Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Algebraic and Geometric Topology

Proceedings, Rutgers 1983

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PREFACE

This volume was conceived as the proceedings of a conference on surgery theory held at Rutgers University in July, 1983. The editors have taken the opportunity to considerably expand the subject matter.

The articles in this volume present original research on a wide range of topics in modern topology. They include important new material on the algebraic K-theory of spaces (Waldhausen, Vogell), the algebraic obstructions to surgery and finiteness (Cappell and Shaneson, Milgram, Pedersen and Weibel, Ranicki, Sondow), geometric and chain complexes (Davis, Quinn, Smith, Weinberger), characteristic classes (Levitt), and transformation groups (Assadi and Vogel).

A paper of J.Levine on homotopy spheres, written in 1969 ~~as~~ the sequel to the classic work of Kervaire and Milnor but never published, is also included.

Andrew Ranicki

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November, 1984

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SEMIFREE FINITE GROUPS ACTIONS ON COMPACT MANIFOLDS

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INTRODUCTION

One of the classical problems in transformation groups has been to study the properties of the stationary point sets of actions on manifolds, and to characterize them whenever possible. P. A. Smith theory in combination with various other topological considerations provide a number of necessary conditions to be satisfied by the stationary point sets of some restricted classes of actions. In the case of smooth actions of a compact Lie group G on a manifold W, the stationary point set, say F, is a manifold and its normal bundle in W, say ν, is a G-bundle which determines the action in a (tubular) neighborhood of F.

For a complete characterization (of the diffeomorphism type) of F, one needs to show that the above mentioned necessary conditions are sufficient as well, in the following sense. Assuming that the submanifold F of the prescribed manifold W, and the G-bundle ν given, one tries to find an action on W which would restrict to the given action in the tubular neighborhood of F provided by the G-bundle ν. Special cases of such problems have been considered under various circumstances by various authors: [J1], [J2], [A1], [A2], [A3], [A-B 1], [A-B 1], [L], [D-R], [S] to mention a few. In these and other related contexts, a common hypothesis is that W is simply-connected and this assumption is indispensable for the techniques and the arguments to be applicable.

In the following, we consider this and some other relevant questions in the case of non-simply connected compact manifolds on which a finite group G has a "simple semifree action," i.e. where action is free outside of the stationary point set, and a certain localized Borel construction becomes fibre homotopy trivial. Although semifree actions comprise a restricted class, their understanding seems essential in developing general theories with more complicated isotropy group structures. The further restriction of "simplicity" of actions has been imposed to bring the homotopy-theoretic constructions and algebraic

(*) Partially supported by an NSF grant.

calculations within reach, as well as to provide a satisfactory answer to the above-mentioned questions in the form of less-complicated necessary and sufficient conditions.

In the presence of the fundamental group of the ambient manifold — on which the desired G-action is to be constructed — much of the methods and results of the simply-connected cases (in their various forms and contexts) are inapplicable. Thus, one is led to construct a new obstruction group and a new invariant (depending on both $\pi_1 W$ and G) whose vanishing is one of the necessary conditions for the existence of such actions. The obstruction group fits in a five-term exact sequence relating various Whitehead groups, and conceivably it can be defined as the fundamental group of the fibre of a transfer map between two Whitehead spaces involved in the problem, although its definition given below is in purely algebraic terms. The above-mentioned invariant is related to a certain Reidemeister torsion-type invariant.

If $\pi = 1$, Wh_1^T becomes simply $\tilde{K}_0$. This functor takes into account the interaction between $\tilde{K}_0$ (the finiteness obstruction in the presence of G-actions) and Wh_1 (the Whitehead torsion involving the fundamental group $\pi_1 = \pi$) in a way which is necessary to study the above mentioned problems. Thus, in the geometric context, Wh_1^T plays the same role in the study of finite group actions on non-simply connected compact manifolds that $\tilde{K}_0$ does in the simply-connected case.

The organization of the paper is as follows. In Section I we introduce Wh_1^T and state some of its algebraic properties which are used subsequently to detect the (combined) finiteness and Whitehead torsion type obstructions as the image of a Reidemeister torsion type invariant. Section II illustrates some computations of Wh_1^T. (The details of the results in these sections will appear elsewhere.) Section III considers semifree simple actions and gives necessary and sufficient conditions for existence of simple actions in this context. The problem of characterization of the stationary point sets of simple semi-free actions on compact bounded manifolds and an extension theorem for free simple actions are reduced to the homotopy theoretic problem of constructing appropriate Poincaré complexes, which are carried out using mixing the localizations of diagrams of spaces involved. Section IV gives an indication of the proofs of the theorems of Section III. Section V gives some useful theorems on constructing free simple actions either by extending a given action on a subspace or by pulling back actions from a given space, thus formalizing and generalizing the constructions needed in Section III. Although these are non-simply connected versions of analogous results in [A2] and [A3] where free actions are constructed from homotopy actions on simply-connected spaces (which are not simple in general), there is little overlap in scope or the methods.

There is somewhat of an overlap between some of the results obtained independently by S. Cappell and S. Weinberger [CW] as well as S. Weinberger [W], P. Löffler [L], P. Löffler and M. Raußen [LR]. The papers of L. Jones [J] and F. Quinn [Qu] also deal with related problems.

SECTION 1. Let Λ be a ring and $P(\Lambda)$ denote the category of finitely generated projective Λ-modules. In the sequel, G will denote a finite group, and π a discrete group which denotes as well the subgroup $\pi \times \{1\} \subset \pi \times G$ for simplicity of notation. Consider the set $A = \{(P,B) \mid P \in P(\mathbb{Z}(\pi \times G)),\ B = \mathbb{Z}\pi\text{-basis for } P\}$. The operation of direct sum of modules and disjoint union of $\mathbb{Z}\pi$-bases in the given order gives A the structure of a monoid with neutral element $(0,\emptyset)$. We introduce the equivalence relation $(P,B) \sim (P',B')$ among the elements of A if there exists a $\mathbb{Z}(\pi \times G)$-linear isomorphism $\alpha : P \xrightarrow{\cong} P'$ such that $\tau_\pi(\alpha) = 0$ with respect to B and B', where $\tau_\pi(\alpha) \in Wh_1(\pi)$ is the Whitehead torsion. The set of equivalence classes $A' = A/\sim$ inherits the monoid structure of A, and contains the submonoid "of trivial elements"; namely, (P,B) represents a trivial element in A' if $\underline{P}$ is $\mathbb{Z}(\pi \times G)$-free, and B is induced by a $\mathbb{Z}(\pi \times G)$-basis. The quotient monoid A' modulo the submonoid of trivial elements is seen to be an abelian group and is denoted by $Wh_1^T(\pi \subset \pi \times G)$. We have an obvious homomorphism $\alpha : Wh_1^T(\pi \subset \pi \times G) \longrightarrow \tilde{K}_0(\mathbb{Z}(\pi \times G))$ induced by the forgetful map $(P,B) \longrightarrow P \in \tilde{K}_0(\mathbb{Z}(\pi \times G))$. There is a further homomorphism $\beta : Wh_1(\pi) \longrightarrow Wh_1^T(\pi \subset \pi \times G)$ which is induced by the operation of "twisting the standard basis;" namely, let $x \in Wh_1(\pi)$ be represented by $\phi : (\mathbb{Z}\pi)^n \longrightarrow (\mathbb{Z}\pi)^n$. After stabilization, we have a π-linear homomorphism $\phi \oplus id : (\mathbb{Z}(\pi \times G))^m \longrightarrow (\mathbb{Z}(\pi \times G))^m$. Let B be the image of the standard basis of $(\mathbb{Z}(\pi \times G))^m$ under the $\mathbb{Z}\pi$- linear map $\phi \oplus id$. Then B is a $\mathbb{Z}\pi$-basis for $(\mathbb{Z}(\pi \times G))^m$ and $((\mathbb{Z}(\pi \times G))^m, B)$ represents $\beta(x) \in Wh_1^T(\pi \subset \pi \times G)$.

1.1 Theorem. There is an exact sequence

$$Wh_1(\pi \times G) \xrightarrow{Tr} Wh_1(\pi) \xrightarrow{\beta} Wh_1^T(\pi \subset \pi \times G) \xrightarrow{\alpha} Wh_0(\pi \times G) \xrightarrow{tr} Wh_0(\pi)$$

in which Tr and tr are transfer homomorphisms and $Wh_0 \equiv \tilde{K}_0$.

The homomorphism $\mathbb{Z}\pi \to \mathbb{Z}_q\pi$ induces a homomorphism $Wh_1(\pi) \to Wh_1(\pi;\mathbb{Z}_q) \overset{def}{\equiv} K_1(\mathbb{Z}_q\pi)/\{\pm\pi\}$ where $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. One has a further map $\gamma : Wh_1(\pi;\mathbb{Z}_q) \to Wh_1^T(\pi \subset \pi \times G)$ defined as follows. Let $GL_n'(\mathbb{Z}\pi)$ be the monoid of $(n\times n)$-matrices which have an inverse in $GL_n(\mathbb{Z}_q\pi)$. Given $\phi \in GL_n'(\mathbb{Z}\pi)$, one has an exact sequence

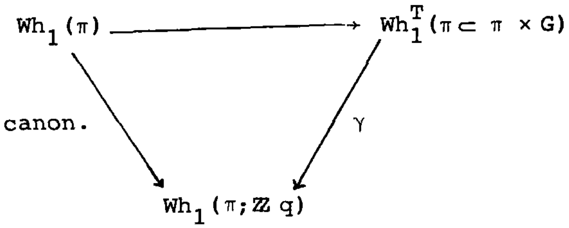
$$(C_*) \ : \ 0 \to (\mathbb{Z}\pi)^n \xrightarrow{\phi} (\mathbb{Z}\pi)^n \to M \to 0$$

Thus $M_q = M \otimes \mathbb{Z}_q = 0$. It follows that proj $\dim_{\mathbb{Z}(\pi\times G)} M \leq 1$, and we may take a short projective resolution over $\mathbb{Z}(\Pi \times G)$ for M, where order $(G) = q$:

$$(C_*') \ : \ 0 \to C_1' \to C_0' \to M \to 0$$

such that C_1' is free and C_0' is projective over $\pi \times G$. There is a $\mathbb{Z}\pi$-linear chain homotopy equivalence $\zeta : C_* \to C_*'$. Since the finiteness obstruction of C_* over $\mathbb{Z}\pi$ vanishes, C_0' is stably trivial over $\mathbb{Z}\pi$ also. After stabliization, we choose $Z\pi$-basis for C_1' and C_0', say $\mathcal{B}_1'$ and $\mathcal{B}_2'$. If we choose the "standard bases $\mathcal{B}_1$ and $\mathcal{B}_0$ in the resolution (C_*) above for $C_1 \cong (\mathbb{Z}\pi)^n$ and $C_0 \cong (\mathbb{Z}\pi)^n$, then it is possible to arrange for the choices of $\mathcal{B}_1'$ and $\mathcal{B}_0'$ so that becomes a simple homotopy equivalence over $\mathbb{Z}\pi$. Let $\gamma(\phi) = [(C_1',\mathcal{B}_1')] - [(C_1',\mathcal{B}_1')]$ in $Wh_1^T(\pi \subset \pi \times G)$. In general, for $\phi \in GL_n(\mathbb{Z}_q\pi)$, we take $\phi = \frac{1}{s}\psi$, where $(s,q) = 1$. Then $s(Id) \in GL_n'(\mathbb{Z}\pi)$ and $\psi \in GL_n'(\mathbb{Z}\pi)$.
Let $\gamma(\phi) = \gamma(s(Id))$.

<u>I.2. Theorem.</u> γ induces a well-defined homomorphism such that the following diagram commutes



Suppose C_* is a chain complex over $\mathbb{Z}\pi$ such that $H_*(C_*\otimes\mathbb{Z}q) = 0$. Then the Reidemeister torsion of C_* is a well-defined element of $Wh_1(\pi;\mathbb{Z}q)$ and is denoted by $\tau(C_*)$. The main algebraic result of this section is the following:

<u>1.3. Theorem</u>. Let A'_* be a finite $\mathbb{Z}\pi$-based chain complex, and A_* be a finite $\mathbb{Z}(\pi \times G)$-based chain complex. Suppose there exists a $\mathbb{Z}\pi$-linear map $f: A'_* \longrightarrow A_*$ which is a $\mathbb{Z}\pi$-chain homotopy equivalence. Further, suppose $H_*(A \otimes \mathbb{Z}_q) = 0$ and that G acts trivially on $H_*(A)$, where order $(G) = q$. Then there is a finite $\mathbb{Z}(\pi \times G)$-based complex B_* and a $\mathbb{Z}(\pi \times G)$-chain homotopy equivalence $h: A_* \to B_*$ such that $hf: A_* \to B_*$ is π-simple if and only if $\gamma(\tau(A_*)) = 0$.

The above algebraic theory has the following application which is crucial in the construction of surgery problems of the next sections.

<u>1.4 Theorem.</u> Suppose we have a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \hookrightarrow & \tilde{X} \\ \beta \downarrow & & \downarrow \alpha \\ Y & \hookrightarrow & X \end{array}$$

with the following properties:

i) $\tilde{X}, \tilde{Y}$ and Y are finite connected CW complexes, and X is a connected CW-complex.

ii) $\pi_1(\tilde{X}) = \pi_1(\tilde{Y}) = \pi, \pi_1(X) = \pi_1(Y) = \pi \times G$.

iii) $\tilde{Y}$ is a covering space of Y and α induces a homotopy equivalence from $\tilde{X}$ to the covering space of X with the fundamental group π.

iv) $H_*(\tilde{X},\tilde{Y};\mathbb{Z}_q[\pi]) = 0$ and the Reidemeister torsion of $(\tilde{X},\tilde{Y})$ is $\tau(\tilde{X},\tilde{Y})$ in $Wh_1(\pi;\mathbb{Z}_q)$.

v) G acts trivially on $H_*(\tilde{X},\tilde{Y});\mathbb{Z}[\pi]) = H_*(X,Y;\mathbb{Z}[\pi \times G])$.

Then there exists a homotopy equivalence from X to a finite complex Z such that the composite map $\tilde{X} \xrightarrow{\alpha} X \longrightarrow Z$ induces a simple homotopy equivalence from $\tilde{X}$ to a covering space of Z, if and only if $\gamma(\tau(\tilde{X},\tilde{Y})) = 0$.

<u>Indication of Proof</u>: Let us denote by $C_*(-;M)$ the cellular chain complex with (twisted coefficients M. We have a π-linear homotopy equivalence $f : C_*(\tilde{X},\tilde{Y};\mathbb{Z}\pi) \longrightarrow (C_*(X,Y;\mathbb{Z}[\pi \times G])$. If there exists such a Z, then we have a π-simple homotopy equivalence

$$C_*(\tilde{X},\tilde{Y};\mathbb{Z}\pi)\longrightarrow C_*(Z,Y;\mathbb{Z}[\pi\times G])$$

from a finite π-based complex to a finite $\pi\times G$-based complex. Hence by Theorem 1.3, $\gamma(\tau(\tilde{X},\tilde{Y}))=0$.

Conversely, suppose that $\gamma(\tau(\tilde{X},\tilde{Y}))$ vanishes. Then there exists a finite $\pi\times G$-based chain complex B_* and a $\pi\times G$-homotopy equivalence g from $C_*(X,Y;\mathbb{Z}[\pi\times G])$ to B_* such that $g\circ f$ is π-simple. This implies that the finiteness obstruction of X vanishes and there exists a homotopy equivalence from X to a finite complex $\tilde{Z}_1$. Moreover, we can add 2-cells and 3-cells to $\tilde{Z}_1$ in order to modify the simple type of Z_1 to obtain a finite complex Z such that the composite map

$B_*\xrightarrow{g^{-1}} C_*(X,Y;\mathbb{Z}[\pi\times G])\longrightarrow C_*(Z,Y;\mathbb{Z}[\pi\times G])$ is a $\pi\times G$-simple homotopy equivalence. It is easy to see that the composite map $\tilde{X}\to X\to Z$ induces a simple homotopy equivalence from $\tilde{X}$ to the covering space of Z with fundamental group π.

<u>SECTION II</u>. Let $A=\mathbb{Z}\pi$ and $\omega=\sum_{g\in G} g$ be the norm of G. For simplicity of notation, let $A[G]/\omega A[G]\equiv A[G]/\omega$, $A/qA\equiv A_q$, and $\mathbb{Z}/2\mathbb{Z}\times M\cong\{+1,-1\}\times M\equiv \pm M$ for any group M. Consider the cartesian diagram:

$$\begin{array}{ccc} A[G] & \xrightarrow{\quad h\quad} & A[G]/\omega \\ \Big\downarrow f & & \Big\downarrow \\ A & \longrightarrow & A_q \end{array} \qquad \text{(C)}$$

where f is the augmentation and all other homomorphisms are cannonically defined quotient morphisms. The associated Mayer-Vietories sequence is:

$$K_1(A[G])\longrightarrow K_1(A)\oplus K_1(A[G]/\omega)\longrightarrow K_1(A_q)\longrightarrow K_0(A[G])\longrightarrow$$
$$K_0(A)\oplus K_0(A[G]/\omega)\longrightarrow K_0(A_q) \qquad \text{(MV)}$$

Corresponding to (MV), one has the following exact sequence if $G\neq\mathbb{Z}_2$

(U) $0\to \pm H_1(\pi)\times H_1(G)\to \pm H_1(\pi)\oplus\pm H_1(\pi)\times H_1(G)\to \pm H_1(\pi)\xrightarrow{0}\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\to 0$

and if $G=\mathbb{Z}_2$, the sequence reads:

$$0\to \pm H_1(\pi)\times H_1(G)\to \pm H_1(\pi)\oplus H_1(\pi)\times H_1(G)\to H_1(\pi)\xrightarrow{0}\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\to 0$$

The sequences (U) and the corresponding homomorphisms are also obtained from the diagram (C). The sequence (U) admits an injective homomorphism into the sequence (MV) and the quotient sequence is the exact sequence of the Whitehead groups below:

$$Wh_1(\pi \times G) \longrightarrow Wh_1(\pi) \oplus K_1(A[G]/\omega)/\pm H_1(\pi) \times H_1(G) \longrightarrow Wh_1(\pi;\mathbb{Z}_q) \xrightarrow{\partial}$$

$$\tilde{K}_0(A[G]) \longrightarrow \tilde{K}_0(A) \oplus \tilde{K}_0(A[G]/\omega) \longrightarrow \tilde{K}_0(A_q)$$

For simplicity of notation we write this sequence in terms of Whitehead groups (by a slight abuse of notation)

$$Wh_1(A[G]) \longrightarrow Wh_1(A) \oplus Wh_1(A[G]/\omega) \longrightarrow Wh_1(A_q) \xrightarrow{\partial} Wh_0(A[G])$$

$$Wh_0(A) \oplus Wh_0(A[G]/\omega) \longrightarrow Wh_0(A_q) \qquad \text{(W)}$$

The boundary map ∂ in the sequence is related to a generalization of the Swan homomorphism $(\mathbb{Z}_q)^x \longrightarrow \tilde{K}_0(\mathbb{Z}\,G)$ in the case of $\pi = 1$, (cf. [Sw] or [M]). We continue to call ∂ the Swan homomorphism.

Let α and γ be as in Theorem I.1. Then the Swan homomorphism is $-\alpha\circ\gamma$. To see this, let $x \in Wh_1(A_q)$ correspond to the isomorphism $\phi : (A_q)^n \longrightarrow (A_q)^n$ induced by the (injective) homomorphism $\phi : A^n \longrightarrow A^n$. As in Section I, it follows that in the exact sequence

$$0 \longrightarrow A^n \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

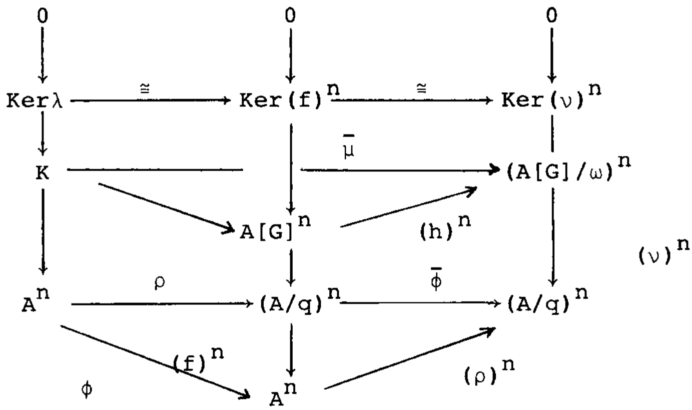
one has $\text{proj dim}_{AG}(M) \leq 1$. Thus one has the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^n & \xrightarrow{\phi} & A^n & \longrightarrow & M & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle (f)^n} & & \uparrow & & \uparrow{\scriptstyle 1} & & \uparrow \\
0 & \longrightarrow & K & \xrightarrow{\mu} & A[G]^n & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where $(f)^n$ is induced by the augmentation f. Thus $\alpha\gamma([\phi]) = -[K]$ and the problem is reduced to show that the following diagram is cartesian:

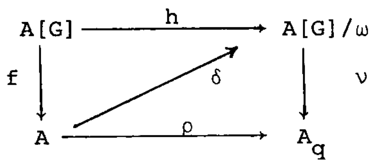
$$\begin{array}{ccc}
K & \xrightarrow{\quad\bar{\mu}\quad} & (A[G]/\omega)^n \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle (\nu)^n} \\
A^n & \xrightarrow{(\rho)^n} (A/q)^n \xrightarrow{\bar{\phi}} & (A/q)^n
\end{array}$$

(Recall the definition of ∂ in the Mayer-Vietories sequence; cf. [M] e.g.). Since $\mathrm{Ker}\lambda \cong \mathrm{Ker}(f)^n \cong \mathrm{Ker}(\nu)^n$ and $(f)^n \circ \mu = \phi \circ \lambda$, one has the diagram:

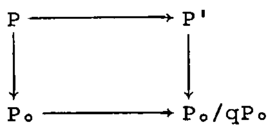


obtained from diagrams (B) and (C) above, and in which $\bar{\phi} \circ (\rho)^n \circ \lambda = (\rho)^n \circ \phi \circ \lambda = (\rho)^n \circ (f)^n \circ \mu = (\nu)^n \circ (h)^n \circ \mu = (\nu)^n \circ \bar{\mu}$. Thus (B) is cartesian.

Next, we identify the transfer $T_i : Wh_1(A[G]) \longrightarrow Wh_i(A)$, $i = 0,1$ in the 5-term exact sequence of Theorem I. Consider the diagram:



where δ is the composite $A \xrightarrow{\cong} A \times \{1\} \longrightarrow AG \longrightarrow AG/\omega$ so that $\nu \circ \delta = \rho$. Let $p \in P(AG)$ be given and tensor P over AG by the diagram (C) to obtain the cartesian diagram:



Thus one obtains four functors from $P(AG)$ to the categories $P(A[G])$, $P(A[G]/\omega)$, $P(A)$, and $P(A_q)$. The above cartesian diagram yields the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & P_o \oplus P' & \longrightarrow & P_o/qP_o & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle 1+0} & & \uparrow{\scriptstyle 1} & & \\
0 & \longrightarrow & P_o & \xrightarrow[(\cdot)\mathrm{x}q]{} & P_o & \longrightarrow & P_o/qP_o & \longrightarrow & 0
\end{array}$$

which yields the exact sequence:

$$0 \longrightarrow P_o \longrightarrow P \oplus P_o \longrightarrow P_o \oplus P' \longrightarrow 0$$

The above sequence defines, in fact, a short exact sequence of the corresponding functors due to the functoriality of all the above constructions. It follows from Quillen's theorem on the additivity of functors (cf. [Q]) that the functor $P \longrightarrow P \oplus P_o$ is the sum of functors $P \longrightarrow P_o$ and $P \longrightarrow P_o \oplus P'$, which in turn implies that induced homomorphisms on K-theory satisfy $Tr = f_* \oplus tr\, h_*$, $Tr : K_*(A[G]) \longrightarrow K_*(A)$ and $tr : K_*(AG/\omega) \longrightarrow K_*(A)$ are transfer homomorphisms. Thus on the level of Whitehead groups, one has the following:

<u>II.1 Lemma</u>. Let $T_i : Wh_i(A[G]) \longrightarrow Wh_i(A)$ and $t_i : Wh_i(A[G]/\omega) \longrightarrow Wh_i(A)$ be transfer homomorphisms. Then $T_i = f_* \oplus t_i h_*$, where f_* and h_* are induced from diagram (C) above.

Let $\rho_i \equiv \rho_* : Wh_i(A) \longrightarrow Wh_i(A) \longrightarrow Wh_i(A_q)$. Specializing to the case $G = \mathbb{Z}_2$, the above calculation is continued to show

<u>II.2 Lemma</u>: The sequence $Wh_1(\pi) \xrightarrow{2\rho_1} Wh_1(\pi;\mathbb{Z}_2) \xrightarrow{\gamma} Wh_1^T(\pi \subset \pi \times \mathbb{Z}_2) \longrightarrow$ $\longrightarrow \mathrm{Ker}\rho_o \longrightarrow 0$ is exact. In particular $\mathrm{Ker}\gamma = \mathrm{Im} 2\rho$.

This characterizes completely the obstructions which are discussed in Section I in terms of the $Wh_1(\pi)$ and the mod 2 reduction $Wh_1(\pi) \longrightarrow Wh_1(\pi;\mathbb{Z}_2)$. To obtain examples of nontrivial obstructions, let $\pi = \mathbb{Z}_8$. Then computations show

$$Wh_1(\mathbb{Z}_8;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$$

<u>II.3 Corollary</u>: $Wh_1^T(\mathbb{Z}_8 \subset \mathbb{Z}_8 \times \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\ker\gamma$ consists of the 2-divisible elements of $Wh_1(\mathbb{Z}_8;\mathbb{Z}_2)$.

Although $Wh_1(\mathbb{Z}_5) \cong \mathbb{Z}$, one can show that in the case $\pi = \mathbb{Z}_5$, $G = \mathbb{Z}_2$, all the obstructions vanish.

II.4 Corollary: $Wh_1(\mathbb{Z}_5;\mathbb{Z}_2) \cong \mathbb{Z}_3$ and $Im(\gamma) \equiv 0$.

Remark: In [Kw] Kwun has shown that the transfer $Wh_1(\mathbb{Z}_2 \times \mathbb{Z}_r) \longrightarrow Wh_1(\mathbb{Z}_r)$ is onto if and only if r = odd or r = 2,4,6. We thank the referee from bringing Kwun's result to our attention.

SECTION III. Let X be a finite dimensional CW complex with $\pi_1(X) = \pi$, and let G be a finite group of order q acting semifreely on X - i.e. the action is free outside of the stationary point set. In general, there is no explicit relationship between $H_*(X)$ and $H_*(X^G)$. The rather implicit information obtained using the localization theorems of Atiyah-Borel-Quillen-Segal type does not seem sufficient to yield a satisfactory characterization of the stationary point set X^G under general hypotheses. In the sequel, we will consider a class of actions which are encountered often in the geometric considerations, and to which it is possible to apply the present techniques of algebraic topology to obtain rather precise information and characterizations of X^G.

Given a connected space X and a subring of rational numbers Λ or $\Lambda = \mathbb{Z}_q$ we denote by X_Λ the localization of X which preserves $\pi_1 X$ and $\pi_i(X_\Lambda) \cong \pi_i(X) \otimes \Lambda$ for $i > 1$. For instance Bousfield-Kan's localization [B-K] applied to the universal covering space $\tilde{X}$ yields $\tilde{X}_\Lambda$ on which $\pi_1(X)$ operates freely and $\tilde{X} \longrightarrow \tilde{X}_\Lambda$ is equivariant. Then X_Λ can be defined as $\tilde{X}_\Lambda/\pi_1(X)$. For $\Lambda = \mathbb{Z}_q$, $\Lambda = \mathbb{Z}_{(q)}$ and $\Lambda = \mathbb{Z}[\frac{1}{q}]$ we can use the notations X_q, $X_{(q)}$ and $X(\frac{1}{q})$ respectively.

The key observation to reconstruct a space (respectively a diagram of spaces) from its localizations (respectively its diagrams of localizations) is the following:

III.1. Lemma. For any connected space X the following diagram is cartesian:

$$\begin{array}{ccc} X & \longrightarrow & X(\frac{1}{q}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X_q & \longrightarrow & X_q(\frac{1}{q}) \end{array}$$

Proof: Since $H_*(X_q, X; \mathbb{Z}/q[\pi]) = 0$ it follows that $H_*(X_q, X; \mathbb{Z}\pi)$ is

$\mathbb{Z}[\frac{1}{q}]$-local. Hence the (homotopy) fibre of f is $\mathbb{Z}[\frac{1}{q}]$-local (Cf. [S]). Since the (homotopy) fibre of f' is also $\mathbb{Z}[\frac{1}{q}]$-local, f and f' has the same fibre (up to homotopy).

<u>Definition</u>. Let X be a connected G-CW complex, where G is a finite group of order q. X is called a simple G-space (and the action is called simple) if $(E_G \times_G X)_q$ is fibre homotopy equivalent to $(BG \times X)_q$.

For instance, if X has trivial mod q homology, then any G-action on X will be simple, or if X has the mod q homology of a sphere and $X^G \neq \emptyset$, the X-{point} has a simple action if we take out a point from X^G.

<u>Proposition</u>. Suppose G is a finite group of order q which has a simple semifree action on the finite dimensional complex X with $\pi_1 X = \pi$. Then $H_*(X, X^G; \mathbb{Z}_q\pi) = 0$, where the homology has local coefficients.

In the case of semifree simple actions on compact manifolds, one obtains further restrictions imposed on X^G. For simplicity, let us consider the case of a smooth semifree G-action on a compact manifold W^n with $\pi_1 W = \pi$. Then the stationary point set $W^G = F^k$ is a submanifold with normal bundle ν which is a G-bundle with a free G-representation at each fibre. Assume that $n-k > 2$. We identify the total space of the disk bundle $D(\nu)$ with a closed G-invariant tubular neighborhood of F. Let C^n = W-interior $D(\nu)$. One can choose an appropriate CW structure for W so that W, C, and $D(\nu)$ become G-CW complexes, and various cellular chain complexes have preferred bases. If the action is simple, then $H_*(W,F;\mathbb{Z}_q\pi) = 0$, and G acts trivially on $H_*(W,F;\mathbb{Z}\pi)$, as well as on $H_*(S(\nu)\ ;\mathbb{Z}[\frac{1}{q}](\pi)) \cong H_*(S(\nu)/G;\mathbb{Z}[\frac{1}{q}](\pi \times G))$. One further observation is that the geometry provides us with the dotted arrow in the following diagram in which $\pi = \pi_1(W)$:

$$\begin{array}{ccc} \pi_1(S(\nu)/G & \dashrightarrow & \pi \\ \uparrow & \nearrow & \\ \pi_1(S(\nu)) & & \end{array}$$

For a pair (W,F) as above, we define an element $\omega(W,F) \in Wh^T(\pi \subset \pi \times G)$ as follows. Given a free finite $\mathbb{Z}\pi$-based chain complex (A'_*, A') and a free $\mathbb{Z}G$-resolution R_* of $\mathbb{Z}$, we form the $\mathbb{Z}(\pi \times G)$-complex $A_* = A'_* \otimes R_*$ which is $\mathbb{Z}\pi$-chain homotopy equivalent to A'_*. Suppose $H_*(A'_* \otimes \mathbb{Z}_q) = 0$.

Then by theorem I.3 there is a finite $\mathbb{Z}(\pi \times G)$-projective complex B_* with a π-basis $\mathcal{B}'$ such that $(B_*, \mathcal{B}')$ is π-simple homotopy equivalent to (A_*', A'). Define $\omega(A_*, A') = \sum(-1)^i [B_i, \mathcal{B}_i] \in Wh^T(\pi \subset \pi \times G)$ which is seen to be well-defined. Now let A_*' be the $\mathbb{Z}\pi$-chain complex of cellular chains of (W,F) with local $\mathbb{Z}\pi$-coefficients and let $\mathcal{R}'$ be the natural preferred bases provided by the cells. Then $\omega(W,F) = \omega(A_*', \mathcal{R}')$ is well-defined. From section I, one can compute that $\omega(A_*', \mathcal{R}') = \gamma\tau(A_*')$.

<u>III.2. Theorem</u>. Let $\phi : G \times W^n \longrightarrow W^n$ be a smooth simple semifree action with $F^k = W^G$, $n-k > 2$, and ν = normal bundle of F in W, $\pi = \pi_1(W)$. Then:

1) $H_*(W,F;\mathbb{Z}_q\pi) = 0$,
2) G acts trivially on $H_*(S(\nu)/G;\mathbb{Z}[\frac{1}{q}](\pi \times G))$,
3) there is a homomorphism ι making the following diagram commute:

$$\begin{array}{ccc} \pi_1(S(\nu)/G) & \xrightarrow{\ \iota\ } & \pi \\ \uparrow & \nearrow & \\ \pi_1(S(\nu)) & & \end{array}$$

4) $\omega(W,F) \in Wh_1^T(\pi \subset \pi \times G)$ vanishes.

Since $C_*(C^n, S(\nu);\mathbb{Z}\pi)$ is $\mathbb{Z}\pi$-chain homotopy equivalent to $C_*(W,F;\mathbb{Z}\pi)$, one verifies that $\omega(W,F)$ is defined under the following more general situation: $F^k \subset W^n$ is a submanifold with normal bundle ν, $n-k > 2$, and ν has G-bundle structure with a free representation on each fibre, and conditions (1) and (3) of Theorem II.2 are satisfied for (W,F).

The main results of this section are the following two theorems.

<u>III.3. Theorem (Characterization of stationary-point sets of simple actions)</u>.

Let W^n be a compact manifold with connected boundary such that $\pi_1(\partial W) \cong \pi_1(W) = \pi$, and let $(F^k, \partial F^k) \subset (W, \partial W)$ be a smooth submanifold with normal bundle ν, $n-k > 2$, $n \geq 6$. Then there is a smooth simple semifree G-action on W^n with $(W^n)^G = F$ if and only if F:

1) ν admits a G-bundle structure over F with a free representation on each fibre.
2) $H_*(W,F;\mathbb{Z}_q\pi) = 0$,
3) $\gamma\tau(W,F) \in Wh_1^T(\pi \subset \pi \times G)$ vanishes.