Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries: Fondazione C.I.M.E., Firenze
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Invariant Theory

Proceedings, Montecatini 1982

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COMPLETE SYMMETRIC VARIETIES

by

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H. Seidel, ges. w. 1,70 *

INTRODUCTION

In the study of enumerative problems on plane conics the following variety has been extensively studied ([6],[7],[15],[17],[18],[19],[20],[23],[25]).

We consider pairs (C,C') where C is a non degenerate conic and C' its dual and call X the closure of this correspondence in the variety of pairs of conics in $\mathbb{P}^2$ and $\check{\mathbb{P}}^2$.

On this variety acts naturally the projective group of the plane and one can see that X decomposes into 4 orbits: X_o open in X; X_1, X_2 of codimension 1 and $X_3 = \bar{X}_1 \cap \bar{X}_2$ of codimension 2. All orbit closures are smooth and the intersection of $\bar{X}_1$ with $\bar{X}_2$ is transversal. This theory has been extended to higher dimensional quadrics ([1],[15],[17],[21]) and also carried out in the similar example of collineations ([16]).

The renewed interest in enumerative geometry (see e.g. [11]) has brought back some interest in this class of varieties ([22],[5] , cf.§6).

In this paper we will study closely a general class of varieties, including the previous examples, which have a significance for enumerative problems.

Let $\bar{G}$ be a semisimple adjoint group, $\sigma: \bar{G} \to \bar{G}$ an automorphism of order 2 and $\bar{H} = \bar{G}^{\sigma}$. We construct a canonical variety X with an action of $\bar{G}$ such that

1) X has an open orbit isomorphic to $\bar{G}/\bar{H}$
2) X is smooth with finitely many $\bar{G}$ orbits
3) The orbit closures are all smooth
4) There is a 1-1 correspondence between the set of orbit closures and the family of subsets of a set I_ℓ with ℓ elements. If $J \subseteq I_\ell$ we denote by S_J the corresponding orbit closure
5) We have $S_I \cap S_J = S_{I \cup J}$ and codim S_I = card I

* We thank the "Lessico intellettuale europeo" for supplying the quotation.

6) Each S_I is the transversal complete intersection of the $S_{\{u\}}$, $u \in I$

7) For each S_I we have a $\bar{G}$ equivariant fibration $\pi_I : S_I \to G/P_I$ with P_I a parabolic subgroup with semisimple Levi factor L, σ stable, and the fiber of π_I is the canonical projective variety associated to L and $\sigma|L$

Using results of Bialynicki Birula [2] we give a paving of X by affine spaces and compute its Picard group. We describe the positive line bundles on X and their cohomology in a fashion similar to that of "Flag varieties".

Next we give a precise algorithm which allows to compute the so called characteristic numbers of basic conditions (in the classical terminology) in all cases. The computation can be carried out mechanically although it is very lengthy.

As an example we give the classical application due to H. Schubert [14] for space quadrics and compute the number of quadrics tangent to nine quadrics in general position.

We should now make three final remarks. First of all our method has been strongly influenced by the work of Semple [15], we have in fact interpeted his construction in the language of algebraic groups. The second point will be taken in a continuation of this work. Briefly we should say that a general theory of group embeddings due to Luna and Vust [13] has been used by Vust to classify all projective equivariant embeddings of a symmetric variety of adjoint type and in particular the ones which have the property that each orbit closure is smooth. We call such embeddings wonderful. It has been shown by Vust that such embeddings are all obtained in most cases from our variety X by successive blow ups, followed by a suitable contraction.

This is the reason why we sometimes refer to X as the minimal compactification, in fact it is minimal only among this special class.

The study of the limit provariety obtained in this way is the clue for a general understanding of enumerative questions on symmetric varieties as we plan to show elsewhere.

Finally we have restricted our analysis to characteristic 0 for simplicity. Many of our results are valid in all characteristics (with the possible exception of 2) and some should have a suitable characteristic free analogue. Hopefully an analysis of this theory may have same applications to representation theory also in positive characteristic.

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1. PRELIMINARIES

In this section we collect a few more or less well known facts.

1.1. Let G be a semisimple simply connected algebraic group over the complex numbers. Let $\sigma: G \to G$ be an automorphism of order 2 and $H=G^\sigma$ the subgroup of G of the elements fixed under σ. The homogeneous space G/H is by definition a symmetric variety and more generally, if G' is a quotient of G by a (finite) σ stable subgroup of the center of G, the corresponding G'/H' will again be a symmetric variety.

Let $\underline{g}$, $\underline{h}$ denote the Lie algebras of G, H respectively. σ induces an automorphism of order 2 in $\underline{g}$ which will again be denoted by σ and $\underline{h}$ is exactly the +1 eigenspace of σ.

We recall a well known fact:

PROPOSITION. Every σ-stable torus in G is contained in a maximal torus of G which is σ stable.

If T is a σ stable torus and $\underline{t}$ its Lie algebra, we can decompose $\underline{t}$ as $\underline{t} = \underline{t}_o \oplus \underline{t}_1$ according to the eigenvalues +1, -1 of σ. $\underline{t}_o$ is the Lie algebra of the torus $T_o = T^\sigma$ while $\underline{t}_1$ is the Lie algebra of the torus $T_1 = \{t \in T | t^\sigma = t^{-1}\}$ such a torus is called anisotropic. The natural mapping $T_o \times T_1 \to T$ is an isogeny, it is not necessarily an isomorphism since the character group of T need not decompose under σ into the sum of the subgroups relative to the eigenvalues ± 1. We indicate still by σ the induced mapping on $\underline{t}^*$ and can easily verify in case T is a maximal torus and $\Phi \subseteq \underline{t}^*$ the root system:

i) If $\underline{t} \oplus \sum_{\alpha\in\Phi} \underline{g}_\alpha$ is the root space decomposition of $\underline{g}$ then $\sigma(\underline{g}_\alpha) = \underline{g}_{\alpha\sigma}$, hence $\sigma(\Phi) = \Phi$.

(ii) σ preserves the Killing form.

We want now to choose among all possible σ stable tori one for which dim T_1 is maximal and call this dimension the rank of G/H, indicated by ℓ.

1.2. Having fixed T and so the root system Φ we proceed now to fix the positive roots in a compatible way.

LEMMA. One can choose the set Φ^+ of positive roots in such a way that: If $\alpha \in \Phi^+$ and $\alpha \not\equiv 0$ on $\underline{t}_1$ then $\alpha^\sigma \in \Phi^-$.

PROOF. Decompose $\underline{t}^* = \underline{t}_o^* \oplus \underline{t}_1^*$; every root α is then written $\alpha = \alpha_o + \alpha_1$ and $\alpha^\sigma = \alpha_o - \alpha_1$. Choose two R-linear forms ϕ_o and ϕ_1 on $\underline{t}_o^*$ and $\underline{t}_1^*$ such that ϕ_o and ϕ_1 are non zero on the non zero components of the roots. We can replace ϕ_1 by a multiple if necessary so that, if $\alpha = \alpha_o + \alpha_1$ and $\alpha_1 \neq 0$ we have $|\phi_1(\alpha_1)| > |\phi_o(\alpha_o)|$. Consider now the R-linear form $\phi = \phi_o \oplus \phi_1$, we have that $\phi(\alpha) \neq 0$ for every root α ; moreover if $\alpha \not\equiv 0$ on t_1, i.e. $\alpha = \alpha_o + \alpha_1$ with $\alpha_1 \neq 0$ the sign of $\phi(\alpha)$ equals the sign of $\phi_1(\alpha_1)$. Thus, setting $\Phi^+ = \{\alpha \in \Phi | \phi(\alpha) > 0\}$ we have the required choice of positive roots. Let us use the following notations

$$\Phi_o = \{\alpha \in \Phi |\ \alpha | t_1 = 0\}, \quad \Phi_1 = \Phi - \Phi_o .$$

Clearly $\Phi_o = \{\alpha \in \Phi | \alpha^\sigma = \alpha\}$ while by the previous lemma σ interchanges Φ_1^+ with Φ_1^-.

Having fixed Φ^+ as in the above lemma we denote by $B \subset G$ the corresponding Borel subgroup and by B^- its opposite Borel subgroup.

1.3. It is now easy to describe the Lie algebra $\underline{h}$ in terms of the root decomposition. We have already noticed that $\sigma(\underline{g}_\alpha) = \underline{g}_{\alpha^\sigma}$.

LEMMA. If $\alpha \in \Phi_o$, σ is the identity on g_α.

PROOF. Let x_α, y_α, h_α be the standard sl_2 triple associated to α. Since $\alpha^\sigma = \alpha$ we have $\sigma(h_\alpha) = h_\alpha$. On the other hand since $\sigma(g_{\pm\alpha}) = g_{\pm\alpha}$ we have $\sigma(x_\alpha) = \pm x_\alpha$. Now if $\sigma(x_\alpha) = -x_\alpha$ we must have also $\sigma(y_\alpha) = -y_\alpha$ since $h_\alpha = [x_\alpha, y_\alpha]$. Now if we consider any element $s \in \underline{t}_1$ we have $[x_\alpha, s] = [y_\alpha, s] = 0$ since α vanishes on $\underline{t}_1$ by hypothesis. This implies, setting $t = x_1 + y_1$, that $\underline{t}_1 + Ct$ is a Toral subalgebra on which σ acts as -1. Since we can enlarge this to a maximal Toral subalgebra, we contradict the choice of T maximizing the dimension of T_1.

PROPOSITION. $\underline{h} = \underline{t}_o + \sum_{\alpha \in \Phi_o} g_\alpha + \sum_{\alpha \in \Phi_1} C(x_\alpha + \sigma(x_\alpha))$.

PROOF. Trivial from the previous lemma.

We may express a consequence of this, the so called Iwasawa decomposition: The subspace $\underline{t}_1 + \sum_{\alpha \in \phi_1^+} Cx_\alpha$ is a complement to $\underline{h}$ and so it projects isomorphically onto the tangent space of G/H at H, in particular since Lie B $\supset t_1 + \sum_{\alpha \in \Phi_1^+} \mathbf{C}x_\alpha$, BH $\subset$ G is dense in G.

COROLLARY. dim G/H = dim $\underline{t}_1 + 1/2|\Phi_1|$.

1.4. If $\Gamma \subset \Phi_+$ is the set of simple roots, let us denote $\Gamma_o = \Gamma \cap \Phi_o$, $\Gamma_1 = \Gamma \cap \Phi_1$ explicitely:

$$\Gamma_o = \{\beta_1,\ldots,\beta_k\}; \qquad \Gamma_1 = \{\alpha_1,\ldots,\alpha_j\}.$$

LEMMA. For every $\alpha_i \in \Gamma_1$ we have that α_i^σ is of the form $-\alpha_k - \Sigma n_{ij}\beta_j$ for some $\alpha_k \in \Gamma_1$ and some non negative integers n_{ij}. Moreover, $\alpha_k^\sigma = -\alpha_i - \Sigma n_{ij}\beta_j$.

PROOF. By Lemma 1.2 we know that $\alpha_i^\sigma \in \Phi^-$ hence we can write $\alpha_i^\sigma = -(\Sigma m_{ik}\alpha_k + \Sigma n_{ij}\beta_j)$ where m_{ik} , n_{ij} are non negative integers.Thus $\alpha_i = \alpha_i^{\sigma\sigma} = \sum_k m_{ik}(\sum_t m_{kt}\alpha_t) + \Sigma m_{ik}\Sigma n_{kj}\beta_j - \Sigma n_{ij}\beta_j$. Since the simple roots are a basis of the root lattice we must have in particular $\Sigma m_{ik}m_{kt} = 0$ for $t \neq i$ and $\Sigma m_{ik}m_{ki} = 1$. Since the m_{ij}'s are non negative integers it follows that only one m_{ik} is non zero and equal to 1 and the m_{ki} is also equal to 1.

Now consider the fundamental weights. Since they form a dual basis of the simple coroots we also divide them:

$$\omega_1,\ldots,\omega_j, \qquad \zeta_1,\ldots,\zeta_k \quad \text{where:}$$

$$(\omega_i , \check\beta_j) = 0, \qquad (\omega_i , \check\alpha_j) = \delta_j^i \quad \text{and similarly for the } \zeta_j\text{'s.}$$

Since σ preserves the Killing form we have:

$$(\omega_i^\sigma , \check\beta_j^\sigma) = (\omega_i^\sigma , \check\beta_j) = 0$$

$$\delta_j^i = (\omega_j^\sigma , \check\alpha_i^\sigma) = (\omega^\sigma , \frac{2}{(\alpha_i , \alpha_i)}(-\alpha_k - \Sigma n_{ij}\beta_j))$$

$$= -(\omega_j^\sigma , \frac{2\alpha_k}{(\alpha_i , \alpha_i)}) = \frac{(\alpha_k , \alpha_k)}{(\alpha_i , \alpha_i)} (\omega_j^\sigma , \check\alpha_k)$$

We deduce that

$$\omega_i^\sigma = -\frac{(\alpha_k , \alpha_k)}{(\alpha_i , \alpha_i)}\omega_k.$$

Now ω_i^σ must be in the weight lattice so $\frac{(\alpha_k , \alpha_k)}{(\alpha_i , \alpha_i)}$ is an integer. Reversing the role of i and k we set that it must be 1 so

$$\omega_i^\sigma = -\omega_k .$$

We can summarize this by saying that we have a permutation $\tilde{\sigma}$ of order 2 in the indices 1,2,...,j such that $\omega_i^\sigma = -\omega_{\tilde{\sigma}(i)}$.

DEFINITION. A dominant weight is special if it is of the form $\Sigma n_i \omega_i$ with $n_i = n_{\tilde{\sigma}(i)}$. A special weight is regular if $n_i \neq 0$ for all i.

Thus we have that a weight λ is special iff $\lambda^\sigma = -\lambda$.

1.5.

LEMMA. Let λ be a dominant weight and let V_λ the corresponding irreducible representation of G with highest weight λ. Then if V_λ^H denotes the subspace of V_λ of H-invariant vectors dim $V_\lambda^H \leq 1$ and if $V_\lambda^H \neq 0$ λ is a special weight.

PROOF. Recall that $BH \subset G$ is dense in G so that H has a dense orbit in G/B. Also $V_\lambda \underset{G}{\cong} H^0(G/B,L)$ for a suitable line bundle L on G/B. So if $s_1, s_2 \in V_\lambda^H - \{0\}$, we have that $\frac{s_1}{s_2}$ is a meromorphic function on G/B constant on the dense H orbit, hence s_1 is a multiple of s_2 and our first claim follows.
Now assume $V_\lambda^H \neq 0$ and let $h \in V_\lambda^H - \{0\}$. Fix an highest weight vector $v_\lambda \in V_\lambda$ and let $U \subset V_\lambda$ be the unique T-stable complement to v_λ. Clearly U is B^- stable and $B^-H \subset G$ is dense in G. Then assume $h \in U$ but an the other hand B^-Hh spans V_λ a contradiction. Hence

$$h = av_\lambda + u \quad , \quad a \in \mathbf{C} - \{0\},\ u \in U$$

Since $T_0 \subset H$ and h is H invariant this implies $\lambda | T_0 = id$ hence λ is special.

1.6. If λ is any integral dominant weight and V_λ the corresponding irreducible representation of G with highest weight λ, we define V_λ^σ to be the space V_λ with G action twisted by σ (i.e. we set $g \circ v$ in V_λ^σ to be $\sigma(g)v$, in V_λ).

LEMMA. If λ is a special weight then V_λ^σ is isomorphic to V_λ^*.

PROOF. V_λ^* can be characterized as the irreducible representation of G having $-\lambda$ as lowest weight. Now let $v_\lambda \in V_\lambda$ be a vector of weight λ, let P be the parabolic subgroup of G fixing the line through v_λ. P is generated by the Borel subgroup B and the root subgroups relative to the negative roots $-\alpha$ for which $\langle \alpha,\lambda \rangle = 0$. Thus the parabolic subgroup P^σ, transformed of P via σ, contains the root subgroups relative to the roots $\pm\beta_i$ and also to the roots α^σ, $\alpha \in \phi_1^+$. Now $\sigma(\phi_1^+) = \phi_1^-$ hence P^σ contains the opposite Borel subgroup B^-. Clearly $v_\lambda \in V_\lambda^\sigma$ is stabilized by P^σ hence v_λ is a minimal weight vector and its weight is $-\lambda$. This proves the claim.

1.7. We have just seen that, if λ is an integral dominant special weight V_λ is isomorphic, in a σ-linear way, to V_λ^*. Under this isomorphism the highest weight vector v_λ is mapped into a lowest weight vector in V_λ^*. We normalize the mapping as follows: In V_λ the line $\mathbb{C}v_\lambda$ has a unique T-stable complement $\bar{V}_\lambda$ we define $v^\lambda \in V_\lambda^*$ by: $\langle v^\lambda, v_\lambda \rangle = 1$, $\langle v^\lambda, \bar{V}_\lambda \rangle = 0$. v^λ is easily seen to be a lowest weight vector in V_λ^*.We thus define $h: V_\lambda^* \to V_\lambda$ to be the (unique) σ-linear isomorphism such that $h(v^\lambda) = v_\lambda$.

REMARK. If $V = \oplus V_{\lambda_i}$ is a G-module, the action of G on $\mathbb{P}(V)$ factors through $\bar{G}$ if and only if the center of G acts on each V_{λ_i} with the same character. This applies in particular when V is a tensor product of irreducible G-modules.

We now analyze the stabilizer in G, $\tilde{H}$; of the line generated by h.

LEMMA. i) $\tilde{H}$ equals the normalizer of H.

ii) We have an exact sequence $H \hookrightarrow \tilde{H} \twoheadrightarrow C$, where C is the subgroup of the center of G formed by the elements expressible as $g\sigma(g^{-1})$ for some $g \in G$.

iii) The stabilizer of the line generated by h in $\bar{G}$ is the subgroup fixed by the order two automorphism induced by σ on $\bar{G}$.

PROOF. Assume ${}^g h = \alpha h$, α a scalar. Since h is σ linear, ${}^g h = ghg^{-1} = g\sigma(g^{-1})h$. Therefore $g\sigma(g^{-1})$ acts on V_λ as a scalar. Since V_λ is irreducible this implies $g\sigma(g^{-1})$ lies in the center of G. Conversely if $g\sigma(g^{-1})$ lies in the center of G, $g \in \tilde{H}$. We claim $g \in N(H)$. In fact putting $\zeta = g\sigma(g^{-1})$ we get for each $u \in H$

$$\sigma(g^{-1}ug) = \sigma(g^{-1})u\sigma(g) = \sigma(g^{-1})\zeta^{-1}u\zeta\sigma(g) = g^{-1}ug.$$

Now assume $g \in N(H)$. To see that $g \in \tilde{H}$ it is sufficient to show that $g\sigma(g^{-1})$ lies in the center of G or equivalently that it acts trivially on $\underline{g}$ = Lie G via the adjoint representation. Decompose $\underline{g} = \underline{h} \oplus \underline{g}_1$. And

consider the subgroup K in Aut($\underline{g}$) generated by adN(H) and σ. Since adN(H) is reductive and has at most index 2 in K(N(H) is clearly σ stable) also K is reductive. We claim that both $\underline{h}$ and $\underline{g}_1$ are K stable. In fact $\underline{h}$ is clearly K stable and the reductivity of K implies that it has a K-stable complement in $\underline{g}$, but the unique σ stable complement of $\underline{h}$ is $\underline{g}_1$ so $\underline{g}_1$ is also K stable.
Now notice that since $g \in N(H)$, for each $u \in H$

$$g^{-1}ug = \sigma(g^{-1})u\sigma(g)$$

so that $g\sigma(g^{-1})$ commutes with H and acts trivially on $\underline{h}$. On the other-hand, if $x \in \underline{g}_1$, we have $\mathrm{ad}g^{-1}(x) \in \underline{g}_1$, since $\underline{g}_1$ is K stable, so

$$-\mathrm{ad}g^{-1}(x) = \sigma(\mathrm{ad}g^{-1}(x)) = -\mathrm{ad}\sigma(g^{-1})(x)$$

and hence $\mathrm{ad}g\sigma(g^{-1})(x) = x$ so $g\sigma(g^{-1})$ acts trivially also on $\underline{g}_1$, and so on $\underline{g}$. This proves i).
ii) is clear from the above.
To see iii) notice that the subgroup fixing the line generated by h in $\bar{G}$ is the image in $\bar{G}$ of $\tilde{H}$. Hence if we denote by σ' the automorphism induced by σ on $\bar{G}$ it consists of the elements such that $g\sigma'(g^{-1}) = id$ which are the elements fixed by σ'.

REMARKS. a) H has finite index in $\tilde{H}$..
b) $\tilde{H}$ is the largest subgroup of G with Lie$\tilde{H} = \underline{h}$.

PROOF. a) follows from part ii) of the previous lemma and b) from the fact that H is connected (cf. [28]).

We complete v_λ to a basis $\{v_\lambda, v_1, v_2, \ldots, v_m\}$ of weight vectors and consider the dual basis $\{v^\lambda, v^1, v^2, \ldots, v^m\}$ in V_λ^*. We have $h(v^\lambda) = v_\lambda$ and, if χ_i is the weight of v_i we have $-\chi_i$ as weight of v^i and so $-\chi_i^\sigma$ as weight of $w_i = h(v^i)$. If we identify $\hom(V_\lambda^*, V_\lambda)$ with $V_\lambda \otimes V_\lambda$ we see that h is identified with the tensor

$$h = v_\lambda \otimes v_\lambda + \sum_{i=1}^{m} w_i \otimes v_i.$$

$v_\lambda \otimes v_\lambda$ has weight 2λ while $w_i \otimes v_i$ has weight $\chi_i - \chi_i^\sigma$.

The fact that h is σ-linear implies in particular that it is an H isomorphism. This in turn means that $\bar{h}$ is fixed under H.
Recall that $v_\lambda \otimes v_\lambda$ generates in $V_\lambda \otimes V_\lambda$ the irreducible module $V_{2\lambda}$.
Now order $\alpha_1, \ldots, \alpha_j$ so that $\alpha_s - \alpha_s^\sigma$ are mutually distinct for $s \leq \ell$ (and of course by 1.4 if $j > \ell$, for each $i > \ell$ there is an index $s \leq \ell$ such that $\alpha_s - \alpha_s^\sigma = \alpha_i - \alpha_i^\sigma$). Call $\bar{\alpha}_s = \frac{1}{2}(\alpha_s - \alpha_s^\sigma)$ $s \leq \ell$ the restricted simple roots.

PROPOSITION. i) If λ is a special weight then $V_{2\lambda}$ contains a non zero element h' fixed under H.
ii) h' is unique up to scalar multiples and can be normalized to be

$$h' = v_{2\lambda} + \sum z_i$$

with $v_{2\lambda}$ a highest weight vector of $V_{2\lambda}$ and the z_i's weight vectors having distinct weights whose weight is of the form $2(\lambda - \sum_{s=1}^{\ell} n_s\bar{\alpha}_s)$, n_i non negative integers.
iii) if λ is a regular special weight then we can assume that the vectors $z_1,\ldots,z_\ell$ have weight $2(\lambda-\bar{\alpha}_1),\ldots,2(\lambda-\bar{\alpha}_\ell)$.

PROOF. If we put h' equal to the image of $\bar{h}$ under the unique G-equivariant projection $V_\lambda \otimes V_\lambda \to V_{2\lambda}$, i) ii) follow from the expression of $\bar{h}$ as a linear combination of weight vectors given above. To see iii) assume λ (and hence 2λ) is a regular special weight. Since h' is fixed under H, $xh' = 0$ for any $x \in \underline{h} = \mathrm{LieH}$. In particular if we let $\bar{\alpha}_s$ be a simple restricted root and $\alpha_s \in \Gamma_1$ be such that $\bar{\alpha}_s = \frac{1}{2}(\alpha_s - \alpha_s^\sigma)$ we have (cf. 1.3)

$$(x_{-\alpha_s} + \sigma(x_{-\alpha_s}))h' = 0, \quad x_{-\alpha_s} \in g_{-\alpha_s}.$$

But

$$(x_{-\alpha_s} + \sigma(x_{-\alpha_s}))v_{2\lambda} = x_{-\alpha_s}v_{2\lambda}$$

since $\sigma(x_{-\alpha_s}) \in g_{-\alpha_s^\sigma}$ and $-\alpha_s^\sigma \in \Phi_1^+$. Also by the regularity of 2λ $x_{-\alpha_s}v_\lambda$ is a non zero weight vector of weight $2\lambda - \alpha_s$. It follows that for some z_i, $\sigma(x_{-\alpha_s})z_i = -x_{-\alpha_s}v_{2\lambda}$ so that z_i has weight $2(\lambda - \bar{\alpha}_s)$ proving the claim.

The analysis just performed does not exclude that V_λ itself may contain a non zero H-fixed vector h_λ. In this case we have seen that we can normalize h_λ : $h_\lambda = v_\lambda + \sum u_i^1$, u_i^1 lower weight vectors. It follows that $h_\lambda \otimes h_\lambda$ must project to h in $V_{2\lambda}$ (by uniqueness of h).

Now the dominant λ's for which $\dim V_\lambda^H = 1$ have been determined completely [9],[24], the result is as follows: Let us indicate Λ^1 such set.

Consider the Killing form restricted to $\underline{t}_1$ and thus to $\underline{t}_1^*$. We look at the restriction of Φ_1 to $\underline{t}_1$, if $\alpha \in \Phi$, let us indicate $\bar{\alpha}$ the restriction of α to $\underline{t}_1$.

If $\mu \in \underline{t}_1^*$ let us indicate by $\tilde{\mu}$ its extension to $\underline{t}$ by setting it 0 to $\underline{t}_0$.

Then the theorem in [9] is:
Consider the set of $\mu \in \underline{t}_1^*$ such that

$$\frac{(\mu,\bar{\alpha})}{(\bar{\alpha},\bar{\alpha})} \text{ is a positive integer for all } \alpha \in \phi$$

Then the set of weights $\tilde{\mu}$ of $\underline{t}$ so obtained is exactly the set Λ^1 of λ for which dim $V_\lambda^H = 1$. One can understand this theorem in a more precise way. If $\alpha \in \phi$, then $\bar{\alpha}$ is exactly $\frac{1}{2}(\alpha - \alpha^\sigma)$, and $(\bar{\alpha};\bar{\alpha}) = (\tilde{\bar{\alpha}},\tilde{\bar{\alpha}})$. Now also a weight ω is of the form $\tilde{\mu}$ if and only if $\omega = \frac{1}{2}(\omega - \omega^\sigma)$. For such weights of course $(\omega,\beta_j) = 0$. Thus we see immediately that Λ^1 is contained in the positive lattice generated by the weights ω_i if $\sigma(i) = i$ and $\omega_i - \omega_{\tilde{\sigma}(i)}$ if $\tilde{\sigma}(i) \neq i$.

To understand exactly the nature of Λ^1 we must see if

$$\frac{(\omega_i,\bar{\alpha})}{(\bar{\alpha},\bar{\alpha})} \quad \left(\text{resp. } \frac{(\omega_i - \omega_{\tilde{\sigma}(i)},\bar{\alpha})}{(\bar{\alpha},\bar{\alpha})}\right)$$

is an integer.

Since in any case for such special weights λ we have $2\lambda \in \Lambda^1$ one knows at least that these numbers are half integers. It follows in any case that Λ^1 is the positive lattice generated by the previous weights or their doubles. i.e.

$$\Lambda^1 = \sum_{i=1}^{\ell} n_i\mu_i, \quad n_i \geq 0 \quad \text{and} \quad \mu_i = \omega_i \quad \text{or}$$

$2\,\omega_i$ (resp. $\omega_i - \omega_{\tilde{\sigma}(i)}$ or $2(\omega_i - \omega_{\tilde{\sigma}(i)})$). Recall that $\ell = \mathrm{rk}\,\Lambda^1$ is also the rank of the symmetric space.

2. THE BASIC CONSTRUCTION

2.1. We consider now a regular special weight λ and all the objects of the previous paragraph V_λ, $h' \in V_{2\lambda}$. Let now $\mathbb{P}_{2\lambda} = \mathbb{P}(V_{2\lambda})$ be the projective space of lines in $V_{2\lambda}$ and $\tilde{h} \in \mathbb{P}_{2\lambda}$ be the class of h'. The basic object of our nalysis is the orbit $G \cdot \tilde{h}$ of $\tilde{h}$ in $\mathbb{P}_{2\lambda}$ and its closure $\underline{\bar{X}} = G \cdot \tilde{h}$. By construction $\underline{\bar{X}}$ is a G-equivariant compactification of the homogeneous space $G \cdot \tilde{h}$, furthermore the stabilizer $\tilde{H}$ of $\tilde{h}$ is a group containing the subgroup H.

We will analyze in detail $\tilde{H}$ and in particular will see that H has finite index in $\tilde{H}$. For the moment we concentrate our attention to $\underline{\bar{X}}$. Since $\underline{\bar{X}}$ is closed in $\mathbb{P}_{2\lambda}$ and G stable it contains the unique closed orbit of G acting on $\mathbb{P}_{2\lambda}$, i.e. the orbit of the highest weight vector $v_\lambda \otimes v_\lambda$. Now the following general lemma is of trivial verification:

LEMMA: If X is a G variety with a unique closed orbit Y and V is an open set in X with $Y \cap V \neq \phi$ then $X = \bigcup_{g \in G} gV$.

The use of this lemma for us is in the fact that it allows us to study the singularities of X locally in V.

2.2. Let λ be a regular special weight. Consider a G module $W \simeq V_{2\lambda} \oplus \sum V_{\mu_i}$ with $\mu_i = 2\lambda - \sum n_i 2\bar{\alpha}_i$ some $n_i > 0$. Let $h \in \underset{\sim}{V}$ be an H invariant with component h' in $V_{2\lambda}$. Decompose $V_{2\lambda} = \mathbf{C}v_{2\lambda} \otimes \tilde{V}_{2\lambda}$ in a T stable way and consider the open affine set $A = v_{2\lambda} \otimes \tilde{V}_{2\lambda} \otimes \sum v_{\mu_i} \subseteq \mathbb{P}(W)$. Notice that $h \in A$ and A is B^- stable.

LEMMA: The closure in A of the T^1 orbit T^1h is isomorphic to ℓ dimensional affine space $\mathbb{A}^\ell$. The natural morphism $T^1 \to T^1h \hookrightarrow \mathbb{A}^\ell$ has coordinates $t \to (t^{-2\bar{\alpha}_1}, t^{-2\bar{\alpha}_2}, \ldots, t^{-2\bar{\alpha}_\ell})$. T^1h is identified with the open set of $\mathbb{A}^\ell$ where all coordinates are non zero.

PROOF: By prop. 1.7 we can write $h = v_{2\lambda} + \sum_i z_i'$ with z_i' weight vectors of weights $\chi_i = 2\lambda - \sum m_j^{(i)} 2\bar{\alpha}_j$ (some $m_j > 0$) and $z_1', \ldots, z_\ell'$ of weights $2\lambda - 2\bar{\alpha}_1, \ldots, 2\lambda - 2\bar{\alpha}_\ell$. Let us apply an element $t \in T'$ to h we get $th = t^{2\lambda}v_{2\lambda} + \sum t^{\chi_i} z_i'$ which, in affine coordinates, is

$$v_{2\lambda} + \sum t^{\chi_i - 2\lambda} z_i'.$$

From the previous formula $\chi_i - 2\lambda = \sum_j m_j^{(i)}(-2\bar{\alpha}_j)$, this means that the coordinates of th are monomials in the first ℓ coordinates.

This means that T^1 maps to a closed subvariety of $\bar{A}$, isomorphic to affine space $\mathbb{A}^\ell$, via the coordinates $(t^{-2\bar{\alpha}_1}, \ldots, t^{-2\bar{\alpha}_\ell})$. Since the restricted simple roots are linearly independent the orbit T^1h is the open dense subset of $\mathbb{A}^\ell$ consisting of the elements with non zero coordinates.

REMARK. The stabilizer of h in T^1 is the finite subgroup of the elements $t \in T^1$ with $t^{2\bar{\alpha}_i} = 1$.

2.3. Let us go back to $\bar{X} \subseteq P_{2\lambda}$. Consider the open affine set $A = v_{2\lambda} \oplus \tilde{V}_{2\lambda} \subseteq P_{2\lambda}$ and set $V = A \cap \bar{X}$. Remark that V is B^- stable, it contains $\tilde{h}$ and so also $\mathbb{A}^\ell$, the closure of $T^1\tilde{h}$ in A, hence $v_{2\lambda} \in V$ and therefore V has a non empty intersection with the unique closed orbit or G in $P_{2\lambda}$.

Let U be the unipotent group generated by the root subgroups X_α, $\alpha \in \phi_1^-$. Since U acts on V we have a well defined map $\phi\colon U \times \mathbb{A}^\ell \to V$ by the formula $\phi(u,x) = u \cdot x$.