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Edited by A. Dold and B. Eckmann

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E. B. Saff (Ed.)

Approximation Theory, Tampa

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Editor

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Preface

The Institute for Constructive Mathematics at the University of South Florida had its beginnings in 1985. Its goal is to foster basic research in the variety of mathematical areas that interface with approximation theory, numerical analysis, and pattern recognition. A significant component of this activity has been to provide an atmosphere conducive to research, not only for faculty at U.S.F., but also for distinguished visiting researchers from the U.S. and abroad. In its maiden year, the Institute hosted mathematicians from Canada, England, Israel, Germany, South Africa, Sweden, Switzerland, the People's Republic of China, as well as a variety of universities in the U.S. Measured by any reasonable standard, the individual and joint accomplishments of these visitors have resulted in substantial advancements in approximation theory as well as greater international cooperation and collaboration.

The papers contained in this Proceedings of the Tampa Approximation Seminar serve as a testimonial to the quality and variety of research activities conducted at the Institute. Although the main theme is approximation theory, this collection reflects the individual interests of the visitors to the Institute during the academic year 1985-1986. It is a pleasure to thank the following mathematicians for their contributions to this issue:

P.R. Graves-Morris	J. Nuttall	B. Shekhtman
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E.B. Saff
Director, Institute for
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May 15, 1987

CONTENTS

A FAST ALGORITHM TO SOLVE KALMAN'S PARTIAL REALISATION PROBLEM FOR SINGLE INPUT, MULTI-OUTPUT SYSTEMS

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Abstract A brief review is given of the solution of the scalar partial realisation problem using Padé approximants. The use of simultaneous Padé approximants in the solution of the single input, multi-output partial realisation problem is then discussed. We show how analogues of Frobenius identities are derived for simultaneous Padé approximants of two series, and we give twelve such identities. We show how some of these identities are combined to construct analogues of Baker's and Kronecker's algorithms. These analogues are fast algorithms for simultaneous Padé approximation of two series, and so also for a solution of the single input, two output partial realisation problem.

1. Introduction

In Kailath's book [14] and in the recent reviews by Gragg and Lindquist [10] and by Bultheel and van Barel [9], the authors explain how Padé approximants are used to construct a partial realisation of a single input, single output system. A certain number of Markov parameters of the system are specified as the data, and the Padé method is used to determine the values of the circuit elements. It is also known that simultaneous Padé approximants may be used to construct a partial realisation of a single input, multi-output system in terms of its Markov parameters [13]. This approach provides a solution of Kalman's partial realisation problem for systems [15].

For example, let $h_1, h_2, h_3, \ldots$ be the Markov parameters for a single input, single output system. These parameters formally define the function

(1.1) $$h(z) := h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + \ldots + h_{2N} z^{-2N} + \ldots$$

Ignoring the exceptional, degenerate cases [2, Chap. 2; 3, Chap. 1], an [N/N] type Padé approximant of h(z) takes the form

$$(1.2)\qquad [N/N]_h(z) = \frac{a_1z^{-1} + a_2z^{-2} + \dots + a_Nz^{-N}}{1 + b_1z^{-1} + b_2z^{-2} + \dots b_Nz^{-N}}$$

with the property that

$$(1.3)\qquad [N/N]_h(z) = h(z) + O(z^{-2N-1}) .$$

Many methods exist for computing Padé approximants [2.3]. The resulting parameters $a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N$ are used to design a controller canonical realisation of the system [14], shown in Fig. 1 for the case of N = 3. The first 2N Markov parameters of this system are $h_1, h_2, \dots, h_{2N}$ and in this sense the Padé approximant provides a partial realisation of h(z).

The equivalent example of a single input, p-output system would involve $\underline{h}_1, \underline{h}_2, \dots, \underline{h}_{2N}$ as the Markov parameters to be realised, with $\underline{h}_i \in \mathbb{R}^p$. We formally define

$$(1.4)\qquad \underline{h}(z) = \underline{h}_1z^{-1} + \underline{h}_2z^{-2} + \dots + \underline{h}_{2N}z^{-2N} + \dots$$

For the case of p = 2, for example

$$(1.5)\qquad h(z) = \begin{pmatrix} h_1^{(1)} \\ h_1^{(2)} \end{pmatrix} z^{-1} + \begin{pmatrix} h_2^{(1)} \\ h_2^{(2)} \end{pmatrix} z^{-2} + \dots + \begin{pmatrix} h_{2N}^{(1)} \\ h_{2N}^{(2)} \end{pmatrix} z^{-2N} + \dots,$$

and in this case the simultaneous Padé approximant (SPA) is

$$(1.6)\qquad [N/N]_{\underline{h}}(z) =$$

$$\left(\frac{a_1^{(1)}z^{-1} + a_2^{(1)}z^{-2} + \dots + a_N^{(1)}z^{-N}}{1 + b_1z^{-1} + b_2z^{-2} + \dots + b_Nz^{-N}} , \frac{a_1^{(2)}z^{-1} + a_2^{(2)}z^{-2} + \dots + a_N^{(2)}z^{-N}}{1 + b_1z^{-1} + b_2z^{-2} + \dots + b_Nz^{-N}} \right)$$

Notice the common denominator in (1.6). The direct interpretation of the coefficients of (1.6) as values of the circuit elements is shown in Fig. 2 for the case of N = 2. In this case, the denominator polynomial is given by

$$(1.7)\qquad Q(z) = \begin{vmatrix} 1 & h_3^{(1)} & h_3^{(2)} \\ z^{-1} & h_2^{(1)} & h_2^{(2)} \\ z^{-2} & h_1^{(1)} & h_1^{(2)} \end{vmatrix}$$

up to a (normally irrelevant) constant factor. Of course, Q(z) must

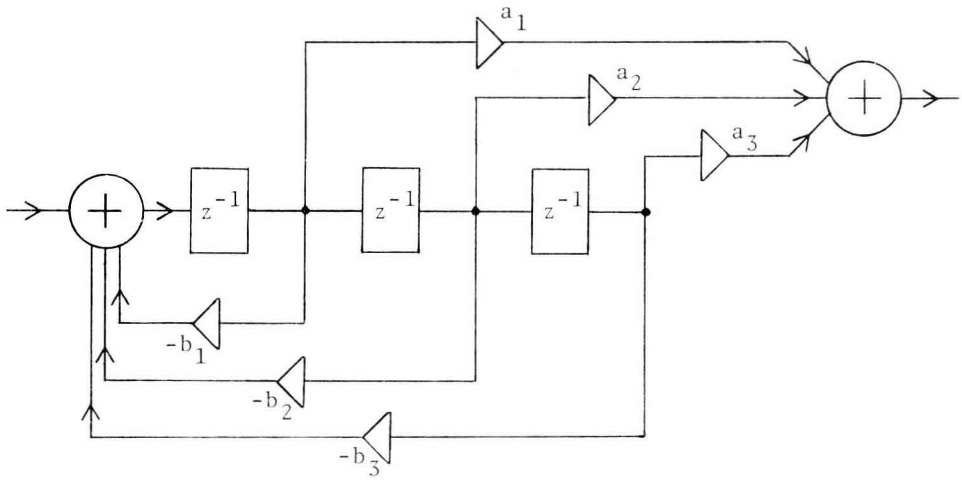

Fig. 1 Controller canonical form of a single input, single output digital system.

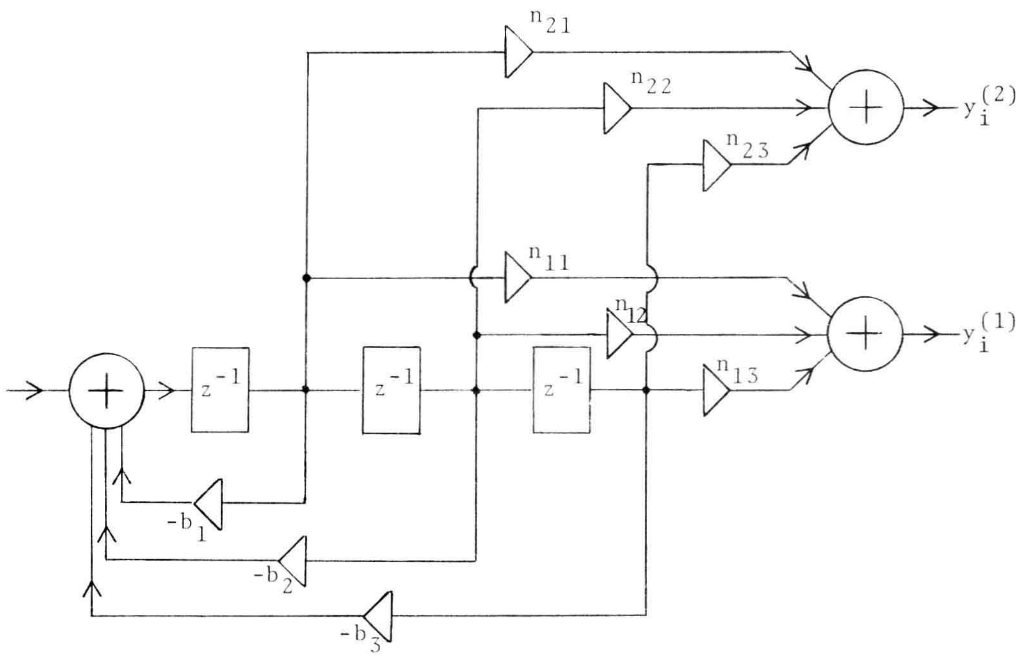

Fig. 2 Controller canonical form of a single input, two output system.

satisfy a stability test. The equivalent general formula, set in its general mathematical context (with $x \equiv z^{-1}$), is given in (2.2). In this paper, we present algorithms suitable for calculating the denominator polynomials and numerator polynomials when the determinants are of high order.

Practical computational methods for this problem have been found by de Bruin [6,7], although his motivation was rather different from ours. Following Padé's approach [19] to Hermite approximation (otherwise called the Latin polynomial approximation problem), de Bruin devised regular algorithms for sequential computation of higher order polynomials from a low order initialisation. In particular, he uses the recurrence

(1.8) $$P_i(k;x) = \alpha_k(x)P_i(k-1;x) + \beta_k(x)P_i(k-2;x) + \gamma_k(x)P_i(k-3;x)$$

(eq (3) of [6]) in which $P_i(k,x)$, $i = 1,2,3$, are the two numerator polynomials and the denominator polynomial constituting $\underline{P}(k;x)$, and $\underline{P}(k;x)$ is the k^{th} vector polynomial in the sequence. In particular, if

(1.9) $$\alpha_k(x) := \alpha_k\ ,\ \beta_k(x) := x\beta_k \text{ and } \gamma_k(x) := x^2\gamma_k,$$

then (1.8), (1.9) and the accuracy-through-order condition (1.4) or (2.3) may be combined to form a regular algorithm for a generalised step-line. In the sequence $\{\underline{P}(k;x),\ k=0,1,2,\ldots\}$, the degrees of the denominator and numerator polynomials have a relative periodical increase of (1, 0, 0) , (0, 1, 0) and (0, 0, 1) respectively.

Here, we adopt a rather different and almost complementary approach. Starting with explicit determinantal formulas (2.2),(2.11) etc., we derive in Section 2 analogues of Frobenius' identities. From these, explicit versions of all the formulas like (1.8) can be derived; even the coefficients can be given explicit determinantal forms which follow from our identities (A) - (L). In Section 2, we develop our theme by deriving "anti-diagonal" regular algorithms, analogous to Baker's algorithm [2, p78] for ordinary Padé approximants. The sequence is a generalised step-line in a four dimensional parameter space.

In Section 3, we develop an algorithm analogous to Kronecker's algorithm for an anti-diagonal sequence in the Padé table. Our analogue, displayed in Figure 6, is based on four term recurrences like (1.8), and so is also a regular algorithm for a generalised (anti-diagonal) stepline.

de Bruin's identity (1.8, 1.9) constitutes a generalisation to single input, two output systems of the Berlekamp algorithm. Other identities of de Bruin [6,7] constitute, in principle, generalisations of Berlekamp's algorithm to single input, multi-output systems. Our algorithm, in Section 3, is a generalisation of Kronecker's algorithm, as discussed by McEliece and Shearer [16]. The contrast between McEliece and Shearer's approach and Berlekamp's approach is described in [17, p369;8]. What is important for our purposes is that Kronecker's algorithm has a simple modification for reliability, as discussed by Warner [20] and McEliece and Shearer [16] and that it has a ready interpretation in terms of the block structure of the Padé table [11]. Our hope is that our algorithm of Section 3 has a simple modification for reliability similar to the Euclidean modification of Kronecker's algorithm. It may also be that de Bruin's approach has a modification for reliability similar to Massey's modification of Berlekamp's algorithm [18].

2. Analogues of Frobenius Identities

The common denominator polynomial for the SPA of type $[N_1, N_2; N_1-m_1, N_2-m_2; m]$ for two series

(2.1) $$f^{(1)}(x) := \Sigma_{i=0}^{\infty} c_i^{(1)} x^i \quad , \quad f^{(2)}(x) := \Sigma_{i=0}^{\infty} c_i^{(2)} x^i$$

is

(2.2) $$Q^{[N_1,\, N_2;\, N_1-m_1,\, N_2-m_2;\, m]}(x)$$

$$= \begin{vmatrix} 1 & c_{N_1}^{(1)} & c_{N_1-1}^{(1)} & \cdots & c_{N_1-m_1+1}^{(1)} & c_{N_2}^{(2)} & c_{N_2-1}^{(2)} & \cdots & c_{N_2-m_2+1}^{(2)} \\ x & c_{N_1-1}^{(1)} & c_{N_1-2}^{(1)} & \cdots & c_{N_1-m_1}^{(1)} & c_{N_2-1}^{(2)} & c_{N_2-2}^{(2)} & \cdots & c_{N_2-m_2}^{(2)} \\ \cdot & & \cdot\ \cdot\ \cdot & & & & \cdot\ \cdot\ \cdot & & \\ \cdot & & \cdot\ \cdot\ \cdot & & & & \cdot\ \cdot\ \cdot & & \\ \cdot & & \cdot\ \cdot\ \cdot & & & & \cdot\ \cdot\ \cdot & & \\ x^m & c_{N_1-m}^{(1)} & c_{N_1-m-1}^{(1)} & \cdots & c_{N_1-m_1-m+1}^{(1)} & c_{N_2-m}^{(2)} & c_{N_2-m-1}^{(2)} & \cdots & c_{N_2-m_2-m+1}^{(2)} \end{vmatrix}$$

up to an arbitrary constant factor, and with the understanding that $c_j^{(1)} = c_j^{(2)} := 0$ for $j < 0$. A result, very similar to (2.2), was originally derived by de Bruin [5]. From (2.2), it is easily verified that polynomials $P^{(1)}(x)$ and $P^{(2)}(x)$ exist for which

(2.3a) $Q^{[\cdot\cdot]}(x)f^{(1)}(x) = P^{(1)}(x) + O\left(x^{N_1+1}\right)$

(2.3b) $Q^{[\cdot\cdot]}(x)f^{(2)}(x) = P^{(2)}(x) + O\left(x^{N_2+1}\right)$

with

(2.4) $\partial P^{(1)} \le N_1-m_1 \ , \ \partial P^{(2)} \le N_2-m_2 \ ,$

(2.5) $\partial Q^{[\cdot\cdot]} \le m \ ,$

and we have used the abbreviation

$$Q^{[\cdot\cdot]}(x) := Q^{[N_1\ N_2;N_1-m_1,N_2-m_2;m]}(x).$$

We view (2.3) as defining $P^{(1)}(x)$, $P^{(2)}(x)$. Formulas (2.2) and (2.3) can be combined to construct determinantal representations of $P^{(1)}(x)$ and $P^{(2)}(x)$. We also define the vector polynomial $\underline{P}^{[\cdot\cdot]}(x) := (P^{(1)}(x), P^{(2)}(x)\ )$. If $Q^{[\cdot\cdot]}(0) \ne 0$, and $N_1 = N_2 = N$,

(2.6) $\underline{P}^{[\cdot\cdot]}(x)/Q(x) = \underline{f}(x) + O(x^{N+1}).$

in which the left-hand side is a vector-valued Padé approximant. With the aid of (2.2) and its companion formulas for $P^{(1)}(x)$ and $P^{(2)}(x)$, we can now derive the analogues of Frobenius' identities for SPAs.

Jacobi's identity [1, p 99] is commonly written as

(2.7) $D_{pq;rs}\, D = D_{p;r}\, D_{q;s} - D_{p;s}\, D_{q;r}\ .$

The notation $D_{p;r}$ in (2.7) denotes the determinant of some original square matrix, of which the p^{th} row and r^{th} column have been deleted; D and $D_{pq;rs}$ have analogous meanings. By applying (2.7) to (2.2) with $(p,q,r,s) = (1,m+1,1,m+1)$, we obtain identity (A):

(2.8) $Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2;\ m]}(x)\ C(N_1,N_2;\ N_1-m_1,N_2-m_2+1;\ m-1)$

$= Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2+1;\ m-1]}(x)\ C(N_1,N_2;\ N_1-m_1,N_2-m_2;m)$

$-xQ^{[N_1-1,N_2-1;\ N_1-m_1-1,N_2-m_2;m-1]}(x)\ C(N_1+1,N_2+1,N_1-m_1+1,N_2-m_2+1;m)$

where

(2.9) $C(N_1,N_2;\ L_1,L_2;\ m) := Q^{[N_1,N_2;\ L_1,L_2;\ m]}(0).$

With $(p,q,r,s) = (1,m+1,1,m_1+2)$ in (2.7), we obtain identity (B):

(2.10) $$Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2;m]}(x)\ C(N_1,N_2-1;\ N_1-m_1,N_2-m_2;\ m-1)$$
$$= Q^{[N_1,N_2-1;\ N_1-m_1,N_2-m_2;m-1]}(x)\ C(N_1,N_2;\ N_1-m_1,N_2-m_2;\ m)$$
$$-xQ^{[N_1-1,N_2-2;\ N_1-m_1-1,N_2-m_2-1;\ m-1]}(x)\ C(N_1+1,N_2+1;\ N_1-m_1+1,N_2-m_2+1;\ m)$$

We regard identities obtained by interchange of the given series $f^{(1)} \leftrightarrow f^{(2)}$ as inessentially different. With this proviso, (2.8) and (2.10) are the only two such identities obtainable from (2.2) using (2.7) directly. However,

(2.11) $$Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2;\ m]}(x)$$
$$= - \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & c^{(1)}_{N_1+1} & c^{(1)}_{N_1} & \cdots & c^{(1)}_{N_1-m_1+1} & c^{(2)}_{N_2} & c^{(2)}_{N_2-1} & \cdots & c^{(2)}_{N_2-m_2+1} \\ x & c^{(1)}_{N_1} & c^{(1)}_{N_1-1} & \cdots & c^{(1)}_{N_1-m} & c^{(2)}_{N_2-1} & c^{(2)}_{N_2-2} & \cdots & c^{(2)}_{N_2-m_2} \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ x^m & c^{(1)}_{N_1-m+1} & c^{(1)}_{N_1-m} & \cdots & c^{(1)}_{N_1-m_1-m+1} & c^{(2)}_{N_2-m} & c^{(2)}_{N_2-m-1} & \cdots & c^{(2)}_{N_2-m_2-m+1} \end{vmatrix}$$

Schwein's identity [1, p 108] follows from Jacobi's identity in the form (2.7) and applied to a determinant such as (2.11) [3, p 85].

With (p, q, r, s) = (1, m+2, 1, m_1+2), we obtain identity (C):

(2.12) $$Q^{[N_1\ N_2;\ N_1-m_1,N_2-m_2;\ m]}(x)\ C(N_1+2,N_2+1;\ N_1-m_1+2,N_2-m_2+1;\ m)$$
$$= Q^{[N_1,N_2;\ N_1-m_1+1,N_2-m_2;\ m-1]}(x)\ C(N_1+2,N_2+1;\ N_1-m_1+1,N_2-m_2+1;m+1)$$
$$+ Q^{[N_1+1,N_2;\ N_1-m_1+1,N_2-m_2;\ m]}(x)\ C(N_1+1,N_2+1;\ N_1-m_1+1,N_2-m_2+1;\ m)$$

With (p,q,r,s) = (1, m+2, 1, m_1+3), we obtain identity (D):

(2.13) $$Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2;\ m]}(x)\ C(N_1+2,N_2;\ N_1-m_1+1,N_2-m_2+1;\ m)$$
$$= Q^{[N_1,N_2-1;\ N_1-m_1,N_2-m_2;\ m-1]}(x)\ C(N_1+2,N_2+1;\ N_1-m_1+1,N_2-m_2+1;\ m+1)$$
$$+ Q^{[N_1+1,N_2-1;\ N_1-m_1,N_2-m_2;\ m]}(x)\ C(N_1+1,N_2+1;\ N_1-m_1+1,N_2-m_2+1;\ m)$$

With (p,q,r,s) = (1, m+2, 1, m+2), we obtain identity (E):

$$(2.14)\quad Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2;\ m]}(x)\ C(N_1+2,N_2+1;\ N_1-m_1+1,N_2-m_2+2;\ m)$$
$$= Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2+1;\ m-1]}(x)\ C(N_1+2,N_2+1;\ N_1-m_1+1,N_2-m_2+1;\ m+1)$$
$$+ Q^{[N_1+1,N_2;\ N_1-m_1,N_2-m_2+1;\ m]}(x)\ C(N_1+1,N_2+1;\ N_1-m_1+1,N_2-m_2+1;\ m)$$

With $(p,q,r,s) = (1,\ 2,\ 1,\ m_1+2)$, we obtain identity (F):

$$(2.15)\quad Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2;\ m]}(x)\ C(N_1+1,N_2;\ N_1-m_1+1,N_2-m_2;\ m)$$
$$= xQ^{[N_1-1,N_2-1;\ N_1-m_1,N_2-m_2-1;\ m-1]}(x)\ C(N_1+2,N_2+1;\ N_1-m_1+1,N_2-m_2+1;\ m+1)$$
$$+\ Q^{[N_1+1,N_2;\ N_1-m_1+1,N_2-m_2;\ m]}(x)\ C(N_1,N_2;\ N_1-m_1,N_2-m_2;\ m)$$

With $(p,q,r,s) = (1,\ 2,\ 1,\ m_1+3)$, we obtain identity (G):

$$(2.16)\quad Q^{[N_1,N_2;\ N_1-m_1;\ N_2-m_2;\ m]}(x)\ C(N_1+1,N_2-1;\ N_1-m_1,N_2-m_2;\ m)$$
$$= xQ^{[N_1-1,N_2-2;\ N_1-m_1-1,N_2-m_2-1;\ m-1]}(x)\ C(N_1+2,N_2+1;\ N_1-m_1+1,\ N_2-m_2+1;\ m+1)$$
$$+\ Q^{[N_1+1,N_2-1;\ N_1-m_1,N_2-m_2;\ m]}(x)\ C(N_1,N_2;\ N_1-m_1,N_2-m_2;\ m)$$

With $(p,q,r,s) = (1,\ 2,\ 1,\ m+2)$, we obtain identity (H):

$$(2.17)\quad Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2;\ m]}(x)\ C(N_1+1,N_2;\ N_1-m_1,N_2-m_2+1;\ m)$$
$$= xQ^{[N_1-1,N_2-1;\ N_1-m_1-1,N_2-m_2;\ m-1]}(x)\ C(N_1+2,N_2+1;\ N_1-m_1+1,N_2-m_2+1;m+1)$$
$$+\ Q^{[N_1+1,N_2;\ N_1-m_1,N_2-m_2+1;\ m]}(x)\ C(N_1,N_2;\ N_1-m_1,N_2-m_2;\ m)$$

Next, we consider the determinant in

$$(2.18)\quad Q^{[N_1,N_2;\ N_1-m_1,N_2-m_2;\ m]}(x)\ .(-1)^{m_1+1}$$
$$= \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & c^{(1)}_{N_1} & \cdots & c^{(1)}_{N_1-m_1+1} & c^{(1)}_{N_1-m_1} & c^{(2)}_{N_2} & \cdots & c^{(2)}_{N_2-m_2+1} \\ x & c^{(1)}_{N_1-1} & \cdots & c^{(1)}_{N_1-m_1} & c^{(1)}_{N_1-m_1-1} & c^{(2)}_{N_2-1} & \cdots & c^{(2)}_{N_2-m_2} \\ \cdot & & \cdots & & & & \cdots & \\ \cdot & & \cdots & & & & \cdots & \\ x^m & c^{(1)}_{N_1-m} & \cdots & c^{(1)}_{N_1-m_1-m+1} & c^{(1)}_{N_1-m_1-m} & c^{(2)}_{N_2-m} & \cdots & c^{(2)}_{N_2-m-m_2+1} \end{vmatrix}$$

With $(p,q,r,s) = (1,\ 2,\ 1,\ m+2)$ in (2.7), (2.18), we obtain identity (I):

$$(2.19)\quad Q^{[N_1,N_2;N_1-m_1,N_2-m_2;m]}(x)\quad C(N_1,N_2;N_1-m_1-1,N_2-m_2+1;m)$$
$$= xQ^{[N_1-1,N_2-1;N_2-m_1-1,N_2-m_2;m-1]}(x)\quad C(N_1+1,N_2+1;N_1-m_1,N_2-m_2+1;m+1)$$
$$+\ Q^{[N_1,N_2;N_1-m_1-1,N_2-m_2+1;m]}(x)\quad C(N_1,N_2;N_1-m_1,N_2-m_2,m)$$

and with (p,q,r,s) = (1, m+2, 1, m+2) in (2.7), (2.18), we obtain identity (J):

$$(2.20)\quad Q^{[N_1,N_2;N_1-m_1,N_2-m_2;m]}(x)\quad C(N_1+1,N_2+1;N_1-m_1,N_2-m_2+2;m)$$
$$= Q^{[N_1,N_2;N_1-m_1,N_2-m_2+1;m-1]}(x)\quad C(N_1+1,N_2+1;N_1-m_1,N_2-m_2+1;m+1)$$
$$+\ Q^{[N_1,N_2;N_1-m_1-1,N_2-m_2+1;m]}(x)\quad C(N_1+1,N_2+1;N_1-m_1+1,N_2-m_2+1;m).$$

There are many other possible values for (p,q,r,s) which yield Frobenius type identities from (2.7), (2.18), but each that we found is a duplicate of one of the preceding identities. However, more identities similar to (A)-(J) follow by elimination of common elements, as happens in the one-dimensional (Padé) case.

From identities (E) and (J), we eliminate $Q^{[N_1,N_2;N_1-m_1,N_2-m_2+1;m-1]}(x)$ and this leads to identity (K):

$$(2.21)\quad Q^{[N_1,N_2;N_1-m_1,N_2-m_2;m]}(x)\quad C(N_1+2,N_2+1;N_1-m_1,N_2-m_2+2;m+1)$$
$$= Q^{[N_1,N_2;N_1-m_1-1,N_2-m_2+1;m]}(x)\quad C(N_1+2,N_2+1;N_1-m_1+1,N_2-m_2+1;m+1)$$
$$+\ Q^{[N_1+1,N_2;N_1-m_1,N_2-m_2+1;m]}(x)\quad C(N_1+1,N_2+1;N_1-m_1,N_2-m_2+1;m+1)$$

Similarly, we put N_2-1 instead of N_2 in (2.14), identity (E), and eliminate $Q^{[N_1,N_2-1;N_1-m_1,N_2-m_2;m-1]}(x)$ from (E) and (D) to obtain identity (L):

$$(2.22)\quad Q^{[N_1,N_2;N_1-m_1,N_2-m_2;m]}(x)\quad C(N_1+2,N_2;N_1-m_1+1,N_2-m_2;m+1)$$
$$= Q^{[N_1,N_2-1;N_1-m_1,N_2-m_2-1;m]}(x)\quad C(N_1+2,N_2+1;N_1-m_1+1,N_2-m_2+1;m+1)$$
$$+\ Q^{[N_1+1,N_2-1;N_1-m_1,N_2-m_2;m]}(x)\quad C(N_1+1,N_2+1;N_1-m_1+1,N_2-m_2;m+1)$$

It may be possible, using the techniques of Padé [19] and de Bruin [5, 6, 7 and references in 7] to show that (A)-(L) are the only Frobenius-type identities for SPAs of two series, but we have not done this here. Just as Frobenius identities are conveniently displayed by diagrams such as $\begin{pmatrix} * & * \\ * & \end{pmatrix}$, so also the identities (A)-(L) should be displayed in a four-dimensional figure. We show the results in two-dimensional sections of figure 4. As an example, identity (A) connects the elements shown in figure 3.

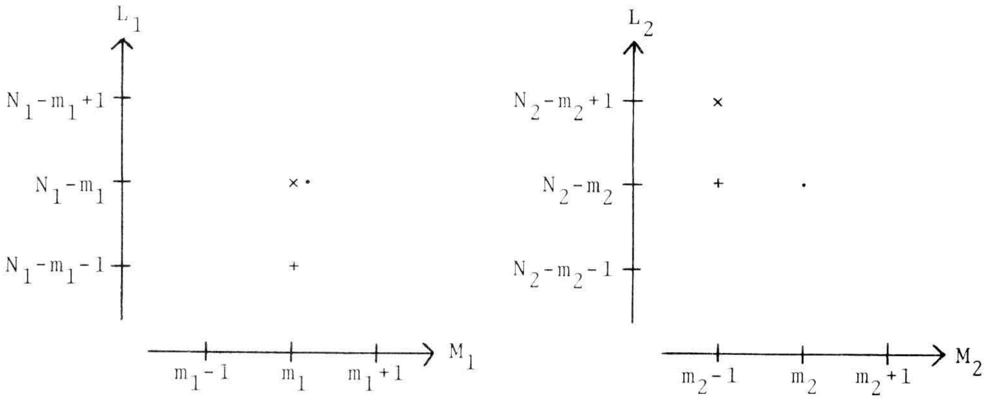

Fig. 3 The locations of the elements of identity (A), denoted by ×, · and +, in the (L_1, M_1) and (L_2, M_2) planes.

The co-ordinates (L_1,M_1), (L_2,M_2) shown in Fig.3 are common to all the identities, and so they are omitted from Fig.4 for conciseness. Fig.4 shows the relative locations of the polynomials in the Frobenius-type identities.

We have expressed the twelve Frobenius type identities among (2.8)-(2.22) in terms of denominator polynomials. It is well known that they also apply to the numerator polynomial $P_1(x)$ (from (2.3a), by truncation) and similarly to $P_2(x)$. Partly to emphasise this point, we define

$$(2.23) \qquad \underline{S}^{[N_1,N_2;L_1,L_2;m]}(x) := \begin{pmatrix} P_1^{[N_1,N_2;L_1,L_2;m]}(x) \\ P_2^{[N_1,N_2;L_1,L_2;m]}(x) \\ Q^{[N_1,N_2;L_1,L_2;m]}(x) \end{pmatrix} .$$

Identity (A) then becomes

$$(2.24) \qquad \underline{S}^{[N_1,N_2;N_1-m_1,N_2-m_2;m]}(x)$$
$$= \alpha \underline{S}^{[N_1,N_2;N_1-m_1,N_2-m_2+1;m-1]}(x)$$
$$+ \beta x \underline{S}^{[N_1-1,N_2-1;N_1-m_1-1,N_2-m_2;m-1]}(x)$$

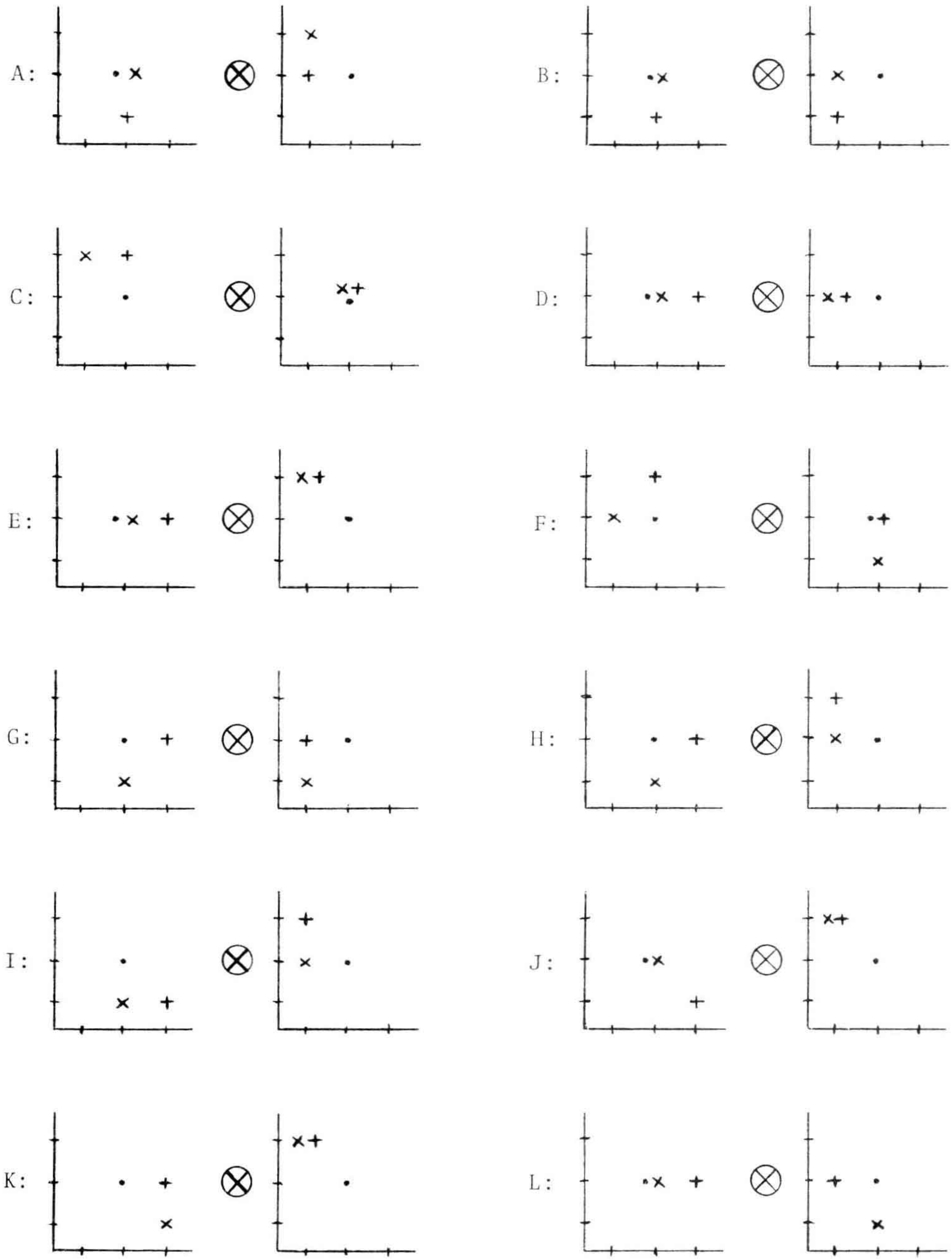

Fig. 4 The relative locations of the polynomials connected by the twelve Frobenius-type identities. Detail of identity A is shown in Fig.3.

for some constants α, β, provided

(2.25) $$C(N_1,N_2;N_1-m_1,N_2-m_2+1;m-1) \neq 0,$$

and similarly for the other Frobenius type identities.

We exploit the result (2.24) as a Euclidean algorithm for the numerators of series 2. We obtain the ratio $\alpha:\beta$ from the leading coefficients of the polynomials on the right-hand side, and use it for the construction of the other two polynomials in $\underline{S}^{[N_1,N_2;N_1-m_1,N_2-m_2;m]}(x)$. We use a prime to denote the interchange $f_1 \leftrightarrow f_2$ in an identity, and present an algorithm based on repeated use of the identities A, E', E; A', C', C.

<u>THE KNIGHT'S MOVE ALGORITHM</u> The name of this algorithm describes the way in which the coefficients of the polynomials are transferred in the construction. We use the notation

(2.26) $$\begin{bmatrix} N_1 & \ell_1 & \\ & & m \\ N_2 & \ell_2 & \end{bmatrix} := S^{[N_1,N_2;\ell_1,\ell_2;m]}(x)$$

to display the construction process in Fig.5. This quantity is well-defined when

(2.27) $$0 \leqslant \ell_1 \leqslant N_1 , \quad 0 \leqslant \ell_2 \leqslant N_2$$

and

(2.28) $$m = (N_1 - \ell_1) + (N_2 - \ell_2).$$

The purpose of the algorithm is to compute the coefficients in the numerator and denominator polynomials of $S^{[N_1,N_2;L_1,L_2;M]}(x)$.

<u>Initialisation</u> We assume that $S^{[N_1,N_2;L_1,L_2;M]}(x)$ is well defined in the sense of (2.27), (2.28). We order the functions $f_1(x)$, $f_2(x)$ so that

(2.29) $$N_1 - L_1 \geqslant N_2 - L_2$$

thereby arranging that the lesser degree reduction (by μ) is assigned to the second series. The degree reduction needed is

(2.30) $$\mu := N_2 - L_2 .$$

We also define