Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

897

Wilfried Buchholz Solomon Feferman
Wolfram Pohlers Wilfried Sieg

Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies

Springer-Verlag
Berlin Heidelberg New York

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

897

Wilfried Buchholz Solomon Feferman
Wolfram Pohlers Wilfried Sieg

Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies

Springer-Verlag
Berlin Heidelberg New York 1981

Authors

Wilfried Buchholz
Mathematisches Institut der Universität München
Theresienstr. 39, 8000 München 2, Germany

Solomon Feferman
Department of Mathematics, Stanford University
Stanford, California 94305, USA

Wolfram Pohlers
Mathematisches Institut der Universität München
Theresienstr. 39, 8000 München 2, Germany

Wilfried Sieg
Department of Philosophy, Columbia University
New York 10027, USA

AMS Subject Classifications (1980): 03 S

ISBN 3-540-11170-0 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-11170-0 Springer-Verlag New York Heidelberg Berlin

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

Lecture Notes in Mathematics

For information about Vols. 1–669, please contact your bookseller or Springer-Verlag.

Vol. 670: Fonctions de Plusieurs Variables Complexes III, Proceedings, 1977. Edité par F. Norguet. XII, 394 pages. 1978.

Vol. 671: R. T. Smythe and J. C. Wierman, First-Passage Perculation on the Square Lattice. VIII, 196 pages. 1978.

Vol. 672: R. L. Taylor, Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces. VII, 216 pages. 1978.

Vol. 673: Algebraic Topology, Proceedings 1977. Edited by P. Hoffman, R. Piccinini and D. Sjerve. VI, 278 pages. 1978.

Vol. 674: Z. Fiedorowicz and S. Priddy, Homology of Classical Groups Over Finite Fields and Their Associated Infinite Loop Spaces. VI, 434 pages. 1978.

Vol. 675: J. Galambos and S. Kotz, Characterizations of Probability Distributions. VIII, 169 pages. 1978.

Vol. 676: Differential Geometrical Methods in Mathematical Physics II, Proceedings, 1977. Edited by K. Bleuler, H. R. Petry and A. Reetz. VI, 626 pages. 1978.

Vol. 677: Séminaire Bourbaki, vol. 1976/77, Exposés 489–506. IV, 264 pages. 1978.

Vol. 678: D. Dacunha-Castelle, H. Heyer et B. Roynette. Ecole d'Eté de Probabilités de Saint-Flour. VII-1977. Edité par P. L. Hennequin. IX, 379 pages. 1978.

Vol. 679: Numerical Treatment of Differential Equations in Applications, Proceedings, 1977. Edited by R. Ansorge and W. Törnig. IX, 163 pages. 1978.

Vol. 680: Mathematical Control Theory, Proceedings, 1977. Edited by W. A. Coppel. IX, 257 pages. 1978.

Vol. 681: Séminaire de Théorie du Potentiel Paris, No. 3, Directeurs: M. Brelot, G. Choquet et J. Deny. Rédacteurs: F. Hirsch et G. Mokobodzki. VII, 294 pages. 1978.

Vol. 682: G. D. James, The Representation Theory of the Symmetric Groups. V, 156 pages. 1978.

Vol. 683: Variétés Analytiques Compactes, Proceedings, 1977. Edité par Y. Hervier et A. Hirschowitz. V, 248 pages. 1978.

Vol. 684: E. E. Rosinger, Distributions and Nonlinear Partial Differential Equations. XI, 146 pages. 1978.

Vol. 685: Knot Theory, Proceedings, 1977. Edited by J. C. Hausmann. VII, 311 pages. 1978.

Vol. 686: Combinatorial Mathematics, Proceedings, 1977. Edited by D. A. Holton and J. Seberry. IX, 353 pages. 1978.

Vol. 687: Algebraic Geometry, Proceedings, 1977. Edited by L. D. Olson. V, 244 pages. 1978.

Vol. 688: J. Dydak and J. Segal, Shape Theory. VI, 150 pages. 1978.

Vol. 689: Cabal Seminar 76–77, Proceedings, 1976–77. Edited by A.S. Kechris and Y. N. Moschovakis. V, 282 pages. 1978.

Vol. 690: W. J. J. Rey, Robust Statistical Methods. VI, 128 pages. 1978.

Vol. 691: G. Viennot, Algèbres de Lie Libres et Monoïdes Libres. III, 124 pages. 1978.

Vol. 692: T. Husain and S. M. Khaleelulla, Barrelledness in Topological and Ordered Vector Spaces. IX, 258 pages. 1978.

Vol. 693: Hilbert Space Operators, Proceedings, 1977. Edited by J. M. Bachar Jr. and D. W. Hadwin. VIII, 184 pages. 1978.

Vol. 694: Séminaire Pierre Lelong – Henri Skoda (Analyse) Année 1976/77. VII, 334 pages. 1978.

Vol. 695: Measure Theory Applications to Stochastic Analysis, Proceedings, 1977. Edited by G. Kallianpur and D. Kölzow. XII, 261 pages. 1978.

Vol. 696: P. J. Feinsilver, Special Functions, Probability Semigroups, and Hamiltonian Flows. VI, 112 pages. 1978.

Vol. 697: Topics in Algebra, Proceedings, 1978. Edited by M. F. Newman. XI, 229 pages. 1978.

Vol. 698: E. Grosswald, Bessel Polynomials. XIV, 182 pages. 1978.

Vol. 699: R. E. Greene and H.-H. Wu, Function Theory on Manifolds Which Possess a Pole. III, 215 pages. 1979.

Vol. 700: Module Theory, Proceedings, 1977. Edited by C. Faith and S. Wiegand. X, 239 pages. 1979.

Vol. 701: Functional Analysis Methods in Numerical Analysis, Proceedings, 1977. Edited by M. Zuhair Nashed. VII, 333 pages. 1979.

Vol. 702: Yuri N. Bibikov, Local Theory of Nonlinear Analytic Ordinary Differential Equations. IX, 147 pages. 1979.

Vol. 703: Equadiff IV, Proceedings, 1977. Edited by J. Fábera. XIX, 441 pages. 1979.

Vol. 704: Computing Methods in Applied Sciences and Engineering, 1977, I. Proceedings, 1977. Edited by R. Glowinski and J. L. Lions. VI, 391 pages. 1979.

Vol. 705: O. Forster und K. Knorr, Konstruktion verseller Familien kompakter komplexer Räume. VII, 141 Seiten. 1979.

Vol. 706: Probability Measures on Groups, Proceedings, 1978. Edited by H. Heyer. XIII, 348 pages. 1979.

Vol. 707: R. Zielke, Discontinuous Čebyšev Systems. VI, 111 pages. 1979.

Vol. 708: J. P. Jouanolou, Equations de Pfaff algébriques. V, 255 pages. 1979.

Vol. 709: Probability in Banach Spaces II. Proceedings, 1978. Edited by A. Beck. V, 205 pages. 1979.

Vol. 710: Séminaire Bourbaki vol. 1977/78, Exposés 507–524. IV, 328 pages. 1979.

Vol. 711: Asymptotic Analysis. Edited by F. Verhulst. V, 240 pages. 1979.

Vol. 712: Equations Différentielles et Systèmes de Pfaff dans le Champ Complexe. Edité par R. Gérard et J.-P. Ramis. V, 364 pages. 1979.

Vol. 713: Séminaire de Théorie du Potentiel, Paris No. 4. Edité par F. Hirsch et G. Mokobodzki. VII, 281 pages. 1979.

Vol. 714: J. Jacod, Calcul Stochastique et Problèmes de Martingales X, 539 pages. 1979.

Vol. 715: Inder Bir S. Passi, Group Rings and Their Augmentation Ideals. VI, 137 pages. 1979.

Vol. 716: M. A. Scheunert, The Theory of Lie Superalgebras. X, 271 pages. 1979.

Vol. 717: Grosser, Bidualräume und Vervollständigungen von Banachmoduln. III, 209 pages. 1979.

Vol. 718: J. Ferrante and C. W. Rackoff, The Computational Complexity of Logical Theories. X, 243 pages. 1979.

Vol. 719: Categorial Topology, Proceedings, 1978. Edited by H. Herrlich and G. Preuß. XII, 420 pages. 1979.

Vol. 720: E. Dubinsky, The Structure of Nuclear Fréchet Spaces. V, 187 pages. 1979.

Vol. 721: Séminaire de Probabilités XIII. Proceedings, Strasbourg, 1977/78. Edité par C. Dellacherie, P. A. Meyer et M. Weil. VII, 647 pages. 1979.

Vol. 722: Topology of Low-Dimensional Manifolds. Proceedings, 1977. Edited by R. Fenn. VI, 154 pages. 1979.

Vol. 723: W. Brandal, Commutative Rings whose Finitely Generated Modules Decompose. II, 116 pages. 1979.

Vol. 724: D. Griffeath, Additive and Cancellative Interacting Particle Systems. V, 108 pages. 1979.

Vol. 725: Algèbres d'Opérateurs. Proceedings, 1978. Edité par P. de la Harpe. VII, 309 pages. 1979.

Vol. 726: Y.-C. Wong, Schwartz Spaces, Nuclear Spaces and Tensor Products. VI, 418 pages. 1979.

Vol. 727: Y. Saito, Spectral Representations for Schrödinger Operators With Long-Range Potentials. V, 149 pages. 1979.

continuation on page 385

TABLE OF CONTENTS

PART C. PROOF THEORY OF THEORIES FOR INDUCTIVE DEFINITIONS REQUIRING THE USE OF THE NOTATION SYSTEM $\overline{\Theta}(\Omega)$.

Preface

HOW WE GOT FROM THERE TO HERE

by

S. Feferman

Preface: How We Got from There to Here

This preface begins with a statement of our main results, in a form suitable for specialists. However, the reader unfamiliar with proof theory and subsystems of analysis will find following that, a gradually unfolding informal explanation of the necessary background which will allow him or her to gain an appreciation of our project as a whole. This accompanies an account tracing developments over the last 20 years, of which the present results are the culmination. We hope that this review will also be of interest to the specialist for putting the work presented here in perspective. The preface concludes with an outline of the contents of the successive chapters. Of these, Chapter I fills in all details of background, so that the work can be read independently of the research literature. (The specialist will find that Ch.I can be skimmed or even skipped.)

The idea for the present volume originated in 1977, during which year each of my co-authors had completed a dissertation on the proof theory of iterated inductive definitions: Wilfried Buchholz and Wolfram Pohlers in their Habilitationsschriften at Munich under the direction of Prof. Kurt Schütte and Wilfried Sieg in his Ph.D. thesis at Stanford under my direction. Following different paths, they had obtained related but in many respects complementary solutions to most of the then outstanding problems in the theory of iterated ID systems (ID_ν):

(1) Supplying the final (previously missing) links in a program for reducing certain subsystems of classical analysis to constructive systems; and

(2) obtaining exact proof-theoretic (ordinal) bounds for the ID_ν for arbitrary ν.

Among the main results of (1) is that

$$(1)^\circ \qquad (\Sigma^1_2\text{-}AC) \equiv ID^i_{<\varepsilon_0}(\mathcal{O}) \equiv T_0(IG\restriction),$$

where $ID^i_{<\varepsilon_0}(\mathcal{O})$ is the intuitionistic theory of the constructive ordinal number classes $\mathcal{O}_\nu$ iterated through all ordinals $\nu < \varepsilon_0$, and $T_0(IG\restriction)$ is a subtheory of my constructive theory T_0 of functions and classes obtained by restricting the inductive generation scheme. (The relation $\equiv$ is that of proof-theoretic equivalence.) Among the main results of (2) is that

$$(2)^{\circ} \qquad |ID_{\nu}| = |ID^{i}_{\nu}(\mathfrak{G})| = \bar{\theta}\varepsilon_{\Omega_{\nu}+1}0 .$$

These results will be described in more detail below and compared with previous knowledge. An interesting variety of methods going beyond predicative proof theory were employed to achieve $(1)^{\circ}$ and $(2)^{\circ}$. While some of the technicalities were quite complicated, they had been made manageable by systematic organization.

The situation as I saw it at the end of 1977 was as follows. On the one hand, a phase in the proof theory of impredicative systems had been dramatically brought to a close by this work of Buchholz, Pohlers and Sieg. The problems (1) and (2) had been grappled with since 1967, and the results finally obtained were conclusive. In the process, our understanding of the ins and outs of theories of inductive definitions had advanced significantly. On the other hand, one did not yet feel that the methods employed had been brought to a definitive form, at least comparable to those of predicative proof theory. It was expected that this might still require a good deal of further research, aimed at making the methods more conceptual. For example, certain collapsing functions in ordinal notation systems played a crucial role in $(2)^{\circ}$, but one had no clear (canonical) meaning for them. Finally, there were open problems that one could hope to attack by an extension of the methods developed but which would require significant additional effort. Foremost among these was the question of finding the proof-theoretic ordinal of Σ^1_2-AC + BI and that of my conjecture that Σ^1_2-AC + BI is reducible to T_0 (the converse reduction having been easily established).

It thus seemed to me to be an opportune moment to present the work of Buchholz, Pohlers and Sieg side-by-side in the spirit of comparing and disseminating approaches and results from an ongoing enterprise, a kind of laboratory of proof-theoretic methods. Moreover, the format of the Springer Lecture Note Series seemed ideally suited for such a cooperative venture. As I saw it, only an additional introductory chapter explaining the background and the common resources of the later chapters would be needed for the general reader; this Sieg and I offered to supply. The proposal was enthusiastically agreed to by all involved. In fleshing out the plan, it was decided to incorporate further closely related unpublished work of Buchholz, Sieg and myself. Even so, it looked reasonable to put a target date of one year for completion of the project.

In fact, it has taken four years from its original conception to bring this work to publication. The reason is quite simple: none of us could bear to let things stand as they were in 1977. Each felt impelled to make improvements, technical and/ or conceptual. Indeed, in Pohlers' case, this led to the development of a major new method, which he calls that of "local predicativity." The result, all told, is a much better volume than if we had stuck to our original plan. In the meantime, the field has not remained still. There have been further important and interesting relevant contributions (of which some indication will be given below). Foremost among these with respect to our own project was the solution by Jäger and Pohlers of the previously mentioned problems concerning Σ_2^1-AC+BI and T_0. Despite this, our joint venture had not lost its timeliness, especially in view of the improvements which had been made. Moreover, many of the reasons for embarking on the project were still valid. Finally, any significant enlargements of the material to be included would require considerable additional effort and cost further loss of time. This explains how we have arrived at the present volume.

As we have said, the reader experienced in modern proof theory can proceed directly to the meat of the volume starting with Chapter II. For the general reader, enough background is supplied in Chapter I to make possible an independent reading of the work as a whole. This background can be enlarged and deepened by judicious choices from the bibliography referred to as one goes along. The following is only intended to hit the main points of what led to the present work and thus to help put that in perspective.

The process of inductive definition is used frequently in mathematics and particularly in mathematical logic. The ubiquitous mathematical example is that of a substructure of a given structure generated by given operations (finitary or infinitary), e.g., of a subgroup of a group or the Borel sets of a space. Examples from logic are: (i) the derivable formulas of a formal system, (ii) partial functions generated by recursive schemata, and (iii) classes of constructive ordinal notations. The first two examples are finitary (for ordinary formal systems, resp. ordinary recursion theory), while the third is infinitary.

Inductive definition is particularly appealing from the constructive point of view, with its genetic conception of the basic structures of mathematics. A constructive theory of countable ordinals (generated successively by countable sums) was developed by Brouwer 1926. A recursive formulation was set up by Church and Kleene 1936 and pursued by Kleene 1938. This provides recursive analogues of the classical (Cantorian) higher ordinal number classes. Modern recursion-theoretic treatments are given in terms of the Kripke-Platek notion of admissible ordinals; cf. Barwise 1975. Our main concern here though is with inductive definitions, in particular those of the ordinal number classes, considered from a strictly constructivist point of view (e.g. that of Brouwer or Bishop - cf. Troelstra 1977).

As it happens, the process of infinitary inductive definition has hardly been applied in constructive mathematics. For one example: Bishop 1967 applied it to develop a theory of measure using Borel sets. But this was superseded by Bishop, Cheng 1972 which dispensed with the use of Borel sets and was otherwise simpler. Indeed, recent investigations (Friedman 1977, Feferman 1979) show that constructive practice of the Bishop school is far from exploiting any but the most elementary constructive principles. (An exception of interest is Richman's 1973 treatment of the Ulm ordinal structure theory of countable Abelian groups.)

It is true that Brouwer's theory of choice sequences has been given a foundation by Kreisel, Troelstra 1970 in the theory of one inductive definition, which is used to generate the class K of (representing functions of) continuous type 2 operations. But the eventual applications of the theory of choice sequences in mathematical analysis are achieved in Bishop's work by much more elementary principles, simply by circumventing the notions and questions which preoccupied Brouwer.

The place where the study of inductive definitions has had its greatest impact is in recursion theory and its generalizations. Here the developments have been extensive and of a high order; cf. particularly Moschovakis 1974 and Barwise 1975. Indeed, the Barwise-Gandy-Moschovakis Theorem analyzes the passage to the next admissible set over a given one in terms of (first-order) inductive definitions over that set. However, the approach there is highly non-constructive.

The study of formal theories featuring inductive definitions in both single and iterated form was initiated by Kreisel 1963. The immediate stimulus was the question of constructive justification of Spector's 1961 consistency proof for analysis. Kreisel 1959 B had extended Gödel's Dialectica interpretation to analysis by the use of continuous (or Kleene "countable") functionals of finite type. Spector had refined this to an interpretation in the so-called bar-recursive functionals of finite type. These were generated by schemata analogous to Brouwer's principle of bar-induction. The use of bar-recursive functionals of type 2 was indeed justifiable constructively (either directly by Brouwer's principle or by working through the inductively generated class K above). Kreisel wanted to see whether iterated inductive definitions (of classes of lawlike operations analogous to K) could serve to model the bar-recursive functionals of higher type. The conclusion was negative, since such a theory of iterated inductive definitions was much weaker than full second order analysis. Indeed, Kreisel thought that even a suitable theory of transfinitely iterated inductive definitions would not go beyond Σ_2^1 - AC.

Proof theory at that time had been pursuing an extension of Hilbert's program, following Gentzen's lead: to reduce subsystems of analysis to extensions of arithmetic based on principles of transfinite induction for constructively recognized ordinals given by "natural" systems of notation. Side results were characterizations of the provably recursive well-orderings and functions of the systems dealt with. Speaking loosely, one measured the exact proof-theoretic strength of these systems in natural ordinal-theoretic terms. This work had been organized most clearly and elegantly by the use of derivations in an infinitary logic with countably long conjunctions and disjunctions $(L_{\omega_1,\omega})$; cf. particularly, Schütte 1951, 1952 (or 1960), Tait 1968 (and more recently Schwichtenberg 1977). Here ordinals make a canonical appearance as a measure of the lengths of proofs as well as of their cut-ranks. Schütte applied this to measure the proof-theoretic strength of systems of ramified analysis. This used the Veblen hierarchy of ordinal functions $\varphi\alpha\beta$ which we here designate $\theta\alpha\beta : \theta 0 \beta = \omega^{\beta}$ and for $\alpha > 0$, $\theta\alpha\beta = \beta^{th}$ common fixed point of all the functions $\theta\gamma$ for $\gamma < \alpha$. The process of cut-elimination for infinitary derivations yields $\theta\alpha\beta$ as upper bound for the length of a cut-free derivation d^{*} obtained from a derivation d of cut-rank $\leq \alpha$ and length $\leq \beta$. Γ_o is defined as the least α with $\theta\alpha 0 = \alpha$. Using the embedding of ramified systems in the infinitary logic, Feferman 1964 and Schütte 1965 independently determined the least impredicative ordinal to be Γ_o - taking the predicative ordinals to be those generated by an autonomous ramified procedure. For this reason, the proof theory of systems which can be interpreted in $L_{\omega_1,\omega}$ is often called predicative (though strictly speaking this is so only for ordinals $< \Gamma_o$).

Spector's striking leap to full classical analysis had not been convincing constructively and had provided no ordinal-theoretic information. The 60's were taken up with efforts to extend ordinally informative proof theory to impredicative subsystems of analysis, but starting back at relatively low levels of the analytic hierarchy. The years 1967-68 constituted a turning point in this program.

First we must say a little more about the systems involved (cf. Chapter I for full details). $\mathfrak{F}$ - CA is the 2nd order system with instances of the comprehension axiom $\exists X\, \forall n[n \in X \leftrightarrow F(n)]$ for all formulas F in $\mathfrak{F}$. Analogously, $\mathfrak{F}$ - AC is a 2nd order system for the (countable) axiom of choice. BI is the principle of bar induction, which allows us to apply proof by transfinite induction (TI) to any recursive well-ordering. We are mainly concerned with the systems Π^1_n-CA and Σ^1_n-AC for $n = 0, 1$ (where Π^1_o is taken to be Π^o_1). Using complete Π^1_n predicates we denote by $(\Pi^1_n\text{-CA})_{\nu}$ the system with $(\Pi^1_n\text{-CA})$ iterated ν times, and $(\Pi^1_n\text{-CA})_{<\nu} = \bigcup_{\mu<\nu} (\Pi^1_n\text{-CA})_{\mu}$.

ID_1 is any first-order system based on axioms of the following kind:

I. $A(P_A, x) \rightarrow P_A(x)$

II. $\forall x[A(B,x) \rightarrow B(x)] \rightarrow \forall x[P_A(x) \rightarrow B(x)]$ for each B,

where $A(P,x)$ is arithmetical in P and has P only in positive occurrences. (The positivity condition assures provable monotonicity $A(P,x) \wedge P \subseteq P' \rightarrow A(P', x)$.) This formalizes an <u>accessibility inductive definition</u> if $A(P,x)$ has the form $A_o(x) \wedge \forall y[(y,x) \in R \rightarrow P(x)]$. In that case $P_A(x)$ is interpreted as the accessible (or well-founded) part of R (hereditarily) in A_o. Speaking mathematically, every accessibility inductive definition is <u>deterministic</u>, i.e. there is a unique "verification tree" for $P_A(x)$ when it is true. The class $\mathcal{O}$ of Church-Kleene constructive notations is given by an accessibility i.d., and the corresponding theory is denoted $ID_1(\mathcal{O})$. A related class we use is W, the class of (codes of) recursive well-founded trees; its theory is denoted $ID_1(W)$. Superscript 'i' is used to indicate restriction to intuitionistic logic, as, e.g., in $ID_1^i(\mathcal{O})$.

The definition of $\mathcal{O}$ may be iterated into the transfinite in two ways, to give classes $\mathcal{O}_a$. One method replaces "recursive" where it appears in the clauses for closure under limit notations by "recursive in $\langle \mathcal{O}_b \rangle_{b < a}$". The second method leaves "recursive" unchanged but builds in regularity of $\mathcal{O}_a$ by requiring closure under recursive limits of $\mathcal{O}_b$-sequences for each $b \prec a$. (Mathematically, these two methods give (ordinally) equivalent results by the work of Richter 1965 and Belyakin 1969.) We use the latter formation method here to specify $ID_\nu(\mathcal{O})$ (the theory of $\mathcal{O}_a$ classes for $a < \nu$) and $ID_{<\nu}(\mathcal{O}) = \bigcup_{\mu < \nu} ID_\mu(\mathcal{O})$. Similarly we can deal with ID_ν and $ID_{<\nu}$ theories for more general iterated closure conditions.

Accessibility inductive definitions enjoy a privileged position in our informal conception of the subject. We have a direct picture of how the members of such i.d. sets are generated, which leads us immediately to recognize the ID axioms for them as correct. This is the picture "from below". Furthermore, we can carry out definition by recursion on accessibility i.d. sets. However the axioms for non-accessibility inductive definitions either need to be justified by impredicative principles "from above" (for the least set satisfying given closure conditions) or require a prior classical theory of ordinals. Among accessibility i.d.'s, those of the constructive number classes $\mathfrak{O}$ occupy a special position - partly for their historical importance but also because each ordinal notation codes its own verification tree. A frequent aim in the following is to reduce classical systems of analysis to accessibility theories ID^i_ν or $ID^i_{<\nu}$; where possible this is carried to $ID^i_\nu(\mathfrak{O})$, resp. $ID^i_{<\nu}(\mathfrak{O})$.

The first significant proof-theoretic results obtained for impredicative systems (after Spector's work) were in Takeuti 1967 for Π^1_1-CA(+BI), later extended to Δ^1_2-CA by Takeuti, Yasugi 1973. For this Takeuti returned to Gentzen's partial cut-elimination method for arithmetic, which only reduced the complexity of derivations of numerical statements. He assigned ordinals to these within special systems of notations which he called <u>ordinal</u> <u>diagrams</u>; these are not (naturally) based on systems of ordinal functions. Takeuti showed the ordinal diagrams to be well-founded, using principles which can be formalized in certain accessibility ID^i_ν theories. Consistency of Π^1_1-CA and (later) Δ^1_2-CA was proved by transfinite induction on suitable classes of ordinal diagrams. Thus the work <u>did</u> provide a reduction of these subsystems of analysis to constructive principles. However, no sharp bound was given for the provably recursive ordinals of these systems, nor were the ordinal diagrams attached in any intrinsic way to proofs (as were the ordinals of derivation trees in Schütte's infinitary approach). Finally, the details of the arguments for Takeuti's method were extremely difficult to follow, and a more perspicuous treatment was much to be hoped for and sought.

The proceedings IPT of the 1968 conference in Buffalo on Intuitionism and Proof Theory (appearing in 1970) contained three papers which contributed to the solution of the problems just mentioned - though much more work had still to be done.

In the first of these, Friedman 1970, it was shown (by a formalized — and a bit tricky — model-theoretic argument) that Σ^1_2-AC is reducible to $(\Pi^1_1\text{-CA})_{<\varepsilon_0}$, the Π^1_1-comprehension axiom iterated any number $<\varepsilon_0$ times. More generally, Σ^1_{n+1}-AC is reducible to $(\Pi^1_n\text{-CA})_{<\varepsilon_0}$.

Feferman 1970 gave a straightforward (formalized model-theoretic) reduction of $(\Pi_1^1\text{-CA})_{<\nu}$ to $ID_{<\nu}(W)$ for various ν, including $\nu = \epsilon_0$. It was conjectured then that ID_ν was (finitistically) reducible to $ID_\nu^i(\mathcal{O})$ and similarly for $ID_{<\nu}$, $ID_{<\nu}^i(\mathcal{O})$. At that time it was known (by previous work of Howard, Kreisel and Troelstra) that ID_1 is reducible to $ID_1^i(\mathcal{O})$, but only by a roundabout argument through a formal theory of choice sequences. All methods known at that point failed to extend to ID_2.

Tait 1970 gave an interpretation of certain theories of iterated inductive definitions in calculi PL_ν of propositional logic with uncountably long conjunctions and disjunctions (ranging over abstract constructive number classes $\mathcal{O}_\alpha$ for $\alpha < \nu$). This was used to interpret $\Sigma_2^1\text{-AC}$ in $PL_{<\epsilon_0}$. The proof made use of the cut-elimination theorem for the PL_ν, which was readily extended from countable logic. However, no ordinals in a natural notation system were attached by this procedure to $(\Sigma_2^1\text{-AC})$. In addition, there was the question whether the work could be made constructive. Tait suggested that this should be possible (it appears using the principles of $ID_{\nu+1}^i$ to treat PL_ν); however, this was not evident nor was it ever carried out by him.

The next main result was due to Howard 1972, who evaluated the least ordinal not provably recursive in $ID_1^i(\mathcal{O})$ (and hence for ID_1 by the reduction mentioned above), in terms of the Bachmann hierarchy of ordinal functions. The latter was an extension of the Veblen hierarchy $\langle\theta\alpha\rangle_{\alpha<\Omega_1}$ to certain uncountable α, in particular to the first ϵ-number $> \Omega_1$, i.e. $\alpha = \epsilon_{\Omega_1+1}$. The crucial new device in Bachmann's system was to diagonalize at α of cofinality Ω_1, e.g. $\theta\Omega_1\beta = \theta\beta 0$, $\theta(\Omega_1+\Omega_1)\beta = \theta(\Omega_1+\beta)0$, etc; this in turn requires an assignment of fundamental sequences of cofinality $\leq \Omega_1$ to a segment of ordinals in the 3^d number class. Howard's ordinal for ID_1^i is $\theta\epsilon_{\Omega_1+1}0$.

Bachmann's method of definition extended the hierarchy $\theta\alpha$ to α in higher number classes Ω_ν. This was done systematically by Pfeiffer 1964 for finite number classes and Isles 1970 for transfinite number classes up to the first inaccessible ordinal. However, since the method requires simultaneous assignment of