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Volume 19

Section: Interpolation and Approximation
G. G. Lorentz, *Section Editor*

Birkhoff Interpolation

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DIA OF MATHEMATICS AND ITS APPLICATIONS

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Birkhoff Interpolation

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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive change of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

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ENCYCLOPEDIA OF MATHEMATICS and Its Applications

Preface

Birkhoff, or lacunary, interpolation appears whenever observation gives scattered, irregular information about a function and its derivatives. First discovered by G. D. Birkhoff in 1906, it received little attention until I. J. Schoenberg revived interest in the subject in 1966. Lacunary interpolation differs radically from the more familiar Lagrange and Hermite interpolation in both its problems and its methods. It could even be described as "non-Hermitian" interpolation. The name *Birkhoff interpolation* is justified also from a historical point of view.

At present, the main definitions and theorems for polynomial Birkhoff interpolation seem to have been found, while the theory for other systems of functions, most notably splines, is in healthy development. Since this material can be found only in research periodicals and proceedings of conferences, it is time for a comprehensive exposition of the material. We have gone to great lengths to unify, simplify, and improve the information already published or in press, and to set the stage for further developments.

The book should be of interest to approximation theorists, numerical analysts, and analysts in general, as well as to computer specialists and engineers who need to analyze functions when their values and those of their derivatives are given in an erratic way. The book could be used as a text for a graduate course, requiring little more than an undergraduate mathematics background.

Many novel ideas and tools have been developed in this theory; interpolation matrices, coalescence of rows in matrices, independent knots, probabilistic methods, diagrams of splines, and the Rolle theorem for splines. There are applications to approximation with constraints, to quadrature formulas, to splines and their zeros, and to the theory of Chebyshev systems.

The book is largely self-contained, at least in its central parts (Chapters 1–8 and 13–14). We begin with the basic definitions and elementary properties of Birkhoff interpolation by linear combinations of smooth functions in Chapters 1 and 2. In Chapters 3 and 4 we introduce coalescence of rows of matrices, and obtain many applications. Rolle theorem methods and independent knots are discussed in Chapter 5; these methods work for interpolation by polynomials in quite general systems of functions. Chapter 6 is concerned with singularity theorems; conditions are given under which the Birkhoff interpolation problem is not solvable.

Chapter 7 returns to the original problem of Birkhoff—to describe the remainders of interpolation formulas by means of an integral involving a kernel function. This naturally leads to the introduction of Birkhoff splines and to the study of their zeros.

In Chapter 8 we investigate a special case that illustrates the complexity of the Birkhoff interpolation problem in even a very simple situation. Selected applications of Birkhoff interpolation to approximation theory and Chebyshev systems are presented in Chapter 9. Many related applications had to be omitted, but they can be traced through the literature cited in the notes. Birkhoff interpolation of functions of several variables—a subject that needs much further investigation—also must have useful applications.

In Chapter 10 we deal with quadrature formulas based on general Birkhoff interpolation matrices. This relatively new theory culminates in theorems about the existence of formulas of Gaussian type.

Lacunary interpolation at special knots has received considerable attention since the work of Turán and his associates in the 1950s. In Chapters 11 and 12 we give a sample of these results; much related material had to be omitted for lack of space.

Chapters 13 and 14 offer an introduction to Birkhoff interpolation by splines. This subject is inherently complicated, and considerable effort has been made to simplify and unify the theory by means of new notation and methods. Applications abound here, and are presented in the last sections of Chapter 14. For the convenience of the reader, we include a Bibliography (with the indication of the sections where different papers are quoted), a Symbol Index, and a Subject Index.

The book has been developed from the report of one of us to the Center of Numerical Analysis, University of Texas, in 1975, as well as from lectures by each of us at our universities and from our recent publications. The authors gratefully acknowledge support of their research activities during the time of writing this book by the National Science Foundation of the United States and the National Science and Engineering Research Council of Canada. We are grateful for generous advice and comments from C. de Boor, T. N. T. Goodman, M. Marsden, C. Micchelli, P. Nevai, A. Sharma, P. W. Smith, and others. We are especially thankful to G. C. Rota for encouraging our participation in the *Encyclopedia*.

G. G. Lorentz
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Approximation and Interpolation in the Last 20 Years

This is the first in a series of books dealing with approximation and interpolation of functions. Many changes have occurred in this theory during the last decades. In what follows, we shall try to describe some of the problems and achievements of this period.

Until about 1955, the leading force in approximation was the Russians, in particular, Bernstein and his school (Ahiezer), Chebyshev, Kolmogorov, and Markov. The development of the subject in Germany, Hungary, and the United States occurred later. The West certainly leads in the number of papers published—see the bulky *Journal of Approximation Theory*. The twelve sections that follow review the newer developments.

The two classical books dealing with approximation and interpolation are those of Natanson [0-N] and Ahiezer [0-A]. Important recent books include two Russian works devoted to special problems: Korneichuk [0-K_2] (see also [0-K_3]) deals with best constants in the trigonometric approximation, while Tihomirov [0-T_1] treats extremal problems, particularly widths and optimization. The book of Butzer and Berens [0-B_2] introduced functional analytic methods into the field; the two books by de Boor [0-B_1] and Schumaker [0-S] deal with splines, an American development rich in practical applications. Karlin and Studden [0-K_1] treat Chebyshev systems exhaustively. Books on general approximation theory are those of Rice [0-R], Lorentz [0-L], Dzyadyk [0-D], and Timan [0-T_2]; the last book contains a wealth of material. Several books will be mentioned in later sections.

What might one recommend to someone who wants to begin a study of the subject? My choice would be the books of Natanson [0-N] and Cheney [0-C], and perhaps also my own [0-L], for the real approximation, and that of Gaier [0-G] for the complex approximation.

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§1. BEST APPROXIMATION; CHEBYSHEV SYSTEMS

We denote by $\mathcal{P}_n$ (or by $\mathcal{T}_n$) the set of all algebraic polynomials P_n (or all trigonometric polynomials T_n) of degree $\leqslant n$ on an interval $[a, b]$ (or the circle $\mathbb{T}$). The degree of approximation of a function f in the L_p norm is

$$E_n(f)_p = \min\|f - P_n\|_p, \qquad E_n^*(f)_p = \min\|f - T_n\|_p \tag{1.1}$$

(the space L_∞ is here interpreted to be C). The polynomial of best approximation to f is one that realizes the minimum in (1.1). The problem of finding or describing polynomials of best approximation is the problem of the qualitative theory of approximation, which has been overshadowed in recent decades by the quantitative theory.

In the case $1 < p < +\infty$, the uniqueness of the polynomial of best approximation follows from the convexity of the space L_p. For spaces C and L_1, we also have unicity, but not trivially. We have:

Let $P_n \neq f \in C[a, b]$. Then P_n is the best approximation to f in $\mathcal{P}_n$ if there exists an alternance, that is, points $a \leqslant x_1 < \cdots < x_{n+2} \leqslant b$ for which

$$f(x_i) - P_n(x_i) = \varepsilon(-1)^i \|f - P_n\|_\infty, \qquad i = 1, \ldots, n+2, \quad \varepsilon = +1 \text{ or } -1. \tag{1.2}$$

This is Chebyshev's theorem. Actually, Chebyshev only proved, using calculus, the existence of $n+2$ points, which are either $x = a$ or $x = b$, or satisfy $f'(x) = P_n'(x)$ where $|f(x) - P_n(x)|$ attains its maximum (see [1-C, p. 284, Theorem 2]). Theorem (1.2) has been gradually developed since then. There is also a different condition of Kolmogorov (see [0-L, p. 18]) which characterizes the polynomial of best approximation. It is much less concrete than (1.2), but implies this condition. Its advantage is high flexibility: It can be adjusted to characterize the best approximation in many other cases. For the practical determination of the polynomial of best approximation we have the famous Remez algorithm.

It is very hard to pinpoint the polynomials of best approximation of a given function. The best example that we have is this. If

$$f(x) = \sum_{k=1}^{\infty} a_k C_{3^k}(x), \qquad \sum |a_k| < +\infty, \tag{1.3}$$

where the C_n are Chebyshev polynomials, then all polynomials of best uniform approximation of f are the partial sums of the series (1.3).

Because of this difficulty, many concrete questions about polynomials P_n of best approximation have not been completely answered. For example, is it true that $f \in C[-1, +1]$ is odd if $P_n(0) = 0$, $n = 0, 1, \ldots$? (Partial affirmative answers are given by Saff and Varga [1-S$_1$].) Or: What can be the highest multiplicity of 0 as a root of P_n? (It can be $\geqslant$ const. $\log n$ for infinitely many n; see [1-L].) A peculiar question has been answered by Borosh, Chui, and Smith [1-B]. We approximate x^{n+1} by a polynomial

$$P_n(x) = \sum_{k=1}^{s} a_k x^{\lambda_k}$$

of length $s \leqslant n$, where the λ_k are integers $\leqslant n$. Which selection of the λ_k produces best results? The answer is that one should take the λ_k as close as possible to n; $\lambda_{s-i} = n - i$, $i = 0, \ldots, s-1$.

· Sometimes best approximation is unique even for infinite-dimensional subspaces. Diliberto and Strauss show this for $f(x, y) \in C(I^2)$, $I = [0,1]$, approximated by functions $g_1(x) + g_2(y)$, $g_i \in C(I)$, $i = 1,2$. However, generalizations of this theorem proved to be difficult (see papers of Cheney, v Golitschek).

It is impossible to achieve the exact degree of approximation $E_n^*(f)$ by means of a linear polynomial operator on C^*. But one can hope for approximation of order $\leqslant$ const. $E_n^*(f)$. This can be realized, both for C^{*r}, $r = 0, 1, \ldots$ (by means of the de la Vallée–Poussin sums) and for the space A_σ^*, $\sigma > 0$, of periodic functions $f(z)$, analytic in $|\mathrm{Re}\, z| < \sigma$ and continuous on the boundary (by means of Fourier partial sums). However, it cannot be done by the same operators for both spaces at once (Dahmen and Görlich [1-D]).

In 1937, Bernstein coined the term *Chebyshev system* for a set of linearly independent functions $g_0, \ldots, g_n$ on $[a, b]$ or $\mathbb{T}$, which has the property that polynomials $P_n = \Sigma_0^n a_k g_k$ interpolate arbitrary data c_i at any set of distinct points x_i, $i = 0, \ldots, n$. In other words, the determinant $\det[g_k(x_i)]_{i,k=0}^n$ must be $\neq 0$, let us say > 0, for each selection of points $x_0 < \cdots < x_n$. Instead of the Lagrange interpolation, we can take Hermite interpolation here, and obtain the extended Chebyshev systems [0-K$_1$]. Very important is Haar's theorem (1911) that $\{g_k\}$ is a Chebyshev system if and only if each continuous function has a unique polynomial of best approximation. The book of Karlin and Studden [0-K$_1$] treats Chebyshev systems exhaustively; Zalik and Zielke (see [1-Z]) also allow discontinuous Chebyshev systems. Newman and Shapiro [1-N$_1$] show that for each Chebyshev system and each $f \in C$ one has, for the polynomial P_n of best approximation to f, and any other polynomial Q_n,

$$\|f - Q_n\| \geqslant \|f - P_n\| + \gamma \|Q_n - P_n\|, \tag{1.4}$$

where $\gamma = \gamma(f) > 0$ is a constant. This is the so-called strong uniqueness theorem. Many authors (Bartelt, Henry, Roulier) have studied the behavior of the constants γ (see, e.g., [1-H]).

A *weak Chebyshev system* is defined by means of the inequality $\det[g_k(x_i)] \geqslant 0$, $x_0 < \cdots < x_n$. In important papers, Nürnberger and Sommer have studied these systems in great detail. We give one of their results [1-N$_2$]. For a weak Chebyshev system, some functions $f \in C$ may have several P_n of best approximation. These P_n form a compact convex subset $\pi(f)$ of the $(n+1)$-dimensional space G spanned by the g_k; the map $f \to \pi(f)$ is called the metric projection. The property of being a weak Chebyshev system is necessary for the existence of a *continuous selection*, that is, of a continuous map $f \to P_n(f)$ of C onto G, with $P_n(f) \in \pi(f)$ for all f. See [1-S$_2$] for literature and other problems.

A related notion is that of *total positivity*. A function $K(x, y)$, $x \in X$, $y \in Y$, where X, Y are linearly ordered, is called totally positive of order r (Karlin [1-K]) if

$$\det\left[K(x_i, y_j)\right] > 0, \quad x_1 < \cdots < x_k, \qquad y_1 < \cdots < y_k, \qquad 1 \leqslant k \leqslant r. \tag{1.5}$$

Properties of total positivity are often very useful (used, e.g., in [1-B], [10-M_2], [10-M_3]); at other times it is difficult to compute all the determinants contained in (1.5).

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§2. MODULI OF SMOOTHNESS; SPACES OF FUNCTIONS; DEGREE OF APPROXIMATION

2.1. Classical Quantitative Theorems

The quantitative theory is the main part of approximation. Here we want good approximation, not the best, mainly because we are seldom able to find the elements of best approximation. In this sense, "best is the enemy of the good." The two main theorems of the quantitative theory are those of Jackson and Bernstein. The first asserts that the degree of approximation $E_n^*(f) = \min_{T_n \in \mathcal{T}_n} \|f - T_n\|_\infty$ satisfies

$$E_n^*(f) \leqslant C_r n^{-r} \omega(f^{(r)}, 1/n) \tag{2.1}$$

if $f \in C^{*r}$. [C^r is the space of r times continuously differentiable functions, $\omega(g, t)$ is the modulus of continuity of g; the asterisk means that the functions are on the circle $\mathbb{T}$.] The theorem of Bernstein states that conversely, $E_n^*(f) \leqslant Cn^{-r-\alpha}$, $0 < \alpha < 1$, implies that $f \in C^{*r}$ and that $f^{(r)} \in \operatorname{Lip} \alpha$. To these two, Butzer likes to add as a third element the theorem of Zamansky, namely, that for $f \in C^*$

$$\|f - T_n\|_\infty = \mathcal{O}(n^{-\beta}),\ \beta > 0, \text{ implies } \|T_n^{(k)}\|_\infty = \mathcal{O}(n^{k-\beta}),\ k > \beta.$$

Since about 1960, new fields of quantitative theory have flourished—constrained approximation, Korovkin theorems, Müntz approximation, incomplete polynomials; these will be reviewed in later sections.

But even in this section progress will be evident. New features are (a) the use of functional analysis (interpolation of linear operators), particularly the use of the K-functionals of Lions and Peetre; (b) some new spaces (e.g., Besov spaces); and (c) problems that refer to two different spaces or to two norms. In §10, some problems of this type can be solved by embedding theorems, but the most of them do not allow this reduction.

2.2. Moduli of Smoothness

We say that $f(x)$, $x \in [a, b]$, has rth derivative $f^{(r)}$ if $f', \ldots, f^{(r-1)}$ exist and are absolutely continuous; then $f^{(r)}$ exists a.e. For functions of several variables, one uses distributional derivatives. The moduli of smoothness of f are given by

$$\omega_r(f, h) = \sup_{t,h} |\Delta_h^r f(t)|, \qquad r = 1, 2, \ldots, \tag{2.2}$$

where $\Delta_h^r f(t)$ is the rth difference of f with step h. The main point in this definition is that Jackson's theorem (2.1) allows an improvement: $E_n^*(f) \leqslant C_r \omega_r(f, 1/n)$. In the space L_p one puts $\omega_r(f, h)_p = \sup_{u \leqslant h} \|\Delta_u^r f(\cdot)\|$, and also has (2.1).

Other moduli of smoothness have been used by Popov [2-P]; for instance, $\tau_r(f, h)_p = \|\omega_r(f, x; \delta)\|_p$, where in $[a, b]$,

$$\omega_r(f, x; \delta) = \sup\left\{ |\Delta_h^r f(t)|, t, t + rh \in \left[x - \frac{r\delta}{2}, x + \frac{r\delta}{2} \right] \right\}. \tag{2.3}$$

This τ_r plays the same role in the one-sided approximation as ω_r in the ordinary Jackson theorem. The one-sided degree of approximation of f is

$$\mathcal{E}_n(f) = \inf\{\|P_n - Q_n\|: P_n(x) \leqslant f(x) \leqslant Q_n(x), a \leqslant x \leqslant b\} \quad (2.4)$$

where P_n, Q_n are two polynomials of degree $\leqslant n$.

2.3. Basic Function Spaces

The basic spaces of functional analysis are C, $L_p(1 \leqslant p \leqslant +\infty)$, the Orlicz spaces and the Lorentz space L_{pq}. For f defined on a finite or infinite interval, let f^* be the decreasing rearrangement of $|f|$, and let $f^{**}(t) = t^{-1}\int_0^t f^*$. The space L_{pq} is given by the norm

$$\|f\|_{pq} = \left[\int_0^\infty (t^{1/p}f^{**})^q \frac{dt}{t}\right]^{1/q}, \qquad 1 \leqslant q < +\infty, \quad (2.5)$$

(with the q norm replaced by sup for $q = +\infty$). This is Calderón's definition. The original definition [2-L] has f^{**} replaced by f^* in (2.5). For $1 \leqslant q \leqslant p < +\infty$ this is equivalent to (2.5), but is not a norm for other values of p, q.

2.4. Besov Spaces

The spaces needed in the approximation theory are derived from these by applying their norm to the important quantities $f^{(r)}$, $\omega(f, h)/h^\alpha$, $\omega_r(f, h)_p$. In this way one obtains spaces $\mathrm{Lip}(\alpha, p) = H_p^\alpha$ for which $\omega(f, h)_p \leqslant Ch^\alpha$, spaces H^ω with $\omega(f, h) \leqslant C\omega(h)$, where ω is some fixed (often concave) modulus of continuity. By W^rE we mean the space of functions f with $f^{(r)} \in E$. The spaces W_p^r (often called Sobolev spaces) consist of functions f with $f^{(r)} \in L_p$. Finally, if we apply to $t^{-1}\omega_r(f, t)_p$ the L_{pq} norm, taking $1 - (1/s) = \theta$, and the original definition (we almost have that $t^{-1}\omega_r$ is positive decreasing, for ω_r is often concave), we obtain the Besov spaces $B_p^{\theta, q}$, $\theta > 0$, with the norm

$$\|f\|_{B_p^{\theta,q}} = \|f\|_p + \left[\int_0^{+\infty} \left(t^{-\theta}\omega_r(f, t)_p\right)^q \frac{dt}{t}\right]^{1/q}, \qquad r = [\theta] + 1. \quad (2.6)$$

(If $q = +\infty$, the q norm is replaced by the supremum norm.)

We shall mention a few facts about Besov spaces. The space $B_p^{\theta,\infty}$, with $\theta = r + \alpha$, is equivalent to W^rH^α; and if $0 < \theta < 1$, then $B_p^{\theta,\infty}$ is the space $\mathrm{Lip}(\theta, p)$. One also has the embedding for $1 \leqslant s \leqslant +\infty$,

$$B_p^{\alpha+\theta, s} \hookrightarrow B_{p_1}^{\alpha, s} \quad \text{if} \quad 1 \leqslant p < p_1 \leqslant +\infty, \quad \theta := \frac{1}{p} - \frac{1}{p_1}. \quad (2.7)$$

The Hardy spaces H_p, and BMO, are also important in functional analysis. The latter has not yet been discovered by approximation theorists.

2.5. *K*-functionals

Interpolation of operators appears in approximation theory mainly in the Peetre form, which is based on K-functionals. A definitive first exposition of this theory was presented in the important book of Butzer and Berens [0-B_2]. Based on this, and the semigroups of operators, the Aachen school of approximation flourished. Compare Butzer [2-B_3] for an exposition of their results until 1973.

Taking the special case, let $X_1 \hookrightarrow X$ be two Banach spaces with a continuous embedding. The K-functional of the spaces X_1, X is the function of t,

$$K(f,t,X_1,X) := \inf_{g \in X_1} \{\|f-g\|_X + t\|g\|_{X_1}\}, \qquad t \geqslant 0. \tag{2.8}$$

There is a standard way of generating intermediate spaces between X_1, X by applying to $K(f,t)$ one of the norms of type (2.5). In this scheme, the Besov space $B_p^{\theta,q}$ is intermediate between W_p^r and L_p.

One of the reasons why the K-functionals are important is the fact, discovered by Johnen [2-J] and Peetre, that they are equivalent to the smoothness moduli:

$$C_1\omega_r(f,t)_p \leqslant K(f,t^r;W_p^r,L_p) \leqslant C_2\omega_r(f,t)_p. \tag{2.9}$$

This allows us, in proving theorems of the Jackson type with ω_r, to assume f to have several derivatives. See DeVore [2-D] for an exposition of the approximation theorems from this point of view.

2.6. Jackson-type Theorems

What is new in theorems of the Jackson type? A function f on $\mathbb{T}$ belongs to $B_p^{\theta,q}$, $\theta < r$, if and only if (Besov [2-B_1])

$$\sum_{n=0}^{\infty} \left(2^{\theta n} E_{2^n}^*(f)\right)^q < +\infty.$$

The influence of the end points in polynomial approximation has been clarified. Timan and Dzyadyk show that $f \in W^r H^\alpha[-1,+1]$ if and only if for some $P_n \in \mathcal{P}_n$,

$$|f(x) - P_n(x)| \leqslant \text{const.}\Delta_n(x)^{r+\alpha}, \qquad \Delta_n(x) := n^{-2} + n^{-1}\sqrt{1-x^2}. \tag{2.10}$$

There is a corresponding result with ω_r. Here, the difficult inverse theorem is due to Brudnyi [2-B_2]. The peculiar fact that (2.10) remains a necessary and sufficient condition if $\Delta_n(x)$ is replaced simply by $n^{-1}\sqrt{1-x^2}$ has been observed by Teljakovskii (see [2-T]). Theorems like (2.10) are also valid in the L_p norm (Oswald [2-O]). In the paper of Butzer and Scherer [2-B_4],