Variational Convergence for Functions and Operators

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H Attouch

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Preface

During the past twenty years (1964-84) new concepts of convergence for sequences of functions and operators have been appearing in mathematical analysis. These concepts are specially designed to approach the limit of sequences of variational problems and are called "variational convergences". With each type of variational problem (minimization, maximization, min-max, the saddle-value problem,...) is associated to a particular concept of convergence.

In this book we focus our attention on *minimization problems* and develop a convergence theory for sequences of functions, called *epi-convergence*, which may be regarded as the "weakest" notion which allows to approach the limit in the corresponding minimization problems. This concept of convergence thus has natural applications in all branches of optimization theory - from stochastic optimization, optimal control, numerical analysis and approximation to calculus of variations and perturbation problems in physics.

The book is divided into three parts. In Chapter 1, epi-convergence is introduced in a general topological setting as the natural concept of convergence that allows to approach the limit of sequences of minimization problems. Relying only on its definition, we show how the technique of epi-convergence may be used to solve various limit problems in analysis. Examples have been chosen which, because of their physical interest and the difficulties they present (in these examples, intuition by itself is not of much help in identifying the right limit problem), have contributed to the development of well-adapted mathematical tools, to which we refer when we speak of variational convergences.

Composite materials (fibred, stratified, porous,...) play an important role in many branches of physical engineering. A good approximation of the macroscopic behaviour of such materials may be obtained by letting the parameter ε, which describes the fineness of structure, approach zero in the equations describing phenomena such as heat conduction and elasticity. This limit analysis process is called *homogenization*. Many examples considered here come from the works of Bensoussan, Lions and Papanicolaou [1], Sanchez-Palencia [9], Cioranescu and Murat [1], Tartar [2], and Marchenko and Hruslov [1]. Epi-convergence provides a precise and flexible tool for such problems. It enhances their topological aspects and can easily be combined with other tools such as convex analysis and measure theory. For some nonlinear problems, such as homogenization of elastoplastic torsion (cf. Attouch [6], Carbone and Salerno [1]) and homogenization of fissured elastic materials (cf. Attouch and Murat [2]) it is the only proof we have, at the present time, of this limit analysis process.

A second type of example comes from perturbation theory: *"singular" perturbation* problems arise naturally when a physical parameter (such as conduction, viscosity, mean free path of a particle, etc), or an economic one (such as a cost) becomes very small or very large with respect to the others. Typical applications are reinforcement problems and shells in mechanical engineering.

Our purpose in Chapter 2 is to give a complete exposition of the topological properties of epi-convergence. A large part of this chapter owes much to De Giorgi and Wets. We pay particular attention to the Moreau-Yosida approximation by inf-convolution which is developed for general real-valued functions defined on a metric space:

$$\forall \lambda > 0 \quad \forall u \in X \quad F_\lambda(u) = \inf_{v \in X} \{F(v) + \frac{1}{2\lambda} d^2(u,v)\}.$$

Epi-limits of sequences of functions can be re-expressed in terms of pointwise limit of their Moreau-Yosida approximates. A topology is naturally attached to the pointwise convergence of these approximates. In the locally compact case (or in an equivalent way when considering uniformly "inf-compact" functions), one can prove that this topology induces epi-convergence. It should be noted that, in general, epi-convergence is not attached to a topology. It is only in certain particular cases (like those described above which, indeed, cover a large number of applications) that there exists a topology $\mathcal{T}$ for which

$$F = \tau\text{-}\lim_e F^n \Longleftrightarrow F^n \overset{\mathcal{T}}{\to} F.$$

In a number of applications such as stochastic optimization a precise approach is to consider functionals as elements of such a topological (compact metric) space (cf. Salinetti and Wets [4], Dal Maso and Modica [3]).

The third and final chapter is devoted to the study of epi-convergence of sequences of convex functions. For simplicity, we restrict our attention to the case of X, a reflexive Banach space. In this case, two topologies play an important role: the strong and the weak topologies.

In this infinite-dimensional framework the continuity property of the Young-Fenchel transformation can be formulated as follows (refer to Theorems 3.7 and 3.9 for precise assumptions):

$$F = w\text{-}\lim_e F^n \iff F^* = s\text{-}\lim_e F^{n*}.$$

Therefore weak and strong topologies are exchanged when considering epi-convergence of convex functions and of their conjugates. These considerations lead to the introduction of the so-called Mosco-convergence, which is epi-convergence for both strong and weak topologies and which, from the above considerations, has the following basic property:

$$F^n \to F \text{ in Mosco sense} \iff F^{n*} \to F^* \text{ in Mosco sense.}$$

This property explains the importance of this concept in the study of stability properties, approximation, etc, in convex optimization. Historically, it appeared (when considering infinite-dimensional spaces) earlier than the more general concept of epi-convergence with respect to a given topology. Let us give its formulation: a sequence $F^n : X \to]-\infty, +\infty]$ Mosco-converges to F if

$$\begin{cases} \text{for every } x \in X, \text{ there exists } (x_n)_{n\in\mathbb{N}} \text{ strongly converging to } x \text{ in } X \\ \text{such that } F^n(x_n) \to F(x); \\ \\ \text{for every weakly converging sequence } x_n \to x,\ F(x) \leq \liminf F^n(x_n). \end{cases}$$

Mosco-convergence is indeed equivalent to the pointwise convergence of the Moreau-Yosida approximates:

$$F^n \to F \text{ in Mosco sense} \iff \text{for every } \lambda > 0, \text{ for every } x \in X\ F^n_\lambda(x) \to F_\lambda(x)$$

where

$$F^n_\lambda(x) = \min_{u\in X} \{F^n(u) + \frac{1}{2\lambda} \|x-u\|^2_X\}.$$

Therefore there exists a topology on the class of closed convex functions called the topology of Mosco-convergence, inducing this convergence. When X is separable, this topological space is a Polish space that is metrizable, separable and complete for a metric inducing the topology.

In convex analysis, in addition to the conjugation operation, another concept plays a fundamental role: this is the subdifferential operation

$$F \to \partial F$$

where $\partial F : X \to X^*$, the subdifferential of F, is given by

$$\partial F = \{(u,f) \in X \times X^* / F(v) \geqslant F(u) + \langle f, v-u \rangle \text{ for every } v \in X\}.$$

A natural question is: given a sequence of closed convex functions $F^n : X \to]-\infty, +\infty]$ which is epi-convergent, what is the corresponding notion of convergence for the sequences of operators $\{\partial F^n : X \to X^*;\ n = 1,2,\ldots\}$? Indeed, historically it was the converse of this question that was asked: subdifferentials of convex functions form an important subclass (nonlinear version of self-adjoint operators) of maximal monotone operators. For such operators a good convergence concept is *graph-convergence* which is equivalent to the pointwise convergence of the *resolvents*. This concept, introduced by Kato [1] for linear monotone operators, has been extended by Browder [2], Brezis [1], [2] to nonlinear maximal monotone operators and by Benilan [1] to accretive operators. It makes it possible to attack convergence of semigroups, approximation and perturbation of evolution equations governed by such operators.

The equivalence between graph-convergence of subdifferential operators $\{\partial F^n;\ n = 1,2,3\ldots\}$ and Mosco-convergence of functions $\{F^n;\ n = 1,2,\ldots\}$ was proved by the author around 1976 (Attouch [2], [4], cf. also Matzeu [1], Zolezzi [4], Sonntag [1]). This links the two theories, convergence of functions and convergence of operators, which in the convex case turn out to be equivalent. Moreover one obtains convergence of elements attached to such operators, such as spectrum (in the linear case), semigroups.

In the last few years many extensions and promising new fields of application of variational convergence have appeared in the literature:

- Convergence of saddle-value and min-sup problems: Attouch and Wets [3], [4], Cavazzuti [1], with a view to applications to critical point problems in economics and mechanics.

- The study of limit analysis problems for systems and higher-order problems in mechanics: Brillard [2] for Stokes equations and Darcy's law in porous media, Aze [1] for elasticity and the dual formulation of the homogenization formula expressed in terms of constraint tensor (cf. also Suquet [2]), Picard [1] for the biLaplacian, Attouch and Murat [2] for homogenization of fissured elastic materials, etc.
- The study of variational convergence in non-reflexive Banach spaces: Picard [1] for minimal surface problems with varying unilateral or bilateral constraints, extensions to capillarity, plasticity etc.
- The study of stochastic optimization problems, for example in statistical decision theory (Salinetti and Wets [4]), in stochastic homogenization (Dal Maso and Modica [3]) etc.
- The study of convergence problems for evolution equations, control problems, rate of convergence (Attouch and Wets [5]) etc.

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H. A.

Introduction

The concept of variational convergence seems to appear for the first time in the work of Wijsman [2] (1964-66) in statistical decision theory, where it arose in the study of the continuity properties of the application

$$C \mapsto s(C)$$

which associates with a closed convex subset of $\mathbb{R}^m$ its support function

$$s(C,x^*) = \sup_{x \in C} \langle x^*,x \rangle.$$

When considering convex subsets C contained in a fixed ball of $\mathbb{R}^m$, Hausdorff metric provides a quite natural measure for perturbations of sets, and

$$C^n \underset{n \to +\infty}{\longrightarrow} C \text{ for Hausdorff metric} \Leftrightarrow \forall x^* \in \mathbb{R}^m \; s(C^n,x^*) \to s(C^n,x^*).$$

However, when considering possible unbounded subsets of $\mathbb{R}^m$, Hausdorff metric is no longer an adequate concept and one has to introduce the more general notion of set convergence, also called Kuratowski convergence (equality between lim inf and lim sup). The question that naturally arose was that of finding the corresponding concept of convergence for the associated support functions. Here it is not the pointwise convergence of the support functions (as in the case of the Hausdorff metric) that gives the correct answer, but rather the epi-convergence (called by Wijsman "infimal convergence").

Epi-convergence thus stems naturally from set convergence theory and has been introduced to study the continuity properties of duality operations. Indeed, epi-convergence of a sequence of functions $F = \lim_e F^n$ is equivalent to the set convergence of their epigraphs, epi $F^n \to$ epi F, where epi F = $\{(x,\lambda) \in X \times \mathbb{R} / \lambda \geq F(x)\}$. This justifies the terminology!

Moreover, the continuous dependence of the support function $s(C,\cdot)$ on C turns out to be a particular case of the following fundamental result: the Young-Fenchel transformation is continuous with respect to epi-convergence

$$F = \lim_e F^* \Leftrightarrow F^* = \lim_e F^{n*}$$

where F^n, F are closed convex functions and $F^*(x^*) = \sup_{x \in \mathbb{R}^m} \{\langle x^*, x\rangle - F(x)\}$.

At this stage the theory has been developed in a satisfactory way but only for the finite-dimensional case. This restriction blurs some important topological features of the theory that cannot be ignored in infinite-dimensional spaces. Active mathematical orientation research, mainly arising from applications to optimization and decision theory (Wets [1], Salinetti and Wets [1]..., Back [1], Vervaat [1], MacLinden [1]) bears a natural relation to the work of Wijsman.

The next step in the development of the theory came from quite a different direction and can be traced to the work of researchers such as Stampacchia and Lions on variational inequalities. In order to study the convergence of solutions of approximations of variational inequalities (such as Galerkin approximation), Mosco [2] (1967-73) and Joly [1] (1970-76) extended the earlier results to infinite-dimensional spaces. The theory was still limited to the case of convex functions and to topologies such as weak and strong topologies on (reflexive) Banach spaces.

The concept in a general topological setting has finally been delineated by De Giorgi [1] (1973-83) and the Italian mathematical school (Spagnolo [1], Carbone and Sbordone [2], Buttazzo [1], Dal Maso [1], Modica [1], Boccardo and Marcellini [1], etc). They were mostly concerned with the study of lower semicontinuity and perturbation problems in calculus of variations. To that end, a convergence theory for functions was developed, called Γ-convergence, (and a corresponding theory for operators, called G-convergence). This contains as a particular case epi-convergence, which can be regarded as Γ-convergence specially adapted to minimization problems. The corresponding concept for maximization problems is hypo-convergence, which can easily be derived from epi-convergence by changing functions F into their opposites in the definitions and statements. Recently the Γ-convergence theory for saddle-value problems, called epi-hypo-convergence (which includes the two above concepts), has been developed by Attouch and Wets [3] and Cavazzuti [1].

As with all Γ-convergence concepts, the definition of epi-convergence only requires a topological structure. Given (X, τ), a topological space (which for simplicity we assume here to be metrizable), and F^n, $F: X \to \bar{\mathbb{R}}$, a sequence of real (extended) valued functions, the sequence $\{F^n; n \to +\infty\}$ is said to be τ-epi-convergent to F at $x \in X$ if the two following conditions hold:

(i) there exists a convergent sequence $x_n \xrightarrow[(n\to+\infty)]{} x$ in (X,τ) such that

$F(x) \geq \limsup_{n\to+\infty} F^n(x_n)$;

(ii) for every convergent sequence $x_n \xrightarrow[(n\to+\infty)]{} x$ in (X,τ), $F(x) \leq \liminf_{n\to+\infty} F^n(x_n)$.

We then write $F(x) = (\tau\text{-}\lim_e F^n)(x)$. When this property holds for every $x \in X$, the sequence $\{F^n; n = 1,2,\ldots\}$ is said to be τ-epi-convergent to F and $F = \tau - \lim_e F^n$.

Let us first notice that when such convergence holds, the limit function F is given by the formula

$$F(x) = \min \{\lim_{n\to+\infty} F^n(x_n);\ x_n \xrightarrow[n\to+\infty]{\tau} x\}.$$

When taking $F^n \equiv F^o$, a stationary sequence, F is equal to the τ-lower semi-continuous regularization of F^o. Thus, epi-convergence includes as a particular case the Γ-closure operation. This is the origin of the above terminology.

The fundamental variational property of epi-convergence can now be formulated: let us take $\{F^n, F:X \to \bar{\mathbb{R}}; n = 1,2,\ldots\}$, a sequence of real (extended) functions which satisfies the condition that there exists a topology τ on X and a τ-relatively compact subset K of X such that, for every $n = 1,2,\ldots,$

$$\inf_{x\in X} F^n(x) = \inf_{x\in K} F^n(x).$$

Then, $F = \tau\text{-}\lim_e F^n$ implies the convergence of the corresponding minimization problems (as $n \to +\infty$):

$$\inf_{x\in X} F^n(x) \xrightarrow[(n\to+\infty)]{} \inf_{x\in X} F(x)$$

and every τ-cluster point x of a minimizing sequence ($x_n \in \text{Argmin } F^n; n=1,2,\ldots$) minimizes F. (When there is uniqueness, there is convergence of the whole sequence.) In general, epi-convergence is not implied by and does not imply pointwise convergence. They are two separate concepts. There is, in fact, one important case in which the two concepts coincide: this is when the sequences of functions are monotonically increasing (or decreasing). This explains the success of all monotone schemes in approximation theory.

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