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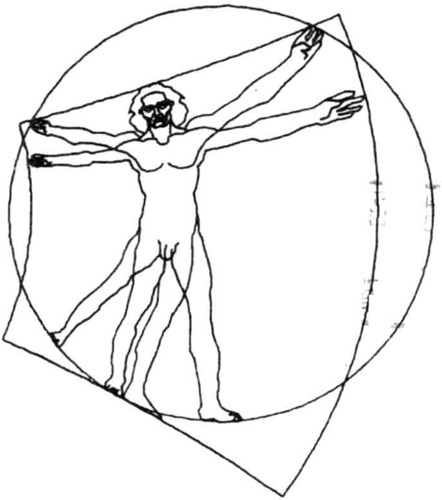

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To

Stefan and Edy Bergman

In Friendship

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The motif on the title page is the da Vinci rendition of the human figure according to the proportions of Vitruvius. It has been subjected to a Möbius transformation leaving the bounding circle invariant. This was kindly provided to me by Professors R. Vitale of Brown and K. Long of the Rhode Island School of Design. The end piece was created by Jonathan Sachs.

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CHAPTER 1

PROLOGUE

In the year 1968–1969, Professor Mary Cartwright was a visiting member of the Division of Applied Mathematics at Brown University. This was a year of turmoil at Brown—particularly curricular turmoil—and in the course of one of our division meetings Miss Cartwright remarked that when she was a student all mathematics majors were required to know a proof of the nine-point circle theorem. Since the nine-point circle now has a distinct flavor of beautiful irrelevance, Miss Cartwright seemed to be telling us that we should not be too dogmatic as to what constitutes a proper mathematics curriculum. Fashion is spinach even in mathematics, and time often works to "nine-point circle-ize" many of our most relevant and sophisticated topics that are now insisted upon.

At the time, I was giving a course in Numerical Analysis entitled "Iteration Theory in Banach Spaces", using notes of L. B. Rall, and I saw that it would be possible—and not too far-fetched—to present several lectures which would trace an unlikely path from the nine-point circle to iteration.

This essay presents such a path. The connecting link is the use of conjugate coordinates and the Schwarz reflection function. The path has been faired, as draftsmen say, to pass in a wide arc near a number of allied topics in complex variable theory that have interested me.

In Chapter Two conjugate coordinates are introduced. In Chapter Three elementary notions of plane analytic geometry are expressed in terms of conjugate coordinates. In Chapter Four this mechanism is used to establish the nine-point circle theorem and Feuerbach's theorem.

In Chapter Five, the Schwarz Function of an analytic arc is defined and numerous examples are given. The remainder of the book is devoted to expounding applications to and connections with a variety of topics, principally in the theory of analytic functions of a complex variable.

Chapter Six develops the relationship of the Schwarz Function to Schwarzian reflection and an important functional identity is adduced. Chapter Seven turns to elementary plane differential geometry and finds relationships between the differential invariants (curvature, etc.), the Schwarz Function and the Schwarzian derivative. Chapter Eight talks about conformal mapping, reflections and symmetry in analytic arcs and invariant curves. A convenient algebraic formalism involving the Schwarz Function is introduced to handle some of the problems and this leads to a number of interesting functional equations.

Chapter Nine relates the Schwarz Function to 2×2 systems of autonomous differential equations. In Chapter Seven, the Schwarz Function is discussed in the small; Chapter Ten takes up the properties in the large, specifically the circumstances when the Schwarz Function is rational.

Chapter Eleven begins by expressing the operators of partial differential equations in terms of conjugate coordinates. Applications are then made to problems of analytic continuation of harmonic functions satisfying nonlinear boundary data as well as to the Cauchy problem for elliptic equations. In the second part of the chapter, passing from derivatives to integrals, Green's theorem, in its complex analytic form, is used to derive conveniently a number of integral identities, some of which are of interest in the theory of approximate quadratures.

Chapter Twelve shows how a number of ideas and principles of two-dimensional flow theory are easily expressed in terms of Schwarz Functions, while Chapter Thirteen relates the solution of the Dirichlet Problem to the Schwarz Function of the boundary of the region in question.

Chapter Fourteen takes up once again properties in the large, dealing specifically with closed curves whose Schwarz Function is meromorphic inside the curve. Further interesting identities and functional equations are derived, some of which connect up with the Bergman Kernel Function.

In the final chapter, analytic functions of one and two real variables are expressed in conjugate coordinates and certain functional iterations suggested by the Schwarz Function are taken up. The Schroeder function is found to fit into this theory nicely and a number of classic problems, including the "function-theoretic center problem" and the "bisection problem for curvilinear angles", are discussed from this point of view.

I have resisted the temptation to expound such things as the integral operator method for the solution of partial differential equations. Conjugate coordinates are crucial in the operator method, but the Schwarz Function plays a minor role. I have also avoided an extensive reworking of inversive geometry and the Poincaré model of non-

Euclidean geometry in the present terms, feeling that these topics are adequately covered from numerous points of view in the text-book literature.

H. A. Schwarz showed us how to extend the notion of reflection in straight lines and circles to reflection in an arbitrary analytic arc. Notable applications were made to the symmetry principle and to problems of analytic continuation. Reflection, in the hands of Schwarz, is an anti-analytic mapping. By taking its complex conjugate, we arrive at an analytic function that we have called here the Schwarz Function of the analytic arc. This function is worthy of study in its own right and this essay presents such a study. In dealing with certain familiar topics, the use of the Schwarz Function lends a point of view, a clarity and elegance, and a degree of generality which might otherwise be missing. I have also found that it opens up a line of inquiry which has yielded numerous interesting things in complex variables; it illuminates some functional equations and a variety of iterations which interest the numerical analyst. The perceptive reader will certainly find here some old wine in relabelled bottles. But one of the principles of mathematical growth is that the relabelling process often suggests a new generation of problems. Means become ends; the medium rapidly becomes the message.

This book is not wholly self-contained. The reader will find that he should be familiar with the elementary portions of linear algebra and of the theory of functions of a complex variable.

CONJUGATE COORDINATES IN THE PLANE

Plane Euclidean geometry and, in particular, many of the topics which normally appear in advanced synthetic or inversive geometry may be expeditiously carried out by working with complex coordinates. The real plane is converted into the complex plane by assigning the complex number $z = x + iy$, $i = \sqrt{-1}$, to the real point (x, y). If we wish to recover x and y from z, it is convenient to introduce the conjugate quantity $\bar{z}$ by means of the equations

$$(2.1) \qquad z = x + iy, \qquad \bar{z} = x - iy,$$

and hence the inverse transformation is given by

$$(2.2) \qquad x = \tfrac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z}).$$

The non-independent quantities z, $\bar{z}$ are called the *conjugate coordinates* of the point (x, y). (If x and y are themselves complex, then z and $\bar{z}$ are independent but are not necessarily conjugate.) In somewhat different contexts involving the geometry of the "complex domain" in which x and y may take complex values, the terms *isotropic* or *minimal* coordinates are used for conjugate coordinates.

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In this book, *we restrict ourself almost entirely to real values of x and y*, i.e., to the usual Gauss plane.

The matrix of the transformation (2.1) is

$$M = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{2.3}$$

We have

$$MM^* = 2I, \tag{2.4}$$

where * designates the conjugate transposed matrix, so that $1/\sqrt{2}M$ is unitary. For some purposes, the corresponding transformation may be somewhat more convenient; indeed, this transformation is widely used in the theory of complex manifolds and of integral operators for the solution of partial differential equations.

CHAPTER **3**

ELEMENTARY GEOMETRIC FACTS

Given two distinct points z_1 and z_2 in the complex plane, the equation

$$z = tz_1 + (1 - t)z_2, \qquad t \text{ real}, \tag{3.1}$$

describes all the points on the straight line joining z_1 and z_2. The point z divides the line segment from z_1 to z_2 in the ratio

$$r = (1 - t)/t. \tag{3.2}$$

The distance from z_1 to z_2 is $\rho = | z_1 - z_2 |$ or

$$\rho = \sqrt{(z_1 - z_2)(\overline{z_1} - \bar{z}_2)}. \tag{3.3}$$

The determinant

$$D = \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix} \tag{3.4}$$

vanishes for $z = z_1$ and for $z = z_2$ and since D is linear in x and y, the equation

$$D = 0 \tag{3.5}$$

must be the equation in conjugate coordinates of the straight line through the points z_1 and z_2. Upon expansion, (3.5) can be written in the following forms:

$$z(\overline{z_1} - \overline{z_2}) - \bar{z}(z_1 - z_2) + z_1\overline{z_2} - z_2\overline{z_1} = 0 \tag{3.6}$$

or

$$\bar{z} = \left(\frac{\overline{z_1} - \overline{z_2}}{z_1 - z_2}\right)(z - z_2) + \overline{z_2}; \qquad \frac{\bar{z} - \overline{z_2}}{z - z_2} = \frac{\overline{z_1} - \overline{z_2}}{z_1 - z_2}; \tag{3.7}$$

$$\bar{z} = \left(\frac{\overline{z_1} - \overline{z_2}}{z_1 - z_2}\right)z + \left(\frac{z_1\overline{z_2} - z_2\overline{z_1}}{z_1 - z_2}\right). \tag{3.8}$$

We can therefore write

$$\bar{z} = Az + B, \tag{3.9}$$

where

$$A = \frac{\overline{z_1} - \overline{z_2}}{z_1 - z_2}, \qquad B = \frac{z_1\overline{z_2} - z_2\overline{z_1}}{z_1 - z_2}, \tag{3.10}$$

as an equation for the line through z_1 and z_2. If we use polar coordinates and write $z_2 = z_1 + \rho e^{i\theta}$ (see Fig. 3.1), then $\overline{z_2} - \overline{z_1} = \rho e^{-i\theta}$ so that

$$A = e^{-2i\theta}, \qquad |A| = 1. \tag{3.11}$$

If we consider the line segment from z_2 to z_1, and if we write

$$\begin{aligned} z_1 &= z_2 + \rho e^{i(\theta+\pi)} \\ &= z_2 - \rho e^{i\theta}, \end{aligned}$$

then we arrive at (3.11) once again. Accordingly, the quantity A gives the "orientation" of the *undirected* line joining z_1 and z_2 and can serve as the complex analogue of

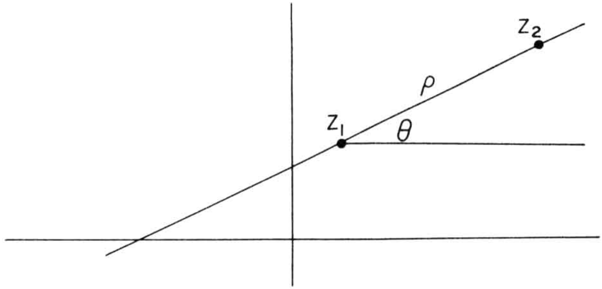

FIG. 3.1

the slope. The quantity A will be called the *clinant* of the line. Note that (3.9) yields

(3.12) $$\frac{d\bar{z}}{dz} = A,$$

which is a direct analog of the real situation

(3.13) $dy/dx = \lambda =$ slope of the line $y = \lambda x + b.$

From (3.7), the equation of the straight line which passes through z_0 and has clinant A is given by

(3.14) $$\bar{z} = A(z - z_0) + \bar{z}_0.$$

The relationship between the clinant A and the slope λ of a straight line is as follows:

(3.15) $$A = \frac{\bar{z}_2 - \bar{z}_1}{z_2 - z_1} = \frac{(x_2 - x_1) - i(y_2 - y_1)}{(x_2 - x_1) + i(y_2 - y_1)}$$

$$= \frac{1 - i\lambda}{1 + i\lambda} = \frac{1 - i\tan\theta}{1 + i\tan\theta} = e^{-2i\theta}.$$

Inversely,

$$(3.16) \qquad \lambda = -i\,\frac{1 - A}{1 + A}.$$

This is a Möbius transformation, mapping the lower half of the complex λ-plane onto the unit disc of the A-plane. We note the particular values:

(3.17)

Slope λ	0	1	∞	-1
Clinant A	1	$-i$	-1	i

The angle ψ from a line with slope λ_1 and clinant A_1 to a line with slope λ_2 and clinant A_2 is given by

$$(3.18) \qquad \tan\psi = \frac{\lambda_2 - \lambda_1}{1 + \lambda_1\lambda_2} = i\,\frac{A_1 - A_2}{A_1 + A_2}.$$

It follows from this that two lines are parallel if and only if

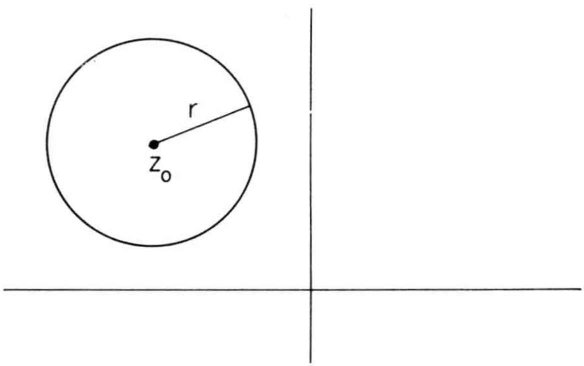

FIG. 3.2

their clinants are equal:

$$A_1 = A_2, \tag{3.19}$$

and two lines are perpendicular if and only if their clinants are negatives:

$$A_1 = -A_2. \tag{3.20}$$

Since for any three numbers z_1, z_2, z_3,

$$\Delta = \begin{pmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.21}$$

by taking determinants we have

$$|\Delta| = -4i \text{ Area } (z_1, z_2, z_3), \tag{3.22}$$

where Area (z_1, z_2, z_3) is the *signed area* of the triangle whose vertices are at z_1, z_2, z_3 in that order.

The circle. The equation of the circle with center at z_0 (see Fig. 3.2) and radius r can be written as $|z - z_0|^2 = r^2$ or $(z - z_0)(\bar{z} - \bar{z}_0) = r^2$. Hence

$$\bar{z} = \bar{z}_0 + \frac{r^2}{z - z_0} \tag{3.23}$$

is its equation in conjugate coordinates.

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