A COURSE IN DESCRIPTIVE GEOMETRY

V. O. GORDON and M. A. SEMENTSOV-OGIEVSKII

ИЗДАТЕЛЬСТВО · МИР ·

КУРС НАЧЕРТАТЕЛЬНОЙ ГЕОМЕТРИИ

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ИЗДАТЕЛЬСТВО «НАУКА»
МОСКВА

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INTRODUCTION

Descriptive geometry is one of the fundamental disciplines making up an engineering education.

It is concerned with setting forth and justifying methods of constructing representations of three-dimensional forms in the plane, as well as methods of solving geometrical problems on the basis of given representations of these forms. As is known, three-dimensional forms can be represented not only in the plane, but on some other surface, for instance, a cylinder or sphere. The latter cases are studied in special branches of descriptive geometry.

The representations constructed according to the rules of descriptive geometry enable us to visualize the shape of objects and their relative positions in space, to determine their dimensions, and to study their geometrical properties.

Descriptive geometry develops the student's three-dimensional imagination by making frequent appeals to it.

Finally, descriptive geometry provides a number of practical means for engineering drawings, ensuring their clarity and accuracy, and, hence, the possibility of manufacturing the represented objects.

The rules for constructing representations, set forth in descriptive geometry are based on the method of projections.

It is standard practice to begin studying the method of projection with the construction of the projections of the *point*, since the construction of the projections of any three-dimensional form involves considering a number of points belonging to this form.

CHAPTER 1

THE METHOD OF PROJECTING

Sec. 1. Central Projections

To obtain *central projections* we must take a plane (the *plane of projection*) and a fixed point not in the plane (the *centre of projection*). The *method of central projection* is illustrated in Fig. 1 showing the plane P and the point S. Taking a point A and drawing through S and A a straight line, we intersect the plane P at point a_p. We then proceed in the same way with the points B and C. The points a_p, b_p, c_p are *central projections* of the points A, B, C on the plane P: they are obtained as the intersections of the *projecting lines* (or rays called the projectors) SA, SB, SC with the plane of projection*.

If for a certain point D (Fig. 1) the projector turns out to be parallel to the plane of projection, then we conventionally consider that they intersect, but at a point at infinity. The point D also has a projection which is an infinitely distant point (d_∞).

Leaving the position of the plane P unchanged and taking a new centre S_1 (Fig. 2), we obtain a new projection of the point A (point a_{p1}). If the centre S_2 is taken on the same projector SA, then the projection a_p remains unchanged.

Hence, given the plane and the centre of projection, we can construct the projection of a point (Fig. 1), but having the projection of a point (for instance, a_p) it is impossible to determine the position of the point A in space, since any point on the projector SA is projected into one and the same point. Obviously, for obtaining the unique solution some additional conditions are required.

The projection of a line can be constructed by projecting a number of its points (Fig. 3), all the projectors generating a conical surface** or being

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*The centre of projection is also called the *pole of projection*, and central projection is termed '*polar projection*'.

**That is why central projections are also called *conical*.

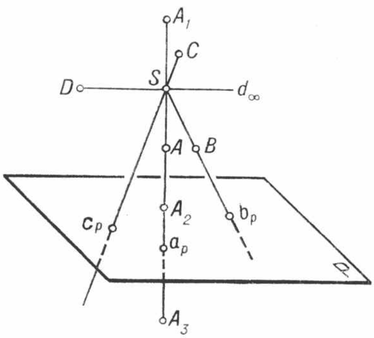

Fig. 1

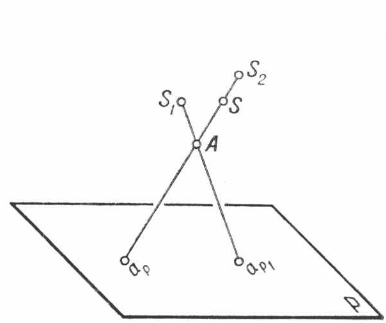

Fig. 2

located in one plane (for instance, when projecting a straight line not passing through the centre of projection, or a polygonal line and a curve all points of which lie in a plane coinciding with the projecting plane).

Obviously, the projection of a line is obtained as the intersection of the projecting surface with the plane of projection (Fig. 3). But, as is shown in Fig. 4, the projection of a line does not determine the line being projected, since the projecting surface may contain a number of lines which are projected on the plane of projection into one and the same line.

From the projecting of points and lines we may pass over to projecting a surface and a solid.

Sec. 2. Parallel Projections

Let us now consider the *method of parallel projection*.

When the centre of projection is a point at infinity, all the projections are parallel. They are drawn in the direction indicated by an arrow (see Fig. 5). *The projections constructed in such a way are called parallel.*

Thus, parallel projection may be considered as a particular case of central projection.

Hence, *the parallel projection of a point is defined as the point of intersection of a projector drawn parallel to a given direction with the plane of projection.*

To obtain a parallel projection of a line it is sufficient to construct projections of a number of its points and to draw through them a line (Fig. 6).

In this case all the projectors form a cylindrical surface, therefore parallel projections are also called *cylindrical*.

In parallel projections, the same as in central projections in general:

(1) for a straight line the projecting surface in the general case is a plane, and therefore a straight line is, in general, projected into a straight line;

(2) any point and line in space have its unique projection;

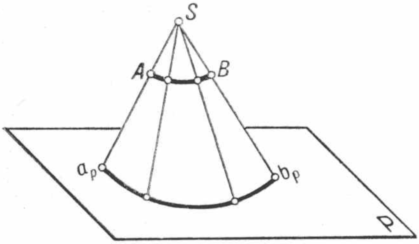

Fig. 3

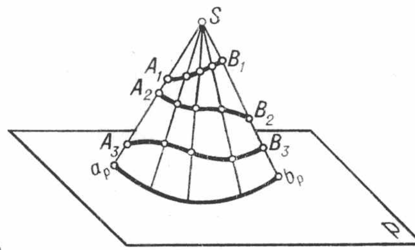

Fig. 4

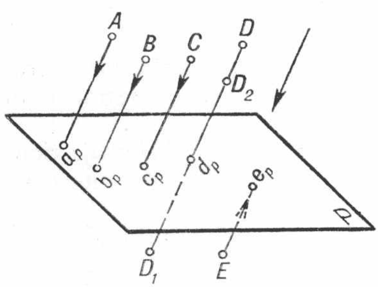

Fig. 5

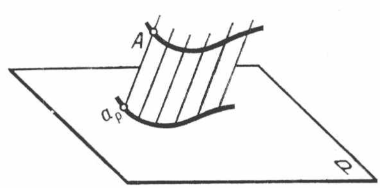

Fig. 6

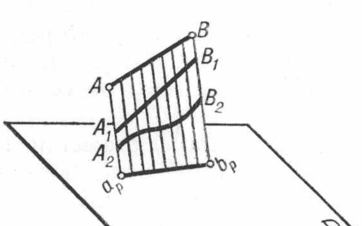

Fig. 7

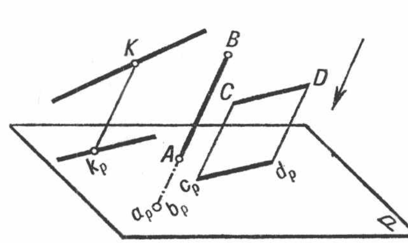

Fig. 8

(3) each point on the projection plane may be the projection of a set of points if they are situated on a straight projector (the point d_p in Fig. 5 is the projection of points D, D_1, D_2);

(4) any line on the projection plane may turn out to be the projection of a set of lines if they are contained in a common projecting plane (Fig. 7: the line segment a_pb_p serves as the projection of line segments AB and A_1B_1 of straight lines and the segment A_2B_2 of a plane curve); obviously, to obtain the unique solution in this case, some additional conditions are required;

(5) to construct the projection of a straight line it is sufficient to project two of its points and to draw a straight line through the obtained projections of these points;

(6) if a point belongs to a straight line, then the projection of the point belongs to the projection of this line (point K in Fig. 8 belongs to a straight line, and the projection k_p belongs to the projection of this line).

In addition to the above listed properties the following is valid for parallel projections:

(7) if a straight line is parallel to the direction of projecting (as, for instance AB in Fig. 8), then the projection of the line (and any of its segments) is a point (a_p, or b_p);

(8) a segment of a straight line parallel to the plane of projection is projected on this plane true length (Fig. 8: CD is equal to c_pd_p as segments of parallel lines between parallel lines).

Later on we shall consider some more properties of parallel projections showing what relationships inherent in objects under considerations are retained in the projections of these objects.

Applying the methods of parallel projection of a point and a line, it is possible to construct parallel constructions of a surface and a solid.

Parallel projections are subdivided into oblique and orthogonal projections. In the first case the direction of projecting forms with the plane of projection an angle not equal to 90°, whereas in the second case the projectors are perpendicular to the plane of projection.

When considering parallel projections the viewer should be imagined as located at an infinite distance from the image. But in reality objects and their images are viewed from a finite distance, and the rays entering viewer's eye form a conical, but not a cylindrical, surface. Hence, a more natural picture is obtained (provided certain conditions are observed) using a central projection, but not a parallel one. Therefore, when it is required to get a representation producing the same visual impression as the object itself, we usually resort to perspective projections which are based on central projecting.

But despite the above mentioned conditionality parallel projecting is widely applied. This is explained by the properties of parallel projections as well as by a comparatively greater simplicity of the constructions involved.

Sec. 3. Monge's Method

Information and methods of construction required for representing space forms in the plane have accumulated gradually since ancient times. During long period of time plane representations were accomplished mostly in a visualized manner. With the development of engineering paramount importance was acquired by the need of developing a method which would ensure accuracy and easiness in measuring graphical representations, i.e. ensure the possibility to locate each point of the representation relative to other points or planes and to determine the dimensions of line segments and figures by simple methods.

The accumulated rules and methods for constructing such representations were systemized and further developed by the great French mathematician G. Monge, the inventor of descriptive geometry, in his work "Essais sur les Géométrie déscriptive" issued in 1779.

Gaspard Monge (1746-1818) is known in history as a great French mathematician, engineer, a public man and a stateman during the period of Revolution of 1789-94 and the rule of Napoleon.

In 1768 Monge became professor of mathematics and in 1771 professor of physics at Mézières; in 1780 he was appointed to a chair of hydrolics at the Lyceum in Paris (held by him together with his appointments at Mézières) and was received as a member of the Académie.

Monge wrote various mathematical and physical papers.

He took an active part in the measures for the establishment of the normal school and of the well-known École Polytechnique (Polytechnic school) and was at each of them professor for descriptive geometry.

Being one of the ministers (Minister of Marine) in the revolutionary government of France, Monge did much for its defence against foreign invaders, as well as for the victory of the revolutionary troops.

For a long time Monge had no possibility to publish his work containing the description of the method elaborated by him. It was considered so valuable that it long was guarded as military secret. Only at the very end of the 18th century the prohibition to publish his book was rescinded by the French government, and in 1799 Monge issued the mentioned work in which he gave a comprehensive description of his method.

On the fall of Napoleon he was deprived, as a Bonapartist, of all his honours and excluded from the list of members of the reconstructed bodies. He was forced to hide, and ends his life in poverty.

The method of parallel projection (with orthogonal projections on two mutually perpendicular planes of projection) invented by Monge was and remains the principal method applied for making engineering drawings, since it ensures obviousness, accuracy, and easiness in measuring representations of various objects in the plane.

As a result of his researches, Monge arrived at that general method of the application of geometry to the arts of construction that later became known as descriptive geometry.

The present course deals preferably with orthogonal projections, which are a particular case of parallel oblique projections. If the latter are used, it will be mentioned each time.

QUESTIONS TO CHAPTER 1

1. How do we construct the central projection of a point?
2. In what case does the central projection of a straight line represent a point?
3. What does the method of parallel projection consist in?
4. How is the parallel projection of a straight line constructed?
5. May the parallel projection of a straight line represent a point?
6. If a point belongs to a given straight line, then what are their relative positions?
7. In what case of parallel projection is a segment of a straight line projected true length?
8. What is the Monge's method?
9. How is the word aorthogonal' deciphered?