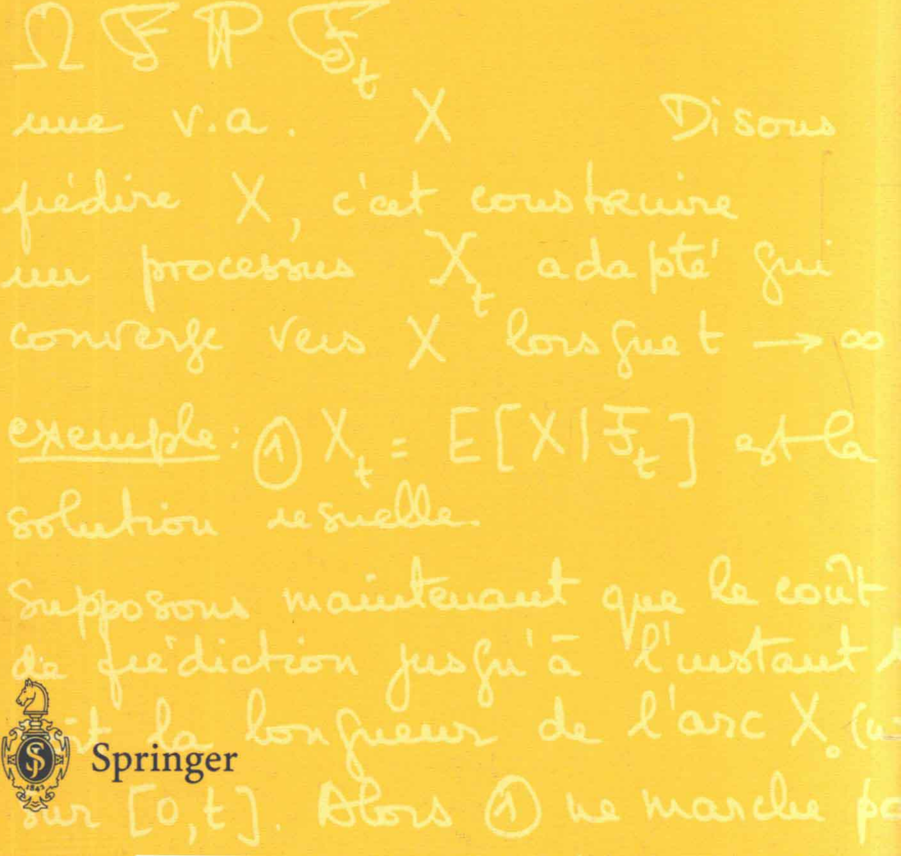
Lecture Notes in Mathematics

1832

J. Azéma M. Émery
M. Ledoux M. Yor (Eds.)

Séminaire de Probabilités XXXVII



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Lecture Notes in Mathematics

Edited by J.-M. Morel, F. Takens and B. Teissier

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1. Lecture Notes aim to report new developments in all areas of mathematics and their applications - quickly, informally and at a high level. Mathematical texts analysing new developments in modelling and numerical simulation are welcome. Manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. They should provide sufficient motivation, examples and applications. There should also be an introduction making the text comprehensible to a wider audience. This clearly distinguishes Lecture Notes from journal articles or technical reports which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character.

2. In general SUMMER SCHOOLS and other similar INTENSIVE COURSES are held to present mathematical topics that are close to the frontiers of recent research to an audience at the beginning or intermediate graduate level, who may want to continue with this area of work, for a thesis or later. This makes demands on the didactic aspects of the presentation. Because the subjects of such schools are advanced, there often exists no textbook, and so ideally, the publication resulting from such a school could be a first approximation to such a textbook.
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   Volume editors and authors should be aware that incomplete or insufficiently close to final manuscripts almost always result in longer evaluation times. They should also be aware that parallel submission of their manuscript to another publisher while under consideration for LNM will in general lead to immediate rejection.

Continued on inside back-cover

Lecture Notes in Mathematics 1832

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Besides the usual "talks" on various topics of interest to the readership of the Séminaires, this volume contains a presentation by A. Lejay of the theory of rough paths, and some of the talks presented in September 2002 at the Journées de Probabilités held in La Rochelle. Other talks given at these Journées will be published in the next volume (volume XXXVIII).

The organizers of the Journées and the Séminaire are thankful to the Conseil Régional de Poitou-Charentes for its support.

The typographical presentation of the contributions has become much more homogeneous than in previous volumes; this meets a demand of our publisher Springer-Verlag, but does not mean that we aim to become a Journal! On the contrary, we are determined to keep using non-scientific acceptance criteria in addition to mathematical ones; the spirit of the Séminaire remains the same: discussing recent results by young (and not so young) researchers, making specialized courses widely available.

This new presentation would not have been possible but for the invaluable TEXnical assistance of Anthony Phan. Many thanks to him for his help and the time and skill he devoted to the finish of this volume.

J. Azéma, M. Émery, M. Ledoux, M. Yor

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Paul André Meyer est décédé subitement à Strasbourg le 30 janvier 2003.

C'est avec une profonde tristesse et une immense reconnaissance que nous saluons la mémoire de celui qui créa et développa sans relâche le Séminaire de Probabilités, et dont les qualités scientifiques et humaines faisaient l'admiration de tous.

Ce volume est dédié à son souvenir.

J. Azéma, C. Dellacherie, M. Émery,
M. Ledoux, M. Weil, M. Yor

An Impression of P. A. Meyer As Deus Ex Machina

Frank B. Knight

When I came to the University of Illinois in 1963 I remember hearing that Meyer had recently spent a semester there, working with J. L. Doob. I believe my first sight of Meyer in person was in 1967 in Madison, Wisconsin at the Chover Symposium on Probability and Potential. Meyer gave a 3-in-1 talk, starting a separate topic on each of the three available blackboards, and ending (no doubt) with a grand unification (but by that time I was lost).

I got the feeling of a sort of Napoleon of probability, both in appearance and in accomplishment. Thus I was agreeably surprised to find at the Dinges and Snell conference in Oberwolfach (1970) that he was in fact quite accessible. There was Meyer standing at the end of a long polished table occupied by 10 or 15 other participants, answering questions in such a calm and resourceful way that I could understand the discussion. It was not long before I, too, was asking questions, and, to make a long story short, we developed a fruitful correspondence. In 1974 I paid my first visit to Strasbourg as a guest of CNRS. That was the year when the volume X of the Séminaire de Probabilités was being aired (and the year of J. L. Doob's 65^{th} birthday), so this is ancient history and we may skip forward to my second visit to Strasbourg in 1982.

I shall recount the incidents preceding publication of the papers [1] and [2], both because even now they seem novel, and also because they are typical of Meyer's skill and generosity in dealing with colleagues. But the main reason for presenting them now is because they complete the proof of an assertion (Theorem 1.4 of [2]) which Meyer himself found "really beautiful" and which may still be relevant to the subject.

At that time my paper [1] had been submitted to the Ann. Sci. Éc. Norm. Sup., and it turned out that Meyer was acting as both editor and referee. I do not know whether he first obtained the paper as the referee or directly from me. In any case he returned it to me for revision, but with special praise for Theorem 1.4. Meanwhile he obtained a very closely related result which was more general insofar as it included the non-Gaussian processes, but not comparable inasmuch as it applied only for $t = \infty$, whereas Theorem 1.4 was for $t < \infty$. Unfortunately neither of us remarked on this distinction until after Meyer made the decision (sufficiently magnanimous to me) to publish my revision followed by a "Remark" by him containing his simplified proof. And that is how it turned out, except for a breakdown at the last moment. Namely, in the proofreading stage, I discovered a flaw in my original proof of Theorem 1.4. Thereupon (there was no other alternative) I deleted my proof entirely and referred instead to [2]. It was after November 23, 1982 and too late to make any more changes, so it was a shock when I finally noticed the gap between $t < \infty$ and $t = \infty$. Not being able to fill it in despite considerable effort, I again called upon Meyer. In a matter of weeks (or days), he came up with the following simple and brilliant solution. (See also [3] Theorem 3.38.)

Dear Frank

I am sorry to answer you with much delay. Here is the answer to your question. I recall the notation. Given $(\mathcal{F}_t)$ and some nice adapted process (X_t), we set

$$Y_t^\lambda = E\left[\int_t^\infty X_s e^{-\lambda s} ds \,\middle|\, \mathcal{F}_t\right] \qquad Z_t^\lambda = \text{martingale part of } Y_t^\lambda.$$

It is proved that if (Z_t^λ) is known on $[0,\infty[$ for all large λ then one can reconstruct (X_t). Your question is

assume (Z_t^λ) is known on $[0,T]$ for some fixed T.

Can one reconstruct (X_t) on $[0,T]$?

The answer is yes, and the proof is simple, though it took me a long time to find it! Set

$$\tilde{\mathcal{F}}_t = \mathcal{F}_{t\wedge T} \qquad \tilde{X}_t = X_t \text{ for } t \le T, \; E[X_t \,|\, \mathcal{F}_T], \, t \ge T$$

Then $\tilde{Y}_t^\lambda = E\left[\int_t^\infty e^{-\lambda s} \tilde{X}_s ds \,\middle|\, \tilde{\mathcal{F}}_t\right] = Y_t^\lambda$ for $t \le T$

and therefore the martingale part is $\tilde{Z}_t^\lambda = Z_t^\lambda$ for $t \le T$,

and since $\tilde{\mathcal{F}}_t = \mathcal{F}_T$ for $t \ge T$ $\tilde{Z}_t^\lambda = Z_T^\lambda$ for $t \ge T$

Therefore if we know (Z_t^λ) on $[0,T]$ we know $(\tilde{Z}_t^\lambda)$ everywhere, so by the above result we know $(\tilde{X}_t)$ everywhere, and hence (X_t) on $[0,T]$.

With kindest regards

André Meyer

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3. Knight (F.B.). Foundations of the Prediction Process. Oxford Studies in Probability 1. Clarendon Press, 1982.

Table des Matières

An Introduction to Rough Paths

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Key words: controlled differential equations, integration against irregular paths, p-variation, stochastic processes, iterated integrals, Chen series, geometric multiplicative functional

Summary. This article aims to be an introduction to the theory of rough paths, in which integrals of differential forms against irregular paths and differential equations controlled by irregular paths are defined. This theory makes use of an extension of the notion of iterated integrals of the paths, whose algebraic properties appear to be fundamental. This theory is well-suited for stochastic processes.

1 Introduction

This article is an introduction to the theory of *rough paths*, which has been developed by T. Lyons and his co-authors since the early '90s. The main results presented here are borrowed from [32, 36]. This theory concerns differential equations controlled by irregular paths and integration of differential forms against irregular trajectories. Here, x is a continuous function from $[0,1]$ to $\mathbb{R}^d$, and the notion of irregularity we use is that of p-variation, as defined by N. Wiener. This means that for some $p \geqslant 1$,

$$\sup_{\substack{k\geqslant 1,\ 0\leqslant t_0\leqslant\cdots\leqslant t_k\leqslant 1\\ \text{partition of } [0,1]}} \sum_{i=0}^{k-1} |x_{t_{i+1}} - x_{t_i}|^p < +\infty.$$

As we will see, the integer $\lfloor p \rfloor$ plays an important role in this theory.

In probability theory, most stochastic processes are not of finite variation, but are of finite p-variation for some $p > 2$. We show in Sect. 10 how to apply this theory to Brownian motion. But the theory of rough paths could be used for many other types of processes, as presented in Sect. 12.

Firstly, we give a meaning to the integral

$$\int_0^t f(x_s)\,\mathrm{d}x_s, \text{ or equivalently, } \int_{x([0,t])} f \tag{1.1}$$

for a differential form

$$f(x) = \sum_{i=1}^d f_i(x)\,\mathrm{d}x^i. \tag{1.2}$$

We are also interested in solving the controlled differential equation

$$\mathrm{d}y_t = f(y_t)\,\mathrm{d}x_t, \tag{1.3}$$

where f is the vector field

$$f(y) = \sum_{i=1}^d f_i(y)\frac{\partial}{\partial x_i}.$$

This will be done using Picard's iteration principle, from the result on integration of one-forms. Using the terminology of controlled differential equations, x is called a *control.*

The theory of rough paths also provided some results on the continuity of the map $x \mapsto y$, where y is given either by (1.1) or (1.3).

The theory of rough paths may be seen as a combination of two families of results:

(1) Integration of functions of finite q-variation against functions of finite p-variation with $1/p + 1/q > 1$ as defined by L.C. Young in [52].
(2) Representation of the solutions of (1.3) using iterated integrals of x: this approach is in fact an algebraic one, much more than an analytical one.

Let us give a short review of these notions.

(1) Young's integral

Let x and y be two continuous functions respectively $1/p$ and $1/q$-Hölder continuous with $\theta = 1/p + 1/q > 1$. Then, Young's integral $\int_s^t y_r\,\mathrm{d}x_r$ of y against x is defined as the limit of $I_{s,t}(\Pi) = \sum_{i=0}^{k-1} y_{t_i}(x_{t_{i+1}} - x_{t_i})$ when the mesh of the partition $\Pi = \{\, t_i \mid s \leqslant t_0 \leqslant \cdots \leqslant t_k \leqslant t \,\}$ of $[s,t]$ goes to zero (see for example [12, 52]). It is possible to choose a point t_j in Π such that

$$|I_{s,t}(\Pi) - I_{s,t}(\Pi \setminus \{\, t_j \,\})| \leqslant \frac{1}{(\mathrm{Card}\,\Pi)^\theta} C|t-s|^\theta$$

for some constant C that depends only on the Hölder norm of x and y. Whatever the size of the partition Π is, $|I_{s,t}(\Pi)| \leqslant |y_s(x_t - x_s)| + |t-s|^\theta \zeta(\theta)$, where

$\zeta(\theta) = \sum_{n\geqslant 1} 1/n^\theta$. The limit of $I_{s,t}(\Pi)$ as the mesh of Π goes to 0 may be considered.

One may be tempted to replace y by $f(x)$, where the regularity of f depends on the irregularity of x. But to apply directly the proof of L.C. Young, one has to assume that f is α-Hölder continuous with $\alpha > p - 1$, which is too restrictive as soon as $p \geqslant 2$. To bypass this limitation, we construct when $x_t \in \mathbb{R}^d$ the integral $\sum_{j=1}^d \int_s^t f_j(x_r)\,\mathrm{d}x_r^j$ as

$$\lim_{\mathrm{mesh}(\Pi)\to 0} \sum_{i=0}^{k-1}\Bigg(\sum_{j=1}^{d} f_j(x_{t_i})(x_{t_{i+1}}^j - x_{t_i}^j) + \sum_{j_1,j_2=1}^{d} \frac{\partial f_{j_1}}{\partial x_{j_2}}(x_{t_i})\mathbf{x}_{t_i,t_{i+1}}^{i,(j_2,j_1)}$$
$$+ \cdots + \sum_{j_1,\dots,j_{\lfloor p\rfloor}=1}^{d} \frac{\partial^{\lfloor p\rfloor -1} f_{j_1}}{\partial x_{j_{\lfloor p\rfloor}}\cdots \partial x_{j_2}}(x_{t_i})\mathbf{x}_{t_i,t_{i+1}}^{\lfloor p\rfloor,(j_{\lfloor p\rfloor},\dots,j_1)}\Bigg) \tag{1.4}$$

with formally

$$\mathbf{x}_{s,t}^{i,(j_i,\dots,j_1)} = \int_{s\leqslant s_i\leqslant\cdots\leqslant s_1\leqslant t} \mathrm{d}x_{s_i}^{j_i}\cdots \mathrm{d}x_{s_1}^{j_1}. \tag{1.5}$$

This expression (1.4) is provided by the Taylor formula on f and the more x is irregular, i.e., the larger p is, the more regular f needs to be.

What makes the previous definition formal is that the "iterated integrals" of x have to be defined, and there is no general procedure to construct them, nor are they unique. The terms $\mathbf{x}^{k,(i_1,\dots,i_k)}$ for $k = 2,\dots,\lfloor p\rfloor$ are limits of iterated integrals of piecewise smooth approximations of x, but they are sensitive to the way the path x is approximated. Due to this property, the general principle in the theory of rough paths is:

> The integral $\sum_{j=1}^d \int_s^t f_j(x_r)\,\mathrm{d}x_r^j$ is not driven by x but, if it exists, by $\mathbf{x} = (\mathbf{x}^{1,(i_1)}, \mathbf{x}^{2,(i_1,i_2)}, \dots, \mathbf{x}^{\lfloor p\rfloor,(i_1,\dots,i_{\lfloor p\rfloor})})_{i_1,\dots,i_{\lfloor p\rfloor}=1,\dots,d}$ corresponding formally to (1.5).

(2) Formal solutions of differential equations

Assume now that x is smooth, and let $\mathbf{x}_{s,t}^{k,(i_1,\dots,i_k)}$ be its iterated integrals defined by (1.5). Given some indeterminates $X^1,\dots,X^d$, we consider the formal non-commutative power series:

$$\Phi([s,t],x) = 1 + \sum_{k\geqslant 1}\sum_{(i_1,\dots,i_k)\in\{1,\dots,d\}^k} X^{i_1}\cdots X^{i_k}\mathbf{x}_{s,t}^{k,(i_1,\dots,i_k)}.$$

As first proved by K.T. Chen in [6], $\Phi([s,t],x)$ fully characterizes the path x, and for all $s \leqslant u \leqslant t$,

$$\Phi([s,u],x)\Phi([u,t],x) = \Phi([s,t],x). \tag{1.6}$$

This relation between iterated integrals is also used to prove that the limit in (1.4) exists. If exp is the non-commutative exponential (defined by a power series), then there exists a formal series $\Psi([s,t],x)$ such that $\Phi([s,t],x) = \exp(\Psi([s,t],x))$ and

$$\Psi([s,t],x) = \sum_{k\geqslant 1} \sum_{(i_1,\dots,i_k)\in\{1,\dots,d\}^k} F_{(i_1,\dots,i_d)}(X^1,\dots,X^d)\mathbf{x}^{k,(i_1,\dots,i_k)}_{s,t}$$

where $F_{(i_1,\dots,i_d)}(X^1,\dots,X^d)$ belongs to the Lie algebra generated by the indeterminates $X^1,\dots,X^d$, i.e., the smallest submodule containing $X^1,\dots,X^d$ and closed under the Lie brackets $[Y,Z] = YZ - ZY$.

If $f = (f_1,\dots,f_d)$ and each of the f_i is linear, i.e., $f_i(y) = C_i y$ where C_i is a matrix, then the solution y of (1.3) is equal to

$$y_t = \exp\Big(\widehat{\Psi}([s,t],x)\Big)y_s,$$

where $\widehat{\Psi}([s,t],x)$ is equal to $\Psi([s,t],x)$ in which X^i was replaced by the matrix C_i. If f is not linear, but is for example a left-invariant vector field on a Lie group, then a similar relation holds, where X^i is replaced by f_i, and the Lie brackets $[\cdot,\cdot]$ are replaced by the Lie bracket between vector fields. Here, the exponential is replaced by the map defining a left-invariant vector field from a vector in the Lie algebra, i.e., the tangent space at 0 (see for example [13]).

This result suggests that when one knows x, he can compute its iterated integrals and then formally solve (1.3) by replacing the indeterminates by f. In fact, when x is irregular, the solution y of (1.3) will be constructed using Picard's iteration principle, i.e., as the limit of the sequence y^n defined by $y^{n+1}_t = y_0 + \int_0^t f(y^n_r)\,\mathrm{d}x_r$. But it corresponds, if $(x^\delta)_{\delta>0}$ is a family of piecewise smooth approximations of x and f is smooth, to

$$y = \lim_{\delta\to 0} y^\delta \text{ with } y^\delta_t = \exp\Big(\widehat{\Psi}([0,t],x^\delta)\Big)y_0.$$

However, in the previous expression, we need all the iterated integrals of x. Yet, even if x is irregular, there exists a general procedure to compute them all, assuming we know $\mathbf{x}$ defined formally by (1.5). However, different families of approximations $(x^\delta)_{\delta>0}$ may give rise to different $\mathbf{x}$. Thus, the solution y of (1.3) given by the theory of rough paths depends also on $\mathbf{x}$ and not only on x, and the general principle stated above is also respected.

Geometric multiplicative functionals

As we have seen, we need to construct an object $\mathbf{x}$ corresponding to the iterated integrals of an irregular path up to a given order $\lfloor p\rfloor$. Since $\mathbf{x}$ may be reached as the limit of smooth paths together with its iterated integrals, $\mathbf{x}$ may be seen as an extension by continuity of the function $x \mapsto \Phi([s,t],x)$ giving the