" one-variable behavior.

I have not given detailed references for everything that is presented here, partly because much of the material is not in exactly the form in which it was originally stated. Basically, Chapter 2 and Appendix II are due to Ryll and Wojtaszczyk; Chapters 3, 4, 9 and most of 10, 11 are due to Aleksandrov. The credit for the Lusin-type theorem of Chapter 15 (which was new even in the disc!) is shared by Aleksandrov and Paula Russo. The striking results of Chapter 8 and 16 are based on the work of Hakim and Sibony.

There are other types of recent constructions in B (for instance, by Globevnik and Stout [**1, 2**], Løw [**3**], Forstnerič [**1**]) that are not included here.

Chapter 18 shows that certain function spaces that are "sufficiently close" to $A(B)$ contain no inner functions.

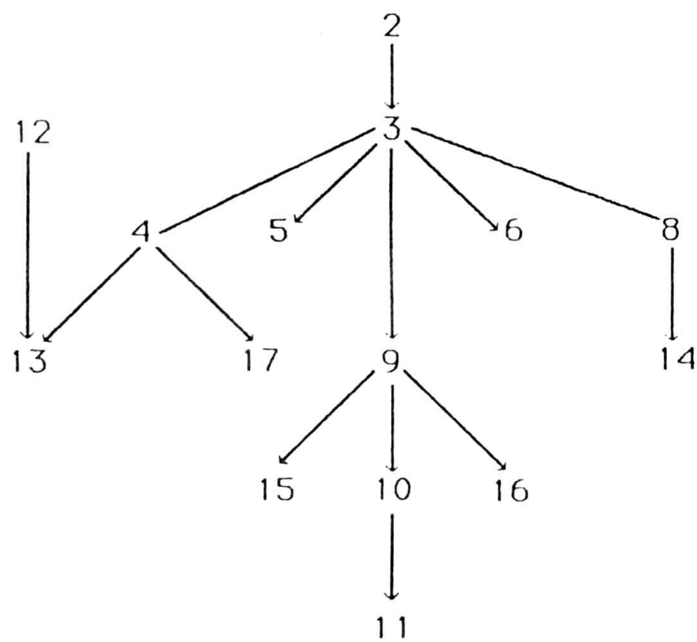
Let me go one step further and end this overly long introduction with a distribution-theoretic "proof" that inner functions don't exist in B when $n > 1$:

Suppose that f is inner in B. Then its boundary function (call it f rather than f^*) satisfies the tangential Cauchy-Riemann equations in the distribution sense. Let L be any one of the tangential C-R operators. Then $Lf = 0$. Since f is inner, $f\bar{f} = 1$ a.e. on S. Thus

$$0 = L(1) = L(f\bar{f}) = fL\bar{f} + \bar{f}Lf = fL\bar{f}.$$

Since $|f| = 1$ a.e., $L\bar{f} = 0$. Thus both f and $\bar{f}$ are holomorphic; hence f is constant. What went wrong?

The following diagram shows how the various chapters depend on each other.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Notation

The terminology and notation used in these notes will be almost exactly as in my book *Function Theory in the Unit Ball of* $\mathbb{C}^n$, which will be referred to by the acronym

$$\mathrm{UB}\mathbb{C}^n.$$

In particular, $\mathbb{C}^n$ is the n-dimensional vector space over the complex field $\mathbb{C}$ with inner product

$$\langle z, w\rangle = \sum_{i=1}^{n} z_i \overline{w}_i,$$

norm $|z| = \langle z, z\rangle^{1/2}$, and corresponding open unit ball

$$B = B_n = \{z \in \mathbb{C}^n : |z| < 1\}$$

whose boundary is the sphere

$$S = \partial B_n = \{\varsigma \in \mathbb{C}^n : |\varsigma| = 1\}.$$

Thus $\overline{B} = B \cup S$ is the closed unit ball.

If $r > 0$, then $rB = \{rz : z \in B\} = \{z : |z| < r\}$.

The term "measure" will refer to finite Borel measures μ. The symbol $\|\mu\|$ denotes the total variation of μ.

σ denotes the unique rotation-invariant probability measure on S. The Lebesgue spaces $L^p(\sigma)$ have their customary meaning, and norms $\|f\|_p$ will refer to σ unless the contrary is stated.

If μ is a measure on S, the symbols

$$\mu \perp \sigma, \quad \mu \ll \sigma$$

mean that μ is singular (or absolutely continuous) with respect to σ.

When $n = 1$, then B, S, σ will usually be replaced by U, T, m.

If $f : B \to \mathbb{C}$ is any function, then

(a) for $0 \le r < 1$, f_r is defined on S by

$$f_r(\varsigma) = f(r\varsigma);$$

(b) for each $\varsigma \in S$, the "slice function" f_ς is defined in U by

$$f_\varsigma(\lambda) = f(\lambda\varsigma);$$

(c) f^* is defined by

$$f^*(\varsigma) = \lim_{r \to 1} f(r\varsigma)$$

at those $\varsigma \in S$ where this limit exists. $H(B)$ is the class of all holomorphic $f: B \to \mathbb{C}$. $f \in H^\infty(B)$ if $f \in H(B)$ and the sup-norm

$$\|f\|_\infty = \sup\{|f(z)|: z \in B\} < \infty.$$

$A(B) = C(\overline{B}) \cap H(B)$ is the ball algebra. $H^p(B)$, for $0 < p < \infty$, is the space of all $f \in H(B)$ that have

$$\sup_{0<r<1} \|f_r\|_p = \sup_{0<r<1} \left\{ \int_S |f_r|^p \, d\sigma \right\}^{1/p} < \infty.$$

$P[f]$ is the Poisson integral of $f \in L^1(\sigma)$, with respect to the ordinary Poisson kernel. Thus $P[f]$ is *harmonic in* B (not M-harmonic, as in UB$\mathbb{C}^n$). Similarly, $P[d\mu]$ is the Poisson integral of the measure μ on S.

$\mathcal{U} = \mathcal{U}(n)$ is the (compact) group of all unitary operators on $\mathbb{C}^n$, with Haar measure dU.

LSC and USC stand for *lower semicontinuous* and *upper semicontinuous*, respectively.

Contents

1. The Pathology of Inner Functions

1.1. INNER FUNCTIONS. By definition, an inner function in B is a nonconstant $f \in H^\infty(B)$ that has $|f^*(\varsigma)| = 1$ a.e. $[\sigma]$ on S.

(The word "nonconstant" is not always included in this definition, but it seems best to say it now rather than have to repeat it every time it is needed.)

This section describes two properties of inner functions in B (when $n > 1$) which, at first glance, almost seem to contradict each other. On the one hand, every inner function is extremely oscillatory near every boundary point; in particular, no inner function extends continuously to even one boundary point. This is a rather weak consequence of Theorem 1.2.

On the other hand, if f is inner and $f(0) = 0$, then its radial limits f^* map S to the unit circle T is such an evenly distributed manner that f^* preserves measure: Explicitly, this means that

$$\sigma((f^*)^{-1}(E)) = m(E)$$

for every Borel set $E \subset T$, where m denotes Lebesgue measure on T normalized so $m(T) = 1$. This is a special case of the (remarkably easy) Theorem 1.3.

1.2. THEOREM. *Suppose that*

(a) Γ *is a nonempty open set in* S,

(b) $r_j \nearrow 1$ *as* $j \to \infty$,

(c) $f \in H^\infty(B)$, f *is not constant*, $|f^*(\varsigma)| = 1$ *a.e. on* Γ.

Then Γ *has a dense* G_δ*-subset* H *such that the set*

$$\{f(r_j\varsigma): j = 1, 2, 3, \ldots\} \tag{1}$$

is dense in the unit disc U *for every* $\varsigma \in H$.

Note that the radii r_j are independent of f.

PROOF. Let A be a countable dense subset of U. For $\alpha \in A$ and positive integers i and k define

$$E_{i,k,\alpha} = \{\varsigma \in \Gamma: |f(r_j\varsigma) - \alpha| \geq 1/k \text{ for all } j > i\}, \tag{2}$$

$$H_\alpha = \Gamma \setminus \bigcup_{i,k} E_{i,k,\alpha} \tag{3}$$

and

(4) $$H = \bigcap_{\alpha \in A} H_\alpha.$$

It is clear that each $E_{i,k,\alpha}$ is (relatively) closed in Γ. If none of them has an interior, then each H_α is a dense G_δ in Γ and so is H. Since $\varsigma \in H_\alpha$ exactly when

(5) $$\liminf_{j \to \infty} |f(r_j\varsigma) - \alpha| = 0,$$

H has the required property.

So let us assume, to reach a contradiction, that some $E_{i,k,\alpha}$ has nonempty interior. Then there is a point $\eta \in \Gamma$ and a $t < 1$ such that $E_{i,k,\alpha}$ contains all $\varsigma \in S$ for which $t < \mathrm{Re}\langle \varsigma, \eta \rangle$.

Put $\Omega = \{z \in B : t < \mathrm{Re}\langle z, \eta \rangle\}$.

Then $|f(z) - a| \geq 1/k$ if $z \in \Omega \cap r_j S$ and $j > i$.

But $f(\Omega \cap r_j B) \subset f(\Omega \cap r_j S)$. A simple winding-number proof of this is almost exactly like the one that occurs in the Introduction, because $\Omega \cap r_j S$ is simply connected.

Hence $|f(z) - \alpha| \geq 1/k$ if $z \in \Omega \cap r_j B$ and $j > i$. Letting $j \to \infty$, we see that

(6) $$|f(z) - \alpha| \geq 1/k \quad \text{for all } z \in \Omega.$$

Now fix a point $p \in \Omega$ close to η. If $\varsigma \in \Gamma$ is such that $\lim f(r\varsigma)$ exists as $r \to 1$, then f has the same limit along the line segment from p to ς; this follows from the Lindelöf-Čirka theorem (Theorem 8.4.4 in UB$\mathbf{C}^n$.) Therefore p lies on a complex line L such that the disc $D = L \cap B$ lies in Ω, and the restriction $f|_D$ of f to D is an inner function (of one complex variable). By (6), $f|_D$ is bounded from α (by $1/k$). Hence $f|_D$ is constant, and this constant has modulus 1.

Thus $|f(p)| = 1$. This holds for all p in some nonempty open subset of Ω. Therefore f is constant, and we have our contradiction.

1.3. THEOREM. *If f is an inner function in B and $f(0) = 0$, then*

(1) $$\int_S (h \circ f^*)\, d\sigma = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{i\theta})\, d\theta$$

for all bounded Borel measurable functions h on T.

If we let h be the characteristic function of a set $E \subset T$, we obtain the statement made in §1.1.

PROOF. First, (1) holds if $h(e^{i\theta}) = e^{ik\theta}$ and $k = 0$. If k is a positive integer, the right side of (1) is 0 and so is the left, since

$$\int_S (h \circ f^*)\, d\sigma = \int_S (f^*)^k\, d\sigma = f(0)^k = 0.$$

Taking complex conjugates, we get the same result for $k < 0$. Hence (1) holds when h is any trigonometric polynomial. The general case follows from the dominated convergence theorem.

1.4. REMARK. If we omit the assumption $f(0) = 0$ in Theorem 1.3, then the result is

$$\int_S (h \circ f^*)\, d\sigma = P[h](f(0)).$$

2. RW-Sequences

The following theorem furnishes one of the tools that will be used in much of our later work.

2.1. THEOREM. *To each dimension n corresponds a constant $c(n) > 0$ with the following property:*

If μ is a positve Borel measure on S, then there exist polynomials W_k ($k = 1, 2, 3, \ldots$) in the variables $z_1, \ldots, z_n$ such that, for every k,

(a) *W_k is homogeneous, of degree k,*

(b) *$|W_k(\varsigma)| \le 1$ for all $\varsigma \in S$,*

(c) *$\int_S |W_k|^2\, d\sigma \ge c(n)$, and*

(d) *$\int_S |W_k|^2\, d\mu \ge c(n) \int_S d\mu$.*

The existence of such sequences $\{W_k\}$ was proved (except for (d), which turns out to be very easy) by Ryll and Wojtaszczyk [**1**]. For brevity, we call them RW-sequences.

The proof of Theorem 2.1 will use the following geometric facts about S.

2.2. LEMMA. *Assume $n > 1$ and define*

(1) $$d(\varsigma, \eta) = (1 - |\langle \varsigma, \eta \rangle|^2)^{1/2}$$

for $\varsigma \in S$, $\eta \in S$ and put

(2) $$E_\delta(\eta) = \{\varsigma \in S : d(\eta, \varsigma) < \delta\} \qquad (0 < \delta \le 1).$$

Then d satisfies the triangle inequality, and

(3) $$\sigma(E_\delta(\eta)) = \delta^{2n-2}.$$

PROOF. Consideration of the orthogonal projection of ς into the space spanned by η shows that

(4) $$d(\varsigma, \eta) = \min\{|\varsigma - \alpha\eta| : \alpha \in \mathbf{C}\}$$

and that $|\alpha| \le 1$ when this minimum is attained. If now ς, μ, ω are in S, choose α, β so that

(5) $$d(\varsigma, \eta) = |\varsigma - \alpha\eta|, \quad d(\eta, \omega) = |\eta - \beta\omega|.$$

Hence

$$d(\varsigma,\omega) \le |\varsigma - \alpha\beta\omega| \le |\varsigma - \alpha\eta| + |\alpha|\,|\eta - \beta\omega|$$
$$\le d(\varsigma,\eta) + d(\eta,\omega).$$

Next, setting $t = (1-\delta^2)^{1/2}$, we see that $\varsigma \in E_\delta(\eta)$ if and only if

$$t < |\langle \varsigma, \eta\rangle| \le 1.$$

Formula 1.4.5(2) of UB $\mathbb{C}^n$ shows therefore that

(6) $$\sigma(E_\delta(\eta)) = 2(n-1)\int_t^1 (1-r^2)^{n-2} r\,dr = \delta^{2n-2}.$$

This proves the lemma. Note that d is not a metric on S, but it may be regarded as a metric on the complex projective space whose points are the circles

(7) $$\Gamma_\varsigma = \{e^{i\theta}\varsigma : -\pi \le \theta \le \pi\} \qquad (\varsigma \in S).$$

2.3. PROOF OF THEOREM 2.1. When $n = 1$, $W_k(z) = z^k$ will do with $c(1) = 1$. So assume $n > 1$, choose $k \ge 1$, pick $\delta > 0$ so that

(1) $$8k\delta^2 = 1,$$

and let $\{\eta_1, \dots, \eta_M\}$ be maximal with respect to having the sets $E_\delta(\eta_j)$ pairwise disjoint. Since d satisfies the triangle inequality, S is then covered by the sets $E_{2\delta}(\eta_j)$. Hence,

$$1 = \sigma(S) \le M\sigma(E_{2\delta}) = M(2\delta)^{2n-2} = M(2k)^{1-n},$$

which gives a lower estimate for M, namely,

(2) $$M \ge (2k)^{n-1}.$$

Next, let $r_1, \dots, r_M$ be Rademacher functions: They are orthonormal on $[0,1]$, and $r_j(t) = \pm 1$. Define

(3) $$Q_t(z) = \sum_{j=1}^{M} r_j(t)\langle z, \eta_j\rangle^k \qquad (0 \le t \le 1,\ z \in \mathbb{C}^n).$$

Proposition 1.4.9 of UB $\mathbb{C}^n$ shows that

(4) $$\int_S |\langle \varsigma, \eta\rangle|^{2k}\, d\sigma(\varsigma) = \frac{(n-1)!k!}{(n-1+k)!}.$$

The orthonormality of $\{r_j\}$ leads therefore to

$$\int_0^1 dt \int_S |Q_t(\varsigma)|^2\, d\sigma(\varsigma) = \int_S \sum_{j=1}^{M} |\langle \varsigma, \eta_j\rangle|^{2k}\, d\sigma(\varsigma)$$
$$= M\frac{(n-1)!k!}{(n-1+k)!}$$
$$\ge \frac{(2k)^{n-1}(n-1)!k!}{(n-1+k)!}$$
$$\ge 1.$$