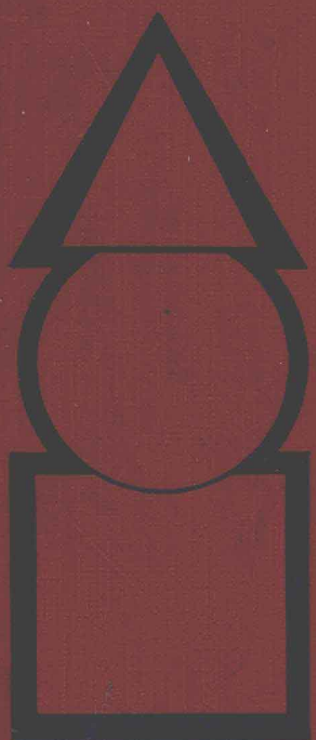
HOW CHILDREN LEARN MATHEMATICS

THIRD EDITION



TEACHING IMPLICATIONS OF PIAGET'S RESEARCH

Richard W. Copeland

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Preface

Purpose

The text has two major purposes. One is diagnostic—to aid in discerning a child's stage of development as a basic for determining the type of mathematics for which he is ready. The second major purpose is to serve the "methods of teaching mathematics" courses in teacher education. The reader will find this book very different from other methods texts because it is based on *how children learn*, not on *how to teach*, and it should enable readers to see mathematics from the standpoint of the child as he progresses through the various stages of development.

The teacher, as described in this book, should be skillful in the interview technique, and many of the quotations included are actual dialogues between children and teachers. Questions are not answered for the children; they are designed so that the child can answer them himself when he is able. Teachers should also be familiar with the type of laboratory materials children need at the concrete operational level in order to learn mathematical concepts. These materials are described in the many illustrations and in the mathematics laboratory chapter.

A Word About the Third Edition

The third edition of this book modifies the second in several ways. There are four new chapters. Three are based on recent books by Piaget.

Chapter 8, "Memory and Mathematics," is based on the book, *Memory and Intelligence*,[1] written in 1973. It demonstrates that memory does not operate in the way many teachers think it does.

Chapter 4, "Knowing Versus Performing Mathematics," is based on the book, *The Grasp of Consciousness*,[2] written in 1976. It demonstrates that **knowing how to perform** mathematics may precede **knowing** or understanding the mathematical process involved by as much as six years.

Chapter 13, "Chance and Probability," based on *The Origin of the Idea of Chance in Children*,[3] is a study of just how difficult such ideas are for children. It is particularly timely because probability topics are now appearing in elementary school math materials.

The fourth new chapter, Chapter 6, "Ordering and Seriation"—developed mainly from Piaget's *The Child's Conception of Number*[4]—treats the idea of number in its ordinal sense separately from the idea of cardinal number, the subject of Chapter 7. Each involves its own distinct psychological and mathematical structures.

Activities are listed by number in each chapter as they occur and also listed in the Appendix by title and page number.

Jean Piaget

Many mathematics educators recognize the importance of Piaget but also recognize that to implement his ideas is a monumental undertaking. One basic problem has been the language barrier. *The Child's Conception of Time*, for example, was not translated into English until 1969 even though it was written in French in 1946. Even in English, Piaget's books are not easy to understand.

A few activities related to such ideas as conservation and seriation are beginning to appear in the elementary schools, but there is as yet no comprehensive program of mathematics instruction that implements the research of Piaget. This book is intended to provide at least a beginning.

R. W. C.

[1] Jean Piaget and Barbel Inhelder. *Memory and Intelligence*. New York: Basic Books, Inc., Publishers, 1973.

[2] Jean Piaget. *The Grasp of Consciousness—Action and Concept in the Young Child*. Cambridge, Mass.: Harvard University Press, 1976.

[3] Jean Piaget and Barbel Inhelder. *The Origin of the Idea of Chance in Children*. New York: W. W. Norton & Company, Inc., 1975.

[4] Jean Piaget. *The Child's Conception of Number*. New York: W. W. Norton & Company, Inc., 1965.

Contents

1 Genetic Epistemology and Mathematics

> He [Piaget] has approached heretofore exclusive philosophical questions in a resolutely empirical fashion and created epistemology as a science, separate from philosophy, but interrelated with all human sciences.[1]

Intelligence in man is of fundamental importance to those concerned with education. In order to educate, teach, or foster learning, one needs to know how intelligence functions.

Intelligence was first defined as "fixed" or static, because it is based on the inherited "genes" of the individual. The IQ (intelligence quotient) was therefore constant. With this assumption went another, that of "predetermined development" based on the growth and maturation of somatic and neural tissues. In this process, intellectual capacity unfolds naturally and automatically. One implication for education if this were told is that the child not be pushed but be allowed to unfold and develop naturally. Rousseau and Pestalozzi were influential in this line of thought and pedagogy.

[1] From citation in a Distinguished Scientific Contribution Award presented to Jean Piaget by the American Psychological Association.

Binet, whose tests have been used so widely to determine this static IQ, did not subscribe to such a view. He said

> We must protest and act against this brutal pessimism. . . . We say that the intelligence of children may be increased. One increases that which constitutes the intelligence of a school child, namely, the capacity to learn, to improve with instruction.[2]

One method of studying the effect of environment on intelligence is through the observation of identical twins raised apart. Because identical twins have the same genes, any significant difference in intelligence must be attributed to environment and not to heredity. In such a study, Newman, Freeman, and Holzinger[3] found differences in IQ as great as 24 points between identical twins.

The realization that environment or experience may play a part in IQ led to the need for new theories. One of these involved the idea of the mind as a set of mental entities or faculties that worked similar to muscles. It was thought that the mind could be expanded by exercise. This led to much rote activity in the schools in the form of mental exercises. It was said that practice in memorization would make it easier for a person to remember and that making careful detailed drawings in science would make one a better observer.

Thorndike and others disagreed with these notions and replaced them with **connectionism**—the stimulus-response sequence in which the mind becomes a network of responses to various stimuli based on the effect of the stimuli on the organism. This somewhat automatic and oversimplified theory gave way to the realization of some conceptual process between stimulus and response that must take place in the mind.

As long ago as 1912, Hunter[4] argued that the mind must be capable of some kind of symbolic process. The idea that the mind contains a central system capable of such a symbolic process was

[2] A. Binet. *Les Idées modernes sur les enfants.* Paris: Ernest Flanarion, 1909. Cited from G. D. Stoddard, "The I.Q.: Its Ups and Downs," *Educational Record,* **20** (1939), pp. 44-57.

[3] H. H. Newman, F. N. Freeman, and K. J. Holzinger. *Twins: A Study of Heredity and Environment.* Chicago: University of Chicago Press, 1937, pp. 18-20, 325-327.

[4] W. S. Hunter. "The Delayed Reaction in Animals and Children," *Behavior Monograph,* 2(1) (1912), pp. 1-85.

characterized by Newell et al. in 1958 as a system for information processing.

> We postulate an information-processing system with large storage capacity that holds among other things complex strategies (programs) that may be evoked by stimuli.[5]

Hunt reports

> They make three postulates concerning the nature of the control system: (a) It has a number of memories containing symbolized information that are inter-connected by various ordering relations; (b) it has a number of processes which operate on the information in the memories; and (c) it has a definite set of rules for combining these processes into programs of processing. . . .[6]

In short, the mind is able to operate in a fashion similar to an electronic computer. There is no shortage of brain cells to do the work because the brain contains some 120 trillion cells.

Neural Development and Experience

Patterns of responses in most animals other than man may be predominately determined by genetic factors (bees communicating directions, bird migrations, and so forth), but the brain of the newborn human is very much uncommitted. The wiring up of the complex electrical computer of the brain depends heavily on learning experiences. Because of this, there are many things that the young child is simply unable to learn.

The relationship between brain development and abilities of the individual is not one-way, but interactive; experiences facilitate neural development, and neural development facilitates higher levels of learning. There are some indications that neural development is not wholly finished before adolescence.[7]

That the brain continues to mature in its function can be assumed from the development of electroencephalogram (EEG) records. The

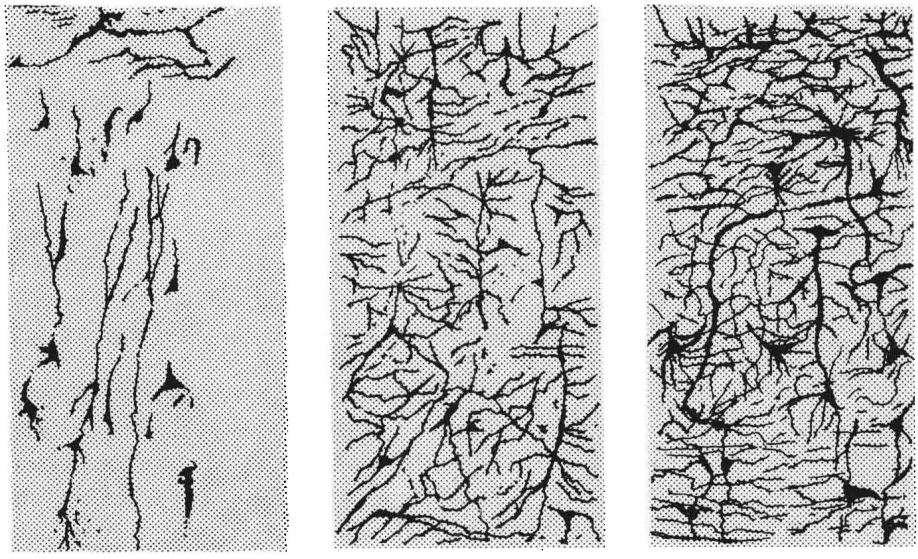
[5] Newell, Shaw, and Simon. "Elements of a Theory of Human Problem Solving," *Psychological Review* (1958), p. 163.

[6] J. M. Hunt. *Intelligence and Experience.* New York: The Ronald Press Company, 1961, pp. 75-76.

[7] John Oates and Ann Floyd. *The Course of Development,* Block 4. Walton Hall, Milton Keynes, England, MK76AA: The Open University Press, 1976, pp. 53-54.

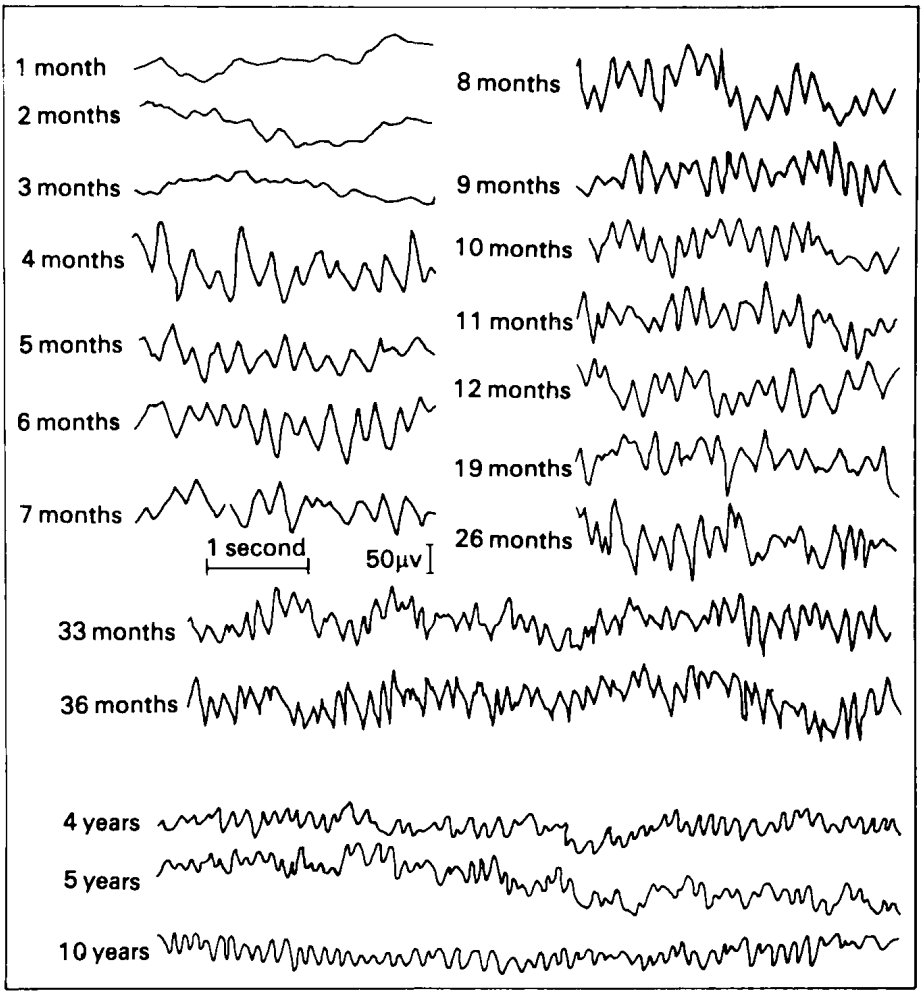
brain, like nerves in other parts of the body, produces electrical impulses, and with sensitive amplifying equipment these impulses can be picked up from electrodes attached to the scalp. The form of this electrical activity is a complex pattern of waves of different frequencies ranging from five cycles per second to bursts of much higher frequencies.[8]

The EEG *full adult pattern*, which has an average frequency of about nine-and-one-half cycles per second, is not reached until between eleven and thirteen years of age, which corresponds with Piaget's formal operational level. At birth, the EEG consists mainly of **delta** waves, i.e. irregular, large amplitude, low frequency (less than seven cycles per second) activity. By the age of five, the **alpha** rhythm predominates (about eight cycles per second), and from then on, higher frequencies gradually develop until the adult value is reached at about eleven years of age.[9]



Development of Neural Interconnections (From *The Course of Development*, Block 4, courtesy of The Open University Press, Walton Hall, Milton Keynes, England, MK76AA.)

[8] Ibid.
[9] Ibid.



Developmental Changes in EEG Rhythms (From *The Course of Development*, Block 4, courtesy of The Open University Press, Walton Hall, Milton Keynes, England, MK76AA.)

Jean Piaget

The preceding summary leads us to the work of the Swiss psychologist, biologist, philosopher, logician, and mathematician Jean Piaget.

The implications of his work for mathematics education will be studied in detail. Concerning Piaget, Hunt reports

> A conception of intelligence as problem-solving capacity based on a hierarchical organization of symbolic representations and information-processing strategies deriving to a considerable degree from past experience has been emerging from several sources. These sources include observations of human behavior in solving problems, the programming of electronic computers, and neuropsychology. It is interesting, therefore, to find such a conception coming also from Piaget's observations of the development of intelligence in children.[10]

Unfortunately, Piaget's early work generated little interest in the United States, because psychology was dominated by associationistic theories of learning and by content-oriented psychometrics. Now there is a cognitive trend in the literature as the search for a theoretical model moves away from the laboratory rat toward the electronic computer.[11]

Piaget's theory is cognitive rather than associationistic. It is more concerned with structures of the mind as related to understanding. This is in contrast to the behavior theorists' concern with the prediction and control of behavior or knowledge by external manipulation.

Genetic Epistemology

The fundamental hypothesis of genetic epistemology is that there is a parallelism between the progress made in logic (or the rational organization of knowledge) and the corresponding formative psychological processes.[12]

An example is Cantor's development of set theory, which is based on the fundamental operation of one-to-one correspondence. Where did the operation of one-to-one correspondence come from? Cantor found it in his thinking, a part of his mental equipment long before he turned to mathematics. The most elementary psychological ob-

[10] Hunt, op. cit., p. 109.

[11] John L. Philips. *The Origins of Intellect—Piaget's Theory*. San Francisco: W. H. Freeman and Company, Publishers, 1969, p. viii.

[12] Jean Piaget. *Genetic Epistemology*. New York: Columbia University Press, 1970, p. 13.

servations reveal that one-to-one correspondence is a primitive operation.[13]

Are there certain mathematical structures basic to all others? The Bourbaki group of mathematicians attempted to isolate the fundamental structures of all mathematics. They established three mother structures: an algebraic structure (the prototype of which is the notion of a group), a structure of ordering, and a topological structure. These were later modified to include the notion of categories.[14]

The question then arises: What is the relation of these fundamental mathematical or logical structures and the psychological structures in the minds of children? A group of mathematicians and psychologists met in Paris for a conference titled "Mental Structures and Mathematical Structures." Piaget reports that

> the mathematician Dieudonne, who was representing the Bourbaki mathematicians, totally mistrusted anything that had to do with psychology. Dieudonne gave a talk in which he described the three mother structures [in mathematics]. Then I gave a talk in which I described the structures that I had found in children's thinking, and to the great astonishment of us both we saw that there was a very direct relationship between these three mathematical structures and the three structures of children's operational thinking. We were, of course, impressed with each other, and Dieudonne went so far as to say to me: "This is the first time that I have taken psychology seriously. It may also be the last, but at any rate it's the first."[15]

Logic and Psychology

The same problem of cooperation was found between logicians and psychologists. In the book *Logic and Psychology* Piaget examines the mathematical and logical basis or model for human thought.

> Theoretically, it is important to ask what sort of correspondence exists between the structures described by logic and the actual thought processes studied by psychology.[16]

[13] Ibid., p. 5.
[14] Ibid., p. 3.
[15] Ibid., p. 26.
[16] Jean Piaget. *Logic and Psychology*. New York: Basic Books, Inc., Publishers, 1960, p. xvii.

> The algebra of logic can therefore help the psychologist, by giving him a precise method of specifying the structures which emerge in the analysis of the operational mechanisms of thought.[17]
>
> Boole, the inventor of the algebra that bears his name, still believed he was describing "The Laws of Thought."[18]

The logician Evert Beth reviewed a work Piaget did on the operational mechanisms of logic and criticized it "very severely." The result ten years later was a book co-authored by Beth and Piaget titled *Mathematical and Epistemology Psychology*.[19]

Basic Mathematical and Psychological Structures

In children's thinking, basic algebraic structures are most readily found in the logic of classes or logical classification (Chapter 5). The psychological structure necessary to properly classify objects in relation to each other is the mathematical or logical relation of **class inclusion.** A six- or seven-year-old may agree that all ducks are birds and that not all birds are ducks, but asked if there are more birds or ducks, he does not know. He does not yet have the necessary algebraic structure.

$$\text{ducks} + \text{other birds} = \text{all birds}$$

$$\text{birds} + \text{other animals} = \text{all animals}$$

Thus a hierarchy of classification is structured, based on the logic of the inclusion relation.

Young children do not have reversibility by negation, such as: If all the birds died, would there be any ducks left; or if all the ducks died, would there be any birds left? (See Chapter 5.)

Second, the psychological structure, **seriation,** necessary for the mathematical structure of **ordering** is not present until approximately seven years of age. For example, before this age, children when asked to arrange a set of 10 sticks in order of length or height will group them in pairs, such as putting two short ones together and two long ones together, or will arrange the sticks in sets of three. This is followed by a trial-and-error approach—trying first one and then another to see which stick is right. There is no over-all coordinated approach.

To be successful in ordering, the child must have the psychological structure necessary to understand the mathematical or logical relation of **transitivity.** If B is longer than A and C is longer than B,

[17] Ibid., p. xviii.

[18] Ibid., p. 2.

[19] Evert Beth and Jean Piaget. *Mathematical Epistemology and Psychology.* Dordrecht, Holland: D. Reidel Publishing Company, 1966, p. xii.

then logically *C* is longer than *A*. Younger children if asked how *A* and *C* compare will say that they have to "look" again, and they put stick *C* by stick *A* to "see" that *C* is longer. (See Chapter 6.)

The third type of basic mathematical structure according to the Bourbaki group is **topological,** even though historically (and in school) the first type of geometry studied was Euclidean. Piaget also finds children able to solve topological problems before Euclidean ones. For example, children can make inside and outside distinctions and differentiate open from closed figures before they can differentiate Euclidean shapes such as circles and squares. (For further discussion see Chapter 16.)

Biology and Intelligence

Piaget, in studying intelligence, adopted the biological model of **adaptation,** a model more popular fifty years ago than now. Intelligence is studied through the general process of adaptation, and adaptation is studied as **behavior.**

Piaget and the rigid behaviorist school of psychology differ sharply in their ideas about behavior. For Piaget, studying behavior involves studying the total organization of the person or other organism. The most relevant aspect of intelligence is not what the person *does* (as an external action) but the rules or organization within the individual that control or govern the action. This is a much broader view than the stimulus-response (S-R) approach, for example, in which the action itself as a response is studied. The more limited mechanistic S-R approach was more conducive to carefully controlled laboratory-type experimentation, but it does not give us the important answers.

Piaget was able, like Darwin or Freud, to systematize his empirical observations of children by discovering structures where others saw nothing but inconsequential childish activities.[20] Einstein summarizes Piaget's developmental approach as "the idea of a genius, such simplicity."[21] It was these very procedures or experiments that were branded "unscientific" and not "controlled" and not large enough "samples" that led to a slow acceptance of Piaget's theories.

Intelligence, biologically speaking, is not a special biological organ but a way of behaving, being determined by the general laws of evolution and adaptation. The intelligence of man, as different from animals and children, is seen in the simple act of driving a nail into a piece of wood. Both the child and the ape may be able to hold and swing the hammer, but they do not have the internal coordination of spacial coordinates (vertical and horizontal) that will allow them to correct their actions as necessary. Later, the child can.

[20] Hans G. Furth. *Piaget and Knowledge.* Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1969, p. 172.

[21] Ibid., p. 6.

2 Jean Piaget and Education

> [Of great concern] is the ignorance in which we still remain with regard to the results achieved by our educational techniques.[1]

Jean Piaget was born in 1896 in Switzerland. He first became a biologist and later a psychologist. At the age of ten he published his first article, concerning a rare albino sparrow, in the *Journal of Natural History of Neuchatel.*

Later, as a university student majoring in biology, he became interested in philosophy and psychology. Because of his mother's poor mental health, he was interested in psychoanalysis and pathological psychology, but he much preferred the study of normalcy and the workings of the intellect to that of the tricks of the unconscious. He studied in the psychological laboratories at the University of Zurich in 1918 and in the experimental laboratory of Alfred Binet in Paris from 1919 to 1921. During this time he attended lectures of Pfister, Jung, and Freud.

[1] Jean Piaget. *Science of Education and the Psychology of the Child.* New York: Orion Press, 1970, p. 5.

From 1929 to 1939 he formulated the psychological concept of **groupings,** which was to tie together his theory of cognitive development. During and following the World War II years Piaget continued his research—lecturing and writing in three capacities as the professor of the History of Scientific Thought at the University of Geneva, as Assistant Director of the Institute of J. J. Rousseau, and as Director of the Bureau of the International Office of Education. Jean Piaget is now Professor of Experimental Psychology and Genetic Epistemology at the University of Geneva. He retired from active teaching in 1972 to devote full time to writing.

Publications

Piaget has probably written more books and had more books written about him than any man living today. Since his primary concern has been the development of logico-mathematical intelligence, many of his books are of interest to the mathematics educator.

For an overview of Piaget's ideas, the books *The Psychology of the Child and Genetic Epistemology* are recent and also less difficult to understand than some of his other works. His views on education are described in *Science of Education and the Psychology of the Child.*

The books of most value in preparing this manuscript were as follows:

	Published in English	*Recent Publisher*
Piaget, *The Child's Conception of Number*	1952	W. W. Norton & Company, Inc., New York
Piaget and Inhelder, *The Child's Conception of Space*	1967	W. W. Norton & Company, Inc., New York
Piaget, Inhelder, and Szeminska, *The Child's Conception of Geometry*	1960	Basic Books, Inc., Publishers, New York
Piaget and Inhelder, *The Early Growth of Logic in the Child*	1964	W. W. Norton & Company, Inc., New York
Beth and Piaget, *Mathematical Epistemology and Psychology*	1966	D. Reidel Publishing Company, Dordrecht, Holland
Piaget and Inhelder, *The Psychology of the Child*	1969	Basic Books, Inc., Publishers, New York