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ABSTRACT

We evaluate the Selberg trace formula for all discrete, irreducible, cofinite subgroups of $PSL_2(\mathbb{R})^n$. In particular, this involves studying the spectral theory of the fundamental domain, and the analysis of the appropriate Eisenstein series. A special role is played by the Hilbert modular groups, both because of their relation to the general case, stemming from a rigidity theorem, and their inherent algebraic number theoretic interest.

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TABLE OF CONTENTS

INTRODUCTION

The Selberg trace formula describes a relation between the geometry of a symmetric Riemannian manifold, obtained as a quotient space, and the spectrum of its invariant differential operators. It has been applied to the study of closed geodesics on the one hand, and to counting eigenvalues on the other hand. For such an application one needs an explicit evaluation of the formula. This, however, has proved to be quite involved, and until recently has only been obtained for spaces of rank one, such as the hyperbolic spaces.

In this paper we develope this theory for the rank n spaces

$$H^n = H \times \cdots \times H$$

where H is the upper half plane with the hyperbolic metric. We consider discrete subgroups Γ of $PSL_2(\mathbb{R})^n$, the connected component of the identity of the group of isometries of H^n, which are irreducible and such that

$$F = H^n/\Gamma$$

is of finite volume, but not compact. Our goal is to derive the Selberg trace formula for all these spaces.

We begin Chapter I by describing the known properties of these groups, and introduce a large family of such in the following way. For a

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totally real algebraic number field K of degree n, let

$$\Gamma_K = \left\{ \left(\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix} \right) \;\middle|\; \begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix} \in PSL_2(\mathcal{O}_K) \right\}$$

be the Hilbert modular group associated to K. A key theorem to our study is Selberg's regidity theorem, which guarantees that for $n \geq 2$ every Γ as above is commensurable with one of the Γ_K's.

We then proceed to give a classification of the elements of Γ. Typically such an element is of a mixed type, some of its components being hyperbolic, others elliptic. The other possible types are totally parabolic, and what we call hyperbolic-parabolic, meaning totally hyperbolic elements some of whose fixed points are also fixed by parabolic elements. The existence of these last two types is due to the noncompactness of F.

In section 2 we introduce the algebra of invariant differential operators of H^n, which is generated by the n Laplacians

$$\Delta_i = y_i^2 \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right) \qquad i = 1,\ldots,n .$$

By an eigenfunction we mean a function u(z) on F such that

$$\Delta_i u(z) + \lambda_i u(z) = 0 \qquad i = 1,\ldots,n .$$

As is usual in this theory, we define a kernel K in terms of Γ, such that the eigenfunctions of the integral operator it defines are the eigenfunctions of the differential operators. To compute its trace, we rewrite K as a sum over the different types of conjugacy classes of Γ, and treat each type separately.

In sections 3 and 4 we do this for the identity and totally elliptic elements. This is a rather straightforward generalization of the $n = 1$ case.

In section 5 we consider a general mixed element γ with m components hyperbolic, $n-m$ elliptic, and look at its centralizer Γ_γ. The problem of determining the structure of Γ_γ is first reduced, using rigidity, to the one for a Hilbert modular group. For $\Gamma = \Gamma_K$, we then recognize Γ_γ as the group of automorphs of a binary quadratic form over $\mathcal{O}_K$

$$ax^2 + bxy + cy^2, \qquad a,b,c \in \mathcal{O}_K,$$

which we identify in terms of units in quadratic extension of K. This in particular implies, that Γ_γ is a free abelian group of rank m.

Now that we know Γ_γ, we can compute the trace of all mixed γ's in section 6. In section 7 we go back to the case of Γ_K and exploit the identification of Γ_γ mentioned above. We show a correspondence between equivalence classes of binary quadratic forms and conjugacy classes of centralizers of mixed elements. This enables us to rewrite the trace entirely in terms of the arithmetic of K. It is the sum over discriminants of a function of the fundamental units, weighted by their class numbers and regulators.

Since F is not compact its spectrum has a continuous part, and Chapter II is devoted to its analysis. The continuous spectrum can be described explicitly, and is furnished by a family of Eisenstein series

$$E(z,s,m) = \sum_{\gamma \in \Gamma/\Gamma_\infty} y^s(\gamma z)\lambda_m(y(\gamma s))$$

where $s \in \mathbb{C}$, $m \in \mathbb{Z}^{n-1}$ and λ_m is an exponential sum similar to a Grössencharakter. One important thing to notice here is that in spite of the arbitrary rank, an Eisenstein series still depends on only one complex parameter.

Since $E(z,s,m)$ is invariant under translations of x, it has a Fourier expansion of the form

$$E(z,s,m) = \sum_{\ell} a_{\ell}(y,s,m)e^{2\pi i\langle x,\ell\rangle}$$

and

$$a_0(y,s,m) = (y_1 \dots y_n)^s \lambda_m(y) + \phi(s,m)(y_1 \dots y_n)^{1-s}\lambda_{-m}(y) .$$

The functions $\phi(s,m)$ play an important role in what follows.

Concentrating again on the Γ_K's, we give in section 2 the explicit formulae for a_{ℓ}. They involve Hecke zeta functions with Grössencharakters and known special functions.

In section 3 we introduce new coordinates, to replace $y_1, \dots, y_n$, which are necessary for the rather delicate analysis that follows. These coordinates bring into the picture the fact that there is a unique geodesic ray connecting a point of F to the cusp. We also compute some of the Riemannian invariants in these coordinates.

In sections 4 and 5 we give a generalization of Selberg's proof of the meromorphic continuation of the E's, which employs the Fredholm theory. We show that $E(z,s,m)$ and $\phi(s,m)$ admit a meromorphic continuation to all of $\mathbb{C}$, and that they satisfy the functional equations:

$$E(z,1-s,-m) = \phi(1-s,-m)E(z,s,m)$$

$$\phi(s,m)\ \phi(1-s,-m) = 1 .$$

This in particular gives the continuation and functional equations for all Hecke zeta functions.

$E(z,s,0)$ always has a pole at $s = 1$, and in section 6 this is used to prove a theorem of Siegel that says that for Γ_K

$$\mathrm{vol}(F) = (2/\pi^n)D^{3/2}\zeta_K(2) .$$

The main relation between Eisenstein series is the Maass-Selberg relation, which is proved in section 7. This is used in section 8 to obtain a decomposition of $L^2(F)$

$$L^2(F) = C \oplus R \oplus E$$

where E is generated in the L^2 sense by the $E(z,\frac{1}{2}+it,m)$, R is generated by the residues of the finitely many poles of $E(z,s,0)$ in $(\frac{1}{2},1]$, and C is the space of cuspidal functions, i.e., those L^2 functions whose zero Fourier coefficient is zero.

Using the Eisenstein series, we construct in section 9 a new kernel H, and prove that $K - H$ is a kernel of an integral operator of trace class. This implies that $C \oplus R$ has a basis of eigenfunctions. The ones in C are called <u>cusp forms</u>.

In Chapter III we go back to the computation of the trace. We begin with the contribution that comes from the kernel H, for which we use the Maass-Selberg relations. We can then proceed to evaluate the contribution of the parabolic and hyperbolic-parabolic elements, which we do in sections 2 and 3. The latter is particularly involved, since we need to cut the domain of integration at the two ends that correspond to the two parabolic fixed points of such an element.

Up to this point we assume for the sake of simplicity that F has only one cusp, and that the cusp is at ∞. In section 4 we describe the changes one needs to make in order to treat the general case of an arbitrary number of cusps.

Finally, in section 5 we collect our results, to give the complete trace formula for Γ.

The possibility of deriving these trace formulae was indicated by Selberg in his fundamental paper [16]. See also P. Zograf [23].

Our point of view in this paper is directed toward applications. Thus, in the second part of this work ([6]) we use the trace formula to establish Weyl's law for all groups Γ as above, with $n \geq 2$:

Theorem: Let $N_\Gamma(T)$ be the number of eigenvalues $(\lambda_1,\ldots,\lambda_n)$ for Γ that lie in the ball of radius T . Then, for $n \geq 2$,

$$N_\Gamma(T) = c_\Gamma \cdot T^n + O\left(\frac{T^{n-\frac{1}{2}}}{\log T}\right)$$

(with an explicit constant c_Γ) .

This constitutes a solution to the higher rank analog of the Roelcke-Selberg conjecture.

In a different direction, W. Müller [13] has recently used a more preliminary version of these trace formulae to settle Hirzebruch's conjecture on signature defects.

Finally a word about style. Some of the arguments needed here are similar to the analogous ones in the special case of $n = 1$. This has received a number of treatments in the literature, notably Hejhal's comprehensive work [10]. We have therefore chosen not to repeat these arguments here, as well as to exclude some of the longer computations. Otherwise the paper is self contained.

CHAPTER I

THE COMPACT CONTRIBUTION TO THE TRACE

1. Discrete subgroups acting on H^n

Let H be the upper half plane $\{z = (x,y) \in \mathbb{C} \mid y > 0\}$ with the hypotherbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. Set $H^n = H \times \cdots \times H$ (n times). Then H^n, with the metric induced from H, is a globally symmetric Riemannian manifold, whose line and volume elements are

$$ds^2 = \sum_{i=1}^{n} \frac{dx_i^2 + dy_i^2}{y_i^2} \qquad d\omega = \prod_{i=1}^{n} \frac{dx_i dy_i}{y_i^2}$$

The connected component of the identity of the group of isometries of H^n is $G = PSL_2(\mathbb{R})^n$, which acts componentwise by linear fractional transformations, i.e., if $z = (z_1,\ldots,z_n) \in H^n$, $\sigma = (\sigma^{(1)},\ldots,\sigma^{(n)}) \in G$, with $\sigma^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$, $i = 1,\ldots,n$, then

$$\sigma z = (\sigma^{(1)} z_1,\ldots,\sigma^{(n)} z_n), \qquad \sigma^{(i)} z_i = \frac{a^{(i)} z_i + b^{(i)}}{c^{(i)} z_i + d^{(i)}} \qquad i = 1,\ldots,n .$$

We consider subgroups $\Gamma \subset G$ that satisfy the following three conditions:

i) Γ is a discrete subgroup of G. Thus we can form the fundamental domain for its action on H^n, $F = H^n/\Gamma$, and induce the metric from H^n to F.

ii) F is not compact, but is of finite volume. Thus F contains parts that stretch out to the boundary of H^n. We shall refer to these non-compact parts as the "cusps" of F.

iii) Since H^n is reducible, we wish to guarantee that our study does not reduce to lower dimensions. To this end, we require that Γ is irreducible, in the sense defined below.

Definition 1.1: Γ and Γ' are said to be strictly commensurable if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. They are said to be commensurable if Γ is strictly commensurable with a conjugate of Γ'.

Definition 1.2: Γ is said to be irreducible if Γ is not commensurable with a direct product $\Gamma_1 \times \Gamma_2$, where $\Gamma_1 \subset G_1$, $\Gamma_2 \subset G_2$ are discrete, G_1 and G_2 are not trivial, and $G = G_1 \times G_2$.

The basic reference on groups that act on H^n is Shimizu [18], where in particular, the next theorem is proved:

Theorem 1.3: The following conditions are equivalent:

i) Γ is irreducible

ii) Γ contains no element $\gamma = (\gamma^{(1)},\ldots,\gamma^{(n)})$ such that $\gamma^{(i)} = 1$ for some i and $\gamma^{(j)} \neq 1$ for some j.

iii) There exists no partial product G' of G such that the projection of Γ to G' is discrete.

iv) For every $\gamma \neq 1$ in Γ, G_γ, the centralizer of γ in G, is an abelian group.

Next we introduce a family of such groups, which plays a key role in everything that follows. Let K be a totally real algebraic number field of degree n, and let $K^{(1)} = K, K^{(2)}, \ldots, K^{(n)}$ be a fixed ordering of the n imbeddings of K in $\mathbb{R}$. Let $\mathcal{O}_K$ be the ring of integers of K.

<u>Definition 1.4</u>: Let $PSL_2(\mathcal{O}_K) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \middle| \, a,b,c,d \in \mathcal{O}_K,\ ad - bc = 1\right\}/\{\pm 1\}$. Then the <u>Hilbert modular group</u> associated to K is

$$\Gamma_K = \left\{\left(\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)}\end{pmatrix}, \ldots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)}\end{pmatrix}\right) \middle| \begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)}\end{pmatrix} \in PSL_2(\mathcal{O}_K)\right\} .$$

It is well known that Γ_K is a discrete subgroup of G, and that its number of cusps equals the class number of K (see Siegel [20]). By 1.3, Γ_K is an irreducible subgroup. In addition to the interest in the Hilbert modular groups for their own right, their importance to this theory stems from Selberg's rigidity theorem:

<u>Theorem 1.5</u> (Selberg [17]): Let $n \geq 2$ and let Γ satisfy i), ii) and iii) above. Then Γ is commensurable with a Hilbert modular group of some field of degree n.

We now give a classification of the elements of Γ. Recall that $\gamma \in PSL_2(\mathbb{R})$ is called elliptic, parabolic or hyperbolic, if $|tr(\gamma)| < 2$, $|tr(\gamma)| = 2$ or $|tr(\gamma)| > 2$, respectively. Correspondingly, we call a $\gamma \in \Gamma$ <u>totally elliptic</u>, <u>totally parabolic</u> or <u>totally hyperbolic</u>, if all its components are elliptic, parabolic or hyperbolic, respectively.