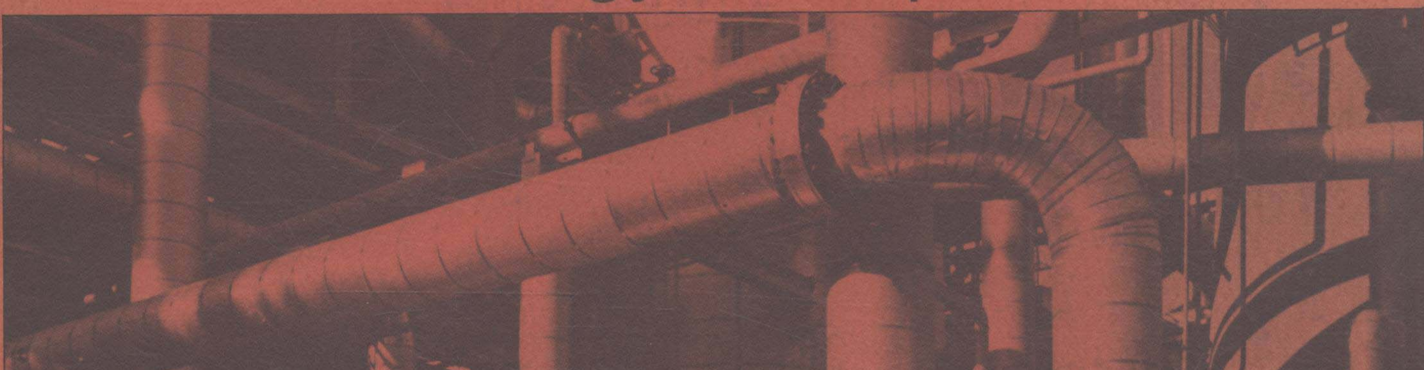
AIChEMI

MODULAR INSTRUCTION

Series C: TRANSPORT

Volume 4: Mathematical Techniques and Energy Transport



AMERICAN INSTITUTE OF CHEMICAL ENGINEERS

AIChEMI MODULAR INSTRUCTION

Series C:

TRANSPORT

Volume 4:

Mathematical Techniques and Energy Transport

R. J. Gordon, Series Editor

AMERICAN INSTITUTE OF CHEMICAL ENGINEERS

American Institute of Chemical Engineers
345 East 47th Street, New York, N.Y. 10017

Library of Congress Cataloging in Publication Data

American Institute of Chemical Engineers.
AIChEMI modular instruction: series C, transport.

Includes bibliographical references.
Contents: v. 1. Momentum Transport and Fluid Flow, v. 2. Momentum Transport, Viscoelasticity and Turbulence, v. 3. Equation of Motion, Boundary Layer Theory, and Measurement Techniques, v. 4. Mathematical Techniques and Energy Transport.
1. Transport Theory. I. Title II. Title: Transport.
TP156.T7A43 1980 660.2′842 80-25573
ISBN 0-8169-0172-4 (v. 1)
ISBN 0-8169-0178-3 (v. 2)
ISBN 0-8169-0210-0 (v. 3)
ISBN 0-8169-0237-2 (v. 4)
ISSN 0270-7632

ISSN 0270-7632/83/$5.00

INTRODUCTION

In 1975 a new venture in education by and for the chemical engineering community was initiated. Prepared by the CACHE Corporation (Computer Aids for Chemical Engineering Education) and under the sponsorship of the National Science Foundation (Grant HES 75-03911), a series of small self-study fundamental concept modules for various areas of chemical engineering were commissioned, Chemical Engineering Modular Instruction, CHEMI.

It has been found in recent studies that modular study is more effective than traditional instruction in both university and continuing education settings. This is due in large measure to the discrete focus of each module, which allows the student to tailor the speed and order of his or her study. In addition, since the modules have different authors, each writing in his or her area of special expertise, they can be produced more quickly, and students may be asured of timely information. Finally, these modules have been tested in the classroom prior to their publication.

The educational effect of modular study is to reduce, in general, the number of hours required to teach a given subject; it is expected that the decreased time and expense involved in engineering education, when aided by modular instruction, will attract a larger number of students to engineering, including those who have not traditionally chosen engineering. For the practicing engineer, the modules are intended to enhance or broaden the skills he or she has already acquired, and to make available new fields of expertise.

The modules were designed with a variety of applications in mind. They may be pursued in a number of contexts: as outside study, special projects, entire university courses (credit or non-credit), review courses, or correspondence courses; and they may be studied in a variety of modes: as supplements to course work, as independent study, in continuing education programs, and in the traditional student/teacher mode.

A module was defined as a self-contained set of learning materials tat covers one or more topics. It should be sufficiently detailed that an outside evaluation could identify its educational objectives and determine a student's achievement of these objectives. A module should have the educational equivalent of a one to three hour lecture.

The CHEMI Project Staff included:

E. J. Henley, University of Houston, Director
W. Heenan, Texas A & I University, Assistant Director
Steering Committee:
L. B. Evans, Massachusetts Institute of Technology
G. J. Powers, Carnegie-Mellon University
E. J. Henley, University of Houston
D. M. Himmelblau, University of Texas at Austin
D. A. Mellichamp, University of California at Santa Barbara
R. E. C. Weaver, Tulane University
Editors:
Process Control: T. F. Edgar, University of Texas at Austin
Stagewise and Mass Transfer Operations: E. J. Henley, University of Houston, J. M. Calo, Brown University
Transport: R. J. Gordon, University of Flordia
Thermodynamics: B. M. Goodwin, Northeastern University
Kinetics: B. L. Crynes, Oklahoma State University
H. S. Fogler, University of Michigan
Material and Energy Balances: D. M. Himmelblau, University of Texas at Austin

Curriculum Analysis: E. J. Henley, University of Houston

The second phase of the project, designed to fill in gaps as well as develop new modules, is under the direction of D. M. Himmelblau, University of Texas at Austin.

Steering Committee:

B. Carnahan, University of Michigan
D. E. Griffith, Oklahoma State University
L. Harrisberger, University of Alabama
D. M. Himmelblau, University of Texas at Austin
V. Slamecka, Georgia Institute of Technology
R. Tinker, Technology Education Research Center

Editors (indicates a new task force head)*:

Process Control: T. F. Edgar, University of Texas at Austin
Stagewise and Mass Transfer Operations: J. M. Calo, Brown University
E. J. Henley, University of Houston
Transport: R. J. Gordon, University of Florida
Thermodynamics: G. A. Mansoori*, University of Ilinois at Chicago Circle
Kinetics: B. L. Crynes, Oklahoma State University
H. S. Fogler, University of Michigan
Material and Energy Balances: E. H. Snider*, University of Tulsa
Design of Equipment: J. C. Beckman, Arizona State University

Volume 1 of each series will appear in 1980; Volume 2 in 1981; and so forth. A tentative outline of all volumes to be produced in this series follows:

SERIES C: TRANSPORT

Volume 1. Momentum Transport and Fluid Flow

C1.1	Simplified One-Dimensional Momentum Transport Problems	G. K. Patterson
C1.2	Friction Factor	R. J. Gordon and N. H. Chen
C1.3	Applications of the Steady-State Mechanical Energy Balance	D. W. Hubbard
C1.4	Flow Meters	W. F. Beckwith
C1.5	Packed Beds and Fluidization	W. J. Hatcher, Jr.
C1.6	Multi-Phase Flow	R. A. Greenkorn and D. P. Kessler

Volume 2. Momentum Transport, Viscoelasticity and Turbulence

C2.1	Non-Newtonian Flow I—Characterization of Fluid Behavior	D. V. Boger and A. L. Halmos
C2.2	Non-Newtonian Flow II—Fully Developed Tube Flow	D. V. Boger and A. L. Halmos
C2.3	Viscoelastic Fluid Flow Phenomena	D. V. Boger and R. I. Tanner
C2.4	Turbulence: General Aspects Illustrated by Channel or Pipe Flow	N. S. Berman
C2.5	Turbulent Drag Reduction	G. K. Patterson

Volume 3. Equation of Motion, Boundary Layer Theory and Measurement Techniques

C3.1	Measurements of Local Fluid Velocities	N. S. Berman and H. Usui
C3.2	Equation of Motion	G. K. Patterson
C3.3	Navier Stokes Equation for Steady One-Directional Flow	G. C. April
C3.4	Boundary Layer Theory	R. J. Gordon
C3.5	Boundary Layer Theory: Approximate Solution Techniques	R. L. Cerro
C3.6	Diffusivity Measurement Techniques in Liquids	V. L. Vilker

Volume 4. Mathematical Techniques and Energy Transport

C4.1	Mathematical Techniques I—Separation of Variables	R. S. Subramanian
C4.2	Mathematical Techniques II—Combination of Variables	R. S. Subramanian
C4.3	Elementary Steady-State Heat Conduction	W. J. Hatcher

C4.4	Natural Convection	R. D. Noble
C4.5	Unsteady-State Heat Conduction	K. I. Hayakawa
C4.6	Differential Energy Balance	R. D. Noble

Volume 5. Mass Transport

C5.1	Unsteady-State Diffusion	S. Uchida
C5.2	Mass Transfer in Laminar Flow	S. H. Ibrahim
C5.3	Turbulent Mass Transfer	S. H. Ibrahim

Volume 6. Transport Phenomena—Special Topics

C6.1	Bubble Dynamics: An Illustration of Dynamically Coupled Rate Processes	T. G. Theofanous
C6.2	Miscible Dispersion	R. S. Subramanian
C6.3	Biomedical Examples of Transport Phenomena I—Coupled Diffusion Effects	R. H. Notter
C6.4	Biomedical Examples of Transport Phenomena II—Facilitated Diffusion	R. H. Notter
C6.5	Mass Transfer in Heterogeneous Media	P. Stroeve
C6.6	Advancing Front Theory	R. Srinivasan and P. Stroeve

Publication and dissemination of these modules is under the direction of Harold I. Abramson, Staff Director, Educational Activities, AIChE. Technical Editor is Lori S. Roth. Chemical engineers in industry or academia who are interested in submitting modules for publication should direct them to H. I. Abramson, Staff Director, Educational Activities, American Institute of Chemical Engineers, 345 East 47th Street, New York, N.Y. 10017.

CONTENTS

Solutions to the Homework Problems are available as a separate reprint from the AIChE Educational Services Dept., 345 East 46th St., New York, NY 10017. The cost is $5.00.

Mathematical Techniques I—Separation of Variables

R. Shankar Subramanian

Department of Chemical Engineering
Clarkson College of Technology
Potsdam, New York 13676

OBJECTIVES

After completion of this module, the student should be able to solve partial differential equations using the method of separation of variables.

PREREQUISITE MATHEMATICAL SKILLS

1. Elementary calculus and differential equations.
2. Previous exposure to Fourier Series would be preferable but not necessary.

PREREQUISITE ENGINEERING AND SCIENCE SKILLS

1. Ability to construct elementary models of transport problems is preferable but not necessary.

Several problems in the area of transport phenomena involve the solution of partial differential equations. Some typical examples are unsteady couette flow, steady heat transfer to a fluid flowing through a pipe, and mass transfer to a falling liquid film. In each case, one can write a partial differential equation describing the process. Such equations are solved by first reducing them to ordinary differential equations using one of several techniques. The ordinary differential equations, then, are solved by well-known methods. This module covers a powerful technique known as the *Separation of Variables.* It is used widely for the solution of linear partial differential equations. Through example, you will learn how and when to use the technique and what its limitations are. The example chosen is a simplified version of the popular *Graetz* problem discussed in many transport phenomena texts.

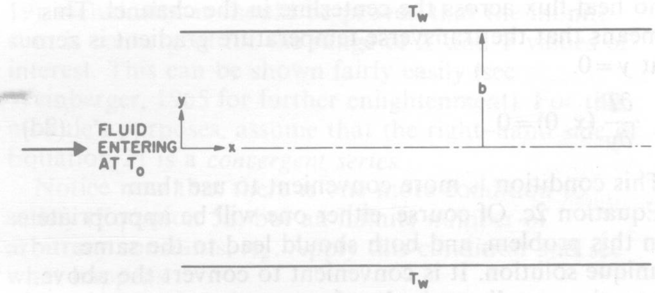


Figure 1. Sketch of the system.

THE MODELING OF THE PROBLEM

Consider a fluid of thermal diffusivity α flowing in a channel formed by the space between two wide parallel plates, a distance 2b apart, as shown in Figure 1.

The fluid enters the channel at a uniform temperature T_o and is heated by the plates which are maintained at some other constant temperature T_w. In the usual version, as posed by Graetz in 1885 (*2*), in a circular tube, the fluid is assumed to be in laminar flow. However, it is easier, especially for the purposes of illustration, to choose the case of steady uniform plug flow in the channel. This assumption allows us to develop solutions in terms of simple well-known trigonometric functions; however, the same procedure can be used with more realistic flows.

If some more simplifying assumptions are made [see any standard textbook on transport phenomena such as Bird et al. 1960 (*1*) or Knudsen and Katz, 1958 (*5*)], the following partial differential equation can be written for the steady temperature field in the fluid.

$$u_o \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (1)$$

Here, u_o is the plug flow velocity, and x and y are the axial and transverse coordinates in the flow channel; these coordinates also are identified on Figure 1.

In order to determine the solution of Equation 1 uniquely, boundary conditions are necessary. From the physical problem, one knows that the temperature everywhere at the inlet is T_o (note that this is strictly not true if one accounts for axial conduction), so that

$$T(0, y)=T_o \tag{2a}$$

Further, the fluid at either wall attains the temperature of the wall at all axial locations. So,

$$T(x, b)=T_w \tag{2b}$$

and

$$T(x, -b)=T_w \tag{2c}$$

Actually, it is quite easy to see from the physics of the problem that the temperature distribution in the bottom half of the channel will be a mirror image of that in the top half. Hence, by symmetry, there can be no heat flux across the centerline in the channel. This means that the transverse temperature gradient is zero at $y=0$.

$$\frac{\partial T}{\partial y}(x, 0)=0 \tag{2d}$$

This condition is more convenient to use than Equation 2c. Of course, either one will be appropriate in this problem, and both should lead to the same unique solution. It is convenient to convert the above equations to dimensionless form so that the number of parameters is minimized.

Defining dimensionless axial and transverse coordinates,

$$X=\frac{\alpha x}{b^2 u_o} \tag{3a}$$

$$Y=\frac{y}{b} \tag{3b}$$

and a dimensionless temperature,

$$\theta(X, Y)=\frac{T(x, y)-T_w}{T_o-T_w} \tag{3c}$$

one may transform Equations 1, 2a, 2b, and 2d to the following form:

$$\frac{\partial\theta}{\partial X}=\frac{\partial^2\theta}{\partial Y^2} \tag{4}$$

$$\theta(0, Y)=1 \tag{5a}$$

$$\theta(X, 1)=0 \tag{5b}$$

$$\frac{\partial\theta}{\partial Y}(X, 0)=0 \tag{5c}$$

Note that Equations 5b and 5c are homogeneous boundary conditions; that is, any constant times θ also will satisfy the same conditions.

It is interesting to note that Equations 4 and 5 also result from the mathematical modeling of several other situations such as unsteady diffusion or heat conduction in a slab.

SEPARATION OF VARIABLES

To solve Equations 4 and 5 using the "separation of variables" technique, one must look for a solution to Equation 4 (ignoring Equations 5 temporarily) in a very specific form—a product of a function of X and a function of Y. That is,

$$\theta(X, Y)=G(X)\phi(Y) \tag{6}$$

If this solution is introduced in Equation 4, and then the resulting equation divided throughout by $G\phi$, then

$$\frac{G'}{G}=\frac{\phi''}{\phi} \tag{7}$$

The primes, of course, refer to differentiation with respect to the argument.

Equation 7 implies that a function of only X, on the left–hand side, is equal to a function of only Y on the right–hand side. The only way for this to be true is for each side to be equal to a constant C. Equating each side to C gives two ordinary differential equations:

$$G'-CG=0 \tag{8a}$$

$$\phi''-C\phi=0 \tag{8b}$$

Equation 8a can be integrated to give

$$G(X)=\beta e^{CX} \tag{9}$$

where β is a constant of integration. Note immediately that C has to be nonpositive, since a positive value of C will result in G becoming very large as X gets larger and this makes no physical sense*. So, if one lets $C=-\lambda^2$, where λ is real, one is always assured that C will be nonpositive. This gives us the following equation for $\phi(Y)$.

$$\phi''+\lambda^2\phi=0 \tag{10}$$

This is a simple second-order ordinary differential equation. Its solution is well-known, and can be obtained by using e^{mY} as the trial solution. It is,

$$\phi(Y)=a_1 \cos \lambda Y+a_2 \sin \lambda Y \tag{11}$$

Quite a few arbitrary constants (β, λ, a_1, a_2) have been introduced in the solution procedure. Now, one must try to determine them.

Let us first write our solution for θ.

$$\theta(X, Y)=e^{-\lambda^2 X}[b_1 \cos \lambda Y+b_2 \sin \lambda Y] \tag{12}$$

Note that β has been eliminated as an arbitrary constant by simply absorbing it into b_1 and b_2; that is, $b_1=a_1\beta$, $b_2=a_2\beta$. Observe that there are three boundary conditions on $\theta(X, Y)$ and three arbitrary constants in the solution, Equation 12. One way to determine these constants is to apply the boundary conditions directly in Equation 12. However, since the

*More precise arguments can be given, but are beyond the scope of this module.

solution is a product of a function of X and a function of Y, one may logically expect the Y-dependent part to satisfy the homogeneous boundary conditions on Y. Applying Equation 5c at $Y=0$ shows immediately that

$$b_2\lambda=0 \tag{13}$$

This can be satisfied by either $\lambda=0$ or $b_2=0$. If $\lambda=0$ in Equation 12, then $\theta(X, Y)\equiv b_1=1$ in order to satisfy Equation 5a. However, this solution will certainly not satisfy the boundary condition at the wall, $Y=1$! So, one discards this possibility and chooses, instead,

$$b_2=0 \tag{14}$$

This makes the solution simpler.

$$\theta(X, Y)=b_1 e^{-\lambda^2 X}\cos\lambda Y \tag{15}$$

Let us try the inlet condition at $X=0$ now.

$$\theta(0, Y)=1=b_1\cos\lambda Y \tag{16}$$

This creates a problem. The only way Equation 16 can be satisfied is if $b_1=1$ and $\lambda=0$. However, this leads again to $\theta(X, Y)\equiv 1$ everywhere, which as we know, is not useful. Before throwing our hands up in despair, let us see if anything can be salvaged using the boundary condition at the wall, $Y=1$. It may seem odd that the actual order of application of the boundary conditions should make any difference, but let us try it anyway. Equation 5b, applied to Equation 15, gives

$$b_1\cos\lambda=0 \tag{17}$$

If $b_1=0$, $\theta(X, Y)\equiv 0$ which is a useless solution since it satisfies the boundary conditions on Y, but not the inlet condition on X. So,

$$\cos\lambda=0 \tag{18}$$

One obvious solution of Equation 18 is $\lambda=\pm\pi/2$. However, it is also clear that $\lambda=\pm 3\pi/2$, $\pm 5\pi/2$ etc. are all solutions or roots of Equation 18. One can verify that $\lambda=+\pi/2$ and $\lambda=-\pi/2$ give exactly the same result in Equation 15 and, therefore, only one of the two need be retained. This is true for all the roots. For convenience, let us choose the positive roots here and designate them with an appropriate subscript. That is,

$$\lambda_n=\frac{2n-1}{2}\pi \tag{19}$$

is a root of Equation 18 for $n=1, 2, 3\ldots$. If one substitutes any one of the values of λ given by Equation 18 into Equation 15, a different solution will result for each value! Since the constant b_1 is arbitrary anyway, and each solution can have its own arbitrary constant, these solutions can be written as

$$\theta_n(X, Y)=A_n e^{-\lambda_n^2 X}\cos\lambda_n Y \tag{20}$$

where A_1, A_2, A_3, etc. are arbitrary constants. Let us see what has been done so far. For any positive integer n, the right–hand side in Equation 20 is *a solution* of the partial differential equation for $\theta(X, Y)$ and the boundary conditions at $Y=0$ and $Y=1$. Let us try and pick one of these solutions as the solution to our problem; that is, one which also will satisfy the inlet condition at $X=0$. Immediately, there is a problem, since for any n, at $X=0$, the solution $\theta_n=A_n\cos\lambda_n Y$ cannot satisfy our inlet condition. However, all is not lost! Since the partial differential equation for $\theta(X, Y)$ (with the homogeneous boundary conditions on Y) is *linear*, the solutions may be added up, and the result will still satisfy the partial differential equation and the same homogeneous boundary conditions. This is a very powerful principle usually referred to as *superposition of solutions.* Let us try writing

$$\theta(X, Y)=\sum_{n=1}^{\infty}\theta_n(X, Y) = \sum_{n=1}^{\infty}A_n e^{-\lambda_n^2 X}\cos\lambda_n Y \tag{21}$$

Adding up a finite number of solutions is *OK*, but when one adds an infinite set of them (since there are an infinite number of roots represented in Equation 19), technically, it should be proven that the infinite sum is convergent for the range of X and Y values of interest. This can be shown fairly easily (see Weinberger, 1965 for further enlightenment). For this module's purposes, assume that the right–hand side of Equation 21 is a *convergent series.*

Notice now that there is one more condition to satisfy (Equation 5a) but an infinite number of arbitrary constants, A_n. Apply this condition and see what happens.

$$\theta(0, Y)=1=\sum_{n=1}^{\infty}A_n\cos\lambda_n Y \tag{22}$$

How does one choose the A_n's to satisfy this equation? Would our choice be unique? The answers can be had with some exposure to the Sturm-Liouville theory or the Fourier series [for details, consult Hildebrand (*3*)]. Sturm-Liouville theory, for instance, provides information on the properties of the solutions of linear second–order homogeneous ordinary differential equations which contain an arbitrary parameter. Usually, an equation which falls into this category would be written in the following standard form [see Hildebrand (*3*) for details on how to do this with any second–order differential equation].

$$\frac{d}{dY}\left[p(Y)\frac{d\phi}{dY}\right]+[q(Y)+\Lambda r(Y)]\phi=0 \tag{23}$$

Equation 10 can be seen to be a special case of Equation 23 when $p(Y)\equiv 1$, $q(Y)\equiv 0$, $r(Y)\equiv 1$ and $\Lambda=\lambda^2$.

When Equation 23 satisfies homogeneous boundary conditions at the two ends of an interval, the differential equation and the boundary conditions are referred to as a Sturm-Liouville system. It can be shown that nontrivial solutions of this system exist, to within an arbitrary multiplicative constant, only when the parameter Λ takes on a certain set of values $\Lambda_n (n=1, 2, 3\ldots)$ called *eigenvalues.* Corresponding to each eigenvalue, the solution ϕ_n is referred to as an *eigenfunction.* In our case, the homogeneous boundary conditions on $\phi(Y)$ are seen to be $\phi'(0)=0$ and $\phi(1)=0$. The eigenvalues are given by Equation 19 as

$$\lambda_n^2=\frac{(2n-1)^2}{4}\pi^2 \quad (24)$$

and the eigenfunctions are

$$\phi_n(Y)=\cos\lambda_n Y \quad (25)$$

It turns out that the eigenfunctions of a Sturm-Liouville system satisfy a very important and useful property known as *orthogonality*. When the product of any two of them is multiplied by $r(Y)$ and integrated over the interval, the result is zero! That is, in the case at hand,

$$\int_0^1 r(Y)\phi_m(Y)\phi_n(Y)dY=\int_0^1 \cos\lambda_m Y\cos\lambda_n Y dY = 0 \quad m\neq n \quad (26)$$

Furthermore, it can be shown that any well-behaved arbitrary function in the interval of interest can be expanded into an infinite series of eigenfunctions of a "proper" Sturm-Liouville system (the definition of "proper" in this context can be found in Hildebrand, 1976) in the form

$$f(Y)=\sum_n A_n\phi_n(Y) \quad (27)$$

and the coefficients A_n may be determined as follows. Multiply both sides of Equation 27 by $r(Y)\phi_m(Y)$ and integrate over the interval (taken to be 0 to 1 in this case).

$$\int_0^1 r(Y)f(Y)\phi_m(Y)dY=\sum_n A_n\int_0^1 r(Y)\phi_m(Y)\phi_n(Y)dY = A_m\int_0^1 r(Y)\phi_m^2(Y)dY \quad (28)$$

All the integrals except one in the infinite series are zero by the orthogonality property from Equation 26. Therefore,

$$A_m=\frac{\int_0^1 r(Y)f(Y)\phi_m(Y)dY}{\int_0^1 r(Y)\phi_m^2(Y)dY} \quad (29)$$

How is all this going to help? Go back and review Equation 22. See that it, indeed, searches for and reveals an expansion of an arbitrary function, in this case, 1, in terms of eigenfunctions of a Sturm-Liouville system. So, Equation 29 can be used to calculate all the unknown constants A_n (usually referred to as "expansion coefficients"). Substituting $r(Y)\equiv 1$, $f(Y)\equiv 1$, and $\phi_m(Y)=\cos\lambda_m Y$, gives

$$A_m=\frac{\int_0^1 \cos\lambda_m Y dY}{\int_0^1 \cos^2\lambda_m Y dY}=2\frac{\sin\lambda_m}{\lambda_m} = \frac{2}{\lambda_m}(-1)^{m-1} \quad (30)$$

or

$$A_n=\frac{2}{\lambda_n}(-1)^{n-1} \quad (31)$$

Therefore, the solution for the temperature field in the channel is

$$\theta(X,Y)=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\lambda_n}e^{-\lambda_n^2 X}\cos\lambda_n Y \quad (32)$$

where λ_n are given by Equation 19.

The Average Temperature

The bulk or cup-mixing average temperature in our system is defined by

$$T_b(x)=\frac{\int_{-b}^{b} Tu_o dy}{\int_{-b}^{b} u_o dy} \quad (33)$$

If one uses the definition of the dimensionless temperature, Equation 3c, in Equation 33, and it is recognized that u_o is a constant, the dimensionless bulk average temperature will be

$$\theta_b(X)=\frac{1}{2}\int_{-1}^{1}\theta(X,Y)dY \quad (34)$$

Note that Equation 32 predicts a temperature distribution which is an even function of Y; that is, the temperature field is symmetric with respect to $Y=0$. The result is that one can rewrite Equation 34 also as

$$\theta_b(X)=\int_0^1\theta(X,Y)dY \quad (35)$$

Substituting for $\theta(X,Y)$ from Equation 32 and performing the necessary calculations finally leads to

$$\theta_b(X)=2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 X}}{\lambda_n^2} \quad (36)$$

Note that as X becomes very large, both $\theta(X,Y)$ and $\theta_b(X)$ approach zero. This is physically correct since the temperature of the fluid everywhere in the channel will approach the wall temperature for large X.

The Nusselt Number

The rate of heat transfer between the walls of the channel and the fluid is of interest in several applications. At any axial location, the heat flux, q, from the walls to the fluid is given by

$$q=-k\left.\frac{\partial T}{\partial y}\right|_{y=-b}=k\left.\frac{\partial T}{\partial y}\right|_{y=+b} \quad (37)$$

where k is the thermal conductivity of the fluid. It is customary to define a heat transfer coefficient h for the system. This is defined here by

$$q=h(T_w-T_b) \quad (38)$$

The Nusselt number is the dimensionless form of the heat transfer coefficient. In this problem it is defined by

$$N_{Nu}=\frac{4hb}{k} \tag{39}$$

By using earlier definitions and the temperature distributions, the following expression may be obtained for the Nusselt number.

$$N_{Nu}(X)=4\frac{\sum_{n=1}^{\infty} e^{-\lambda_n^2 X}}{\sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 X}}{\lambda_n^2}} \tag{40}$$

CONCLUDING REMARKS

This module has covered an extremely useful technique for the solution of partial differential equations often encountered in transport problems. From a computational point of view, the infinite series in Equations 32, 36, or 40 are useful only if they converge fairly rapidly. It can be seen that this will happen if X is sufficiently large; that is, if one is interested in axial stations away from the inlet. Module C4.2 shall cover another technique of solution which is most useful in the inlet region. Aside from computational limitations, the "separation of variables" procedure does not work in problems such as the present one if the partial differential equation or the boundary conditions on Y are nonhomogeneous. It, of course, cannot be used with nonlinear partial differential equations. For complete details on the conditions under which the method can be used, see Weinberger (6).

The example solved in this module treated plug flow. The more realistic case of heat transfer with Poiseuille flow has been analyzed by several workers. You may refer to the book of Kays and Crawford (4) for an exhaustive treatment of this problem.

NOMENCLATURE

a_1, a_2 = constants
A_m = expansion coefficients defined in Equation 29
b = channel half-width
b_1, b_2 = constants
G = function of X (see Equation 6)
h = heat transfer coefficient defined in Equation 38
k = thermal conductivity of the fluid
N_{Nu} = Nusselt number (see Equation 39)
q = heat flux from wall to fluid (see Equation 37)
T = temperature field in fluid
T_b = average temperature (see Equation 33)
T_o = temperature of fluid at the inlet
T_w = wall temperature
u_o = magnitude of plug flow velocity
x = axial coordinate
X = dimensionless axial coordinate defined in Equation 3a
y = transverse coordinate
Y = dimensionless transverse coordinate defined in Equation 3b
α = thermal diffusivity of fluid
β = constant
θ = dimensionless temperature field
θ_b = dimensionless average temperature (see Equation 34)
λ_n = constants defined in Equation 19
ϕ_n = eigenfunctions defined in Equation 25

LITERATURE CITED

1. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," Chapters 9 and 11, John Wiley, New York (1960).
2. Graetz, L., *Ann. Phys. u. Chem.*, *25*, 337 (1885).
3. Hildebrand, F. B., "Advanced Calculus for Applications," Chapter 5, Prentice Hall, New Jersey (1976).
4. Kays, W. M. and M. E. Crawford, "Convective Heat and Mass Transfer," Chapter 8, McGraw-Hill, New York (1980).
5. Knudsen, J. G., and Katz, D. L., "Fluid Dynamics and Heat Transfer," Chapter 13, McGraw-Hill, New York (1958).
6. Weinberger, H. F., "A First Course in Partial Differential Equations," Chapter IV, Xerox, Massachusetts (1965).

STUDY PROBLEMS

1. Solve Equation 8a using any technique and show that Equation 9 is the solution. Why is a positive C physically unrealistic?

2. Solve Equation 10 to obtain Equation 11 as the solution.

3. Show, by direct integration, that Equation 26 is satisfied by the eigenfunctions given in Equation 25.

4. Obtain Equation 36 by performing the integration required in Equation 35.

5. From information available in this module, show the following result.

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$

6. Fill in the details required in the derivation of Equation 40. Can you figure out the behavior of $N_{Nu}(X)$ as $X\to\infty$?

HOMEWORK PROBLEMS

1. Work out the details of the transformations given in Equation 3 in going from Equation 1 to Equation 4.

2. This problem should illustrate the importance of homogeneous boundary conditions in the success of the technique. Attempt to separate the solution of Equations 1 and 2a, 2b, and 2d by writing:

$$T(x, y)=T_1(x)T_2(y)$$

Do you encounter any problems? Where? Now, try writing:

$$\psi(x, y)=T-T_w$$

and derive the corresponding partial differential equation and boundary conditions on $\psi(x, y)$. Then, substitute a separation of variables solution for ψ. That is, write

$$\psi(x, y)=G(x)\phi(y)$$

and derive the differential equations satisfied by $G(x)$ and $\phi(y)$ and the boundary conditions on $\phi(y)$.

3. A related problem is that of heat transfer in plug flow between parallel plates where there is a uniform heat flux q_w at each wall. If one makes the proper assumptions, the same differential equation (Equation 1) can be derived for the temperature field. The boundary conditions, in this case, will be

$$T(0, y)=T_o \quad \text{(H-1)}$$

$$k\frac{\partial T}{\partial y}(x, b)=q_w \quad \text{(H-2)}$$

$$\frac{\partial T}{\partial y}(x, 0)=0 \quad \text{(H-3)}$$

a) By defining $X=(\alpha x)/(b^2u_o)$, $Y=y/b$ and $\theta(X, Y)=k(T-T_o)/(bq_w)$, transform the partial differential equation and the boundary conditions to the following form.

$$\frac{\partial\theta}{\partial X}=\frac{\partial^2\theta}{\partial Y^2} \quad \text{(H-4)}$$

$$\theta(0, Y)=0 \quad \text{(H-5)}$$

$$\frac{\partial\theta}{\partial Y}(X, 1)=1 \quad \text{(H-6)}$$

$$\frac{\partial\theta}{\partial Y}(X, 0)=0 \quad \text{(H-7)}$$

b) In order to use separation of variables, it is first necessary to eliminate the inhomogeneity in Equation H-6. This is done by breaking up the solution for θ as follows.

$$\theta(X, Y)=\theta_\infty(X, Y)-\theta_d(X, Y) \quad \text{(H-8)}$$

where θ_d is supposed to approach zero for large X so that θ_∞ will be the solution for such large X. $\theta_\infty(X, Y)$ can be obtained relatively easily. It is given by

$$\theta_\infty(X, Y)=X+\frac{Y^2}{2}-\frac{1}{6} \quad \text{(H-9)}$$

Using this information, derive the partial differential equation and boundary conditions for $\theta_d(X, Y)$. Solve for $\theta_d(X, Y)$ using separation of variables.

Module C4.2

Mathematical Techniques II—Combination of Variables

R. Shankar Subramanian

Department of Chemical Engineering
Clarkson College of Technology
Potsdam, New York 13676

OBJECTIVES

After completion of this module, the student should be able to solve partial differential equations using the method of combination of variables.

PREREQUISITE MATHEMATICAL SKILLS

1. Elementary calculus and differential equations.

PREREQUISITE ENGINEERING AND SCIENCE SKILLS

1. Elementary modeling of transport problems (preferable, but not necessary).

INTRODUCTION

This module is an introduction to the technique of *Combination Variables* for the solution of partial differential equations commonly encountered in modeling transport problems. It turns out in several practical problems that this technique complements the *separation of variables* solution (See Module C4.1). The latter solution is hard to compute due to slow series convergence in a certain range of variables, and it is precisely in this range that "combination of variables" is most useful and convenient. Let us choose a simple example from the area of heat transfer for illustrating the details of the technique.

MODELING OF THE PROBLEM

Consider a long slab of thermal diffusivity α, shown in Figure 1, which is insulated everywhere except for its ends. Initially, the slab is at a uniform temperature T_o. At time $t=0$, the end $x=0$ is raised to a new temperature T_1, and maintained at this value for all future time. Let us not worry about what happens at the other end at this stage, except to say that the slab is very long. Under these conditions, the cross-sectional geometry of the slab is immaterial—it can be rectangular, cylindrical or any other shape as long as it is uniform. There will be no temperature variations in the slab in the transverse direction at any given axial location; the only spatial variation of temperature will be with the axial coordinate x. Such a problem is usually referred to as an "unsteady one-dimensional conduction." Assuming that the thermal diffusivity is independent of temperature, it can be shown that the temperature in the slab $T(t, x)$ satisfies the partial differential equation,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

Here t is time and x is the axial coordinate indicated in the sketch.

In order to obtain a unique solution of Equation 1 for our problem, it is necessary to formulate *initial* and *boundary* conditions. From the physics of the problem, one may write

$$T(0, x) = T_o \tag{2}$$

and

$$T(t, 0) = T_1 \tag{3}$$

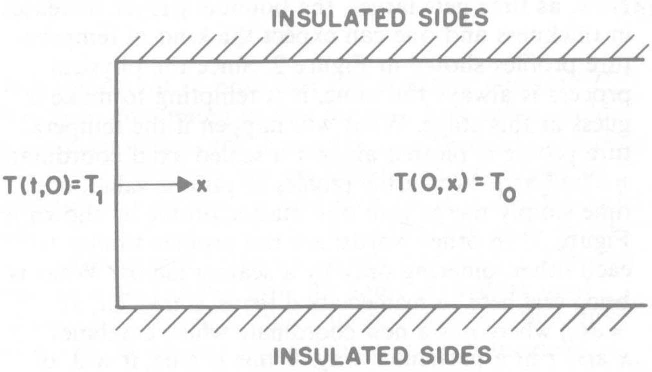


Figure 1. Sketch of the System.

Since there is a second derivative in x in Equation 1, one more boundary condition is needed on this coordinate. If the temperature or the heat flux at the other end of the slab were prescribed, there would, indeed, be such a condition. However, in a very long slab, one would hardly expect the behavior of the temperature field near the end $x=0$ to be influenced by what happens at the other end for relatively small values of time. For such values of time, the other end of the slab might as well be at infinity. Therefore, as the second boundary condition, let us say that the temperature as x approaches infinity is going to remain at its *initial value* of T_o.

$$T(t, \infty)=T_o \qquad (4)$$

Now that the formulation of the mathematical model is complete, one is ready to proceed to solve for the temperature field in the slab. It is worthwhile pointing out at this stage that Equations 1 to 4 also describe several other physical problems. For instance, the concentration distribution for plug flow in a channel near the inlet satisfies an identical model. Also, the so-called "penetration theory" of mass transfer to a falling liquid film results in the same mathematical model for the concentration distribution in the film. In the area of fluid mechanics, the same equations arise when modeling the velocity field in the vicinity of a plate suddenly set in motion in a fluid.

COMBINATION OF VARIABLES

The essence of this solution procedure can be understood if one examines qualitatively, the solution for the temperature field in the slab. Figure 2 shows what may be expected for various values of time. At time 0, the temperature is T_1 at $x=0$ and T_o everywhere else. After a small amount of time t_1 haselapsed, it will change form T_1 at $x=0$ to practically T_o a little distance δ away due to the process of heat conduction into the slab. Think of the region where the temperature changes from T_1 to T_o as a boundary layer. It should be realized that in principle, this region is of infinite length since the temperature is not quite T_o, for $t>0$, anywhere in our slab. There are several different definitions of the thickness of a "boundary layer" in this situation. However, for our qualitative purposes, assume that the "boundary layer" is that region where the temperature changes from T_1 to *practically* T_o. Now, as time gets larger, the boundary layer increases in thickness and one can expect the kind of temperature profiles shown in Figure 2. Since the physical process is always the same, it is tempting to make a guess at this stage. What will happen if the temperature profile is plotted against a scaled axial coordinate, $\eta=(x)/[\delta(t)]$? Would the profiles at various values of time simply merge into one master profile as shown in Figure 3? In other words, are the profiles similar to each other, differing only by a scaling factor? What is being said here, in mathematical terms, is that $T(t, x) = \phi(\eta)$ where η is a new coordinate which combines x and t in a particular way. If this is true, it will, of course, make life simpler since one can reduce the partial differential equation for $T(t, x)$ to an ordinary differential equation for $\phi(\eta)$. Let us start with the assumption that this procedure is going to work and trace through the consequences. Any inconsistency encountered along the way means that the individual profiles cannot be combined into one master profile; that is, they are not similar.

Let us begin transforming from the (t, x) coordinate system to the new coordinate

$$\eta=\frac{x}{\delta(t)} \qquad (5)$$

keeping in mind that $\delta(t)$ has not yet been defined precisely, except that it is only a function of time t. For transforming the differential equation for $T(t, x)$ into one for $\phi(\eta)$, one needs to transform the various derivatives using the chain rule. For instance,

$$\frac{\partial T}{\partial t}=\frac{\partial \eta}{\partial t}\frac{d\phi}{d\eta}=-\frac{x}{\delta^2}\frac{d\delta}{dt}\frac{d\phi}{d\eta}=-\frac{x}{\delta^2}\delta'\phi'$$

where the primes refer to differentiation with respect to the argument.

Similarly,

$$\frac{\partial T}{\partial x}=\frac{\partial \eta}{\partial x}\frac{d\phi}{d\eta}=\frac{1}{\delta}\frac{d\phi}{d\eta}=\frac{\phi'}{\delta}$$

and

$$\frac{\partial^2 T}{\partial x^2}=\frac{\phi''}{\delta^2}$$

Substituting in Equation 1, yields:

$$\phi''+x\frac{\delta'}{\alpha}\phi'=0 \qquad (6)$$

Things look bad because both x and t appear in the equation for ϕ which "apparently" means that ϕ is not a function of only η—the chain rule used earlier to convert derivatives implicitly assumed this to be true.

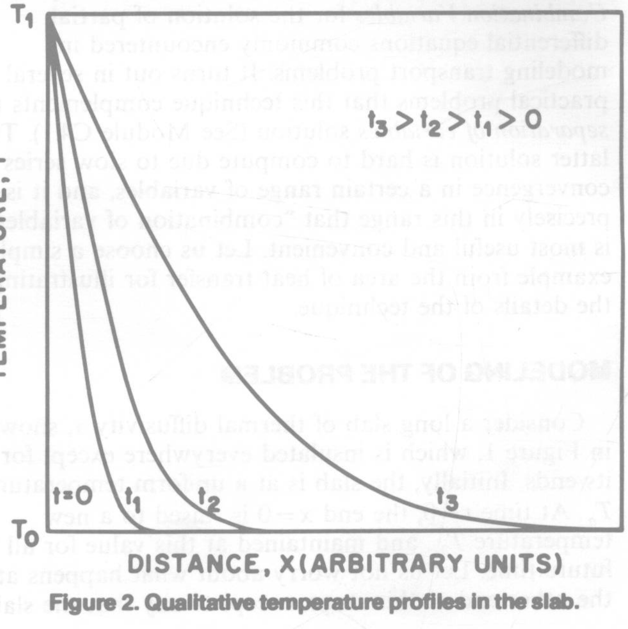


Figure 2. Qualitative temperature profiles in the slab.

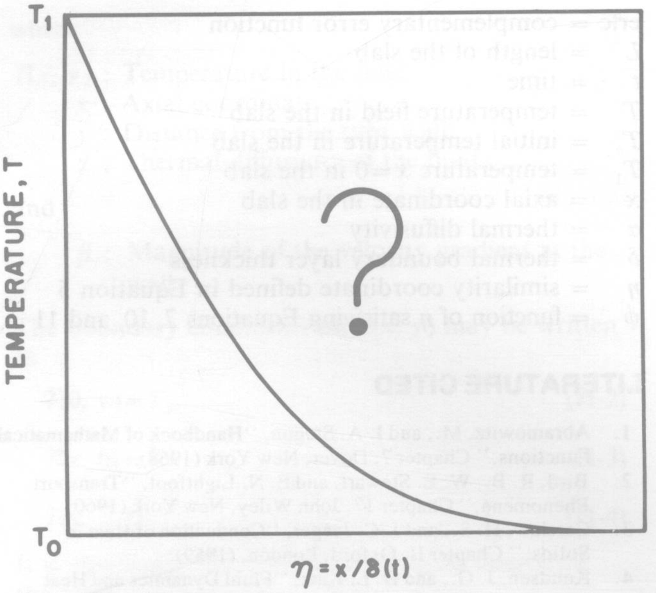


Figure 3. The similarity hypothesis.

But one can use the definition of η given in Equation 5 to eliminate x from Equation 6. Let us see if this helps. Equation 6 can now be rewritten as

$$\phi'' + \left[\frac{\delta}{\alpha}\frac{d\delta}{dt}\right]\eta\phi' = 0 \qquad (7)$$

Now, the equation only has time appearing in it (through $\delta\, d\delta/dt$) besides η. However, since δ *has not been defined precisely so far*, here is a chance to do it and, at the same time, eliminate our problem. Let

$$\frac{\delta}{\alpha}\frac{d\delta}{dt} = C \qquad (8)$$

where C is a constant. This removes the inconsistency encountered before. Equation 7 becomes

$$\frac{d^2\phi}{d\eta^2} + C\eta\frac{d\phi}{d\eta} = 0 \qquad (9)$$

Taking stock now, one has a second-order ordinary differential equation for $\phi(\eta)$ which needs two boundary conditions in order to obtain a unique solution, and a first-order ordinary differential equation for $\delta(t)$ needing one initial condition to solve for δ uniquely. An arbitrary constant C which needs to be determined or eliminated has also been introduced. Let us try to use Equations 2 through 5 to determine the required conditions. At $x=0$, $\eta=0$; therefore,

$$\phi(0) = T(t, 0) = T_1 \qquad (10)$$

As $x \to \infty$, $\eta \to \infty$. So,

$$\phi(\infty) = T(t, \infty) = T_o \qquad (11)$$

Now, let us see what Equation 2 implies. At time zero, $T = T_o$ and, therefore,

$$\phi\left[\frac{x}{\delta(0)}\right] = T_o \qquad (12)$$

If $\delta(0)$ is a nonzero constant, this gives us an inconsistency all over again; that is, one of the boundary conditions on ϕ would still have x appearing in it and ϕ, therefore, would not be a function of only η. The problem is eliminated neatly by the assumption

$$\delta(0) = 0 \qquad (13)$$

which gives an initial condition on $\delta(t)$. This conclusion can also be reached from the physical association of δ with the concept of a thermal boundary layer thickness. At time zero, the boundary layer has to have a zero thickness. Equation 12, therefore, becomes

$$\phi(\infty) = T_o \qquad (14)$$

(since our interest lies only in positive values of x). Note that Equation 14 is identical to Equation 11, and that there are really only two boundary conditions on ϕ, Equations 10 and 11. *This is an essential condition for combination of variables to work.*

Now, it is possible to solve the differential equations for $\phi(\eta)$ and $\delta(t)$, and to apply the conditions given by Equations 10, 11 and 13 in order to arrive at the following results.

$$\delta(t) = \sqrt{2\alpha C t} \qquad (15)$$

$$T(t, x) = \phi(\eta) = T_o + (T_1 - T_o)\ \mathrm{erfc}\left\{\sqrt{\frac{C}{2}}\,\eta\right\} \qquad (16)$$

where erfc $\{\gamma\}$ is the complementary error function given by

$$\mathrm{erfc}(\gamma) = 1 - \mathrm{erf}(\gamma) = 1 - \frac{\int_0^{\gamma} e^{-z^2}dz}{\int_0^{\infty} e^{-z^2}dz} = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\gamma} e^{-z^2}dz \qquad (17)$$

For learning more about the error function (erf) and the complementary error function (erfc), see Abramowitz and Stegun (*1*).

If one rearranges Equation 16, introduces the definition of η from Equation 5, and uses Equation 15, the following final solution is found for the temperature field in the slab.

$$\frac{T - T_o}{T_1 - T_o} = \mathrm{erfc}\left\{\frac{x}{2\sqrt{\alpha t}}\right\} \qquad (18)$$

It is important to note that the arbitrary constant C has been eliminated in this process. Any value could have been chosen for it and the result would have been the same. However, the expression for $\delta(t)$ would be affected as indicated in Equation 15. It is customary to choose $C=2$ in this situation, so that the right-hand side of Equation 16 looks slightly simpler (5). This last reference also shows how the same procedure used above can be used in much more complex situations where the boundary layer thickness δ is a function of time as well as a coordinate.

Concluding Remarks

Since the method outlined here utilizes the concept of similar temperature profiles, it also is referred to as the "similarity solution" technique. It is useful in several steady and unsteady heat and/or mass transfer situations. As mentioned earlier, one of these applications to mass transfer is known as the penetration theory and you can learn more about this from Bird et al. (2). Another classic example in the area of transport phenomena is the Leveque solution for entrance region heat transfer. This is discussed in detail by Knudsen and Katz (4). The Leveque problem is presented in this module as Homework Problem 2.

What are the limitations of this solution procedure? Earlier, it was said that in the slab, the other end is far away and does not "feel" the effect of the change at $x=0$ for small values of time. Therefore, the solution just developed would not be valid for large values of time. How can one decide what is "large" and what is "small?" From a standard table of values of the error function (1), it is known that

$$\mathrm{erf}(2) \approx 0.9953$$

so that

$$\mathrm{erfc}(2) \approx 0.0047$$

Therefore, for a given time t, at $x=4\sqrt{\alpha t}$,

$$\frac{T-T_o}{T_1-T_o} \approx 0.0047$$

That is, the difference between the temperature at that x and the initial temperature is less than 0.5% of the difference between the temperature at $x=0$ and the initial temperature; so, one may arbitrarily say that this is the axial location where the effect of the change at $x=0$ is barely felt. This can be used to make an estimate of the maximum value of time when this solution procedure starts breaking down. For instance, if the slab is L units long and changes in temperature are made *at both ends* at time zero,

$$\frac{L}{2} \approx 4\sqrt{\alpha t_{\max}}$$

or

$$t_{\max} \approx \frac{L^2}{64\alpha} \tag{19}$$

The actual factor of 1/64 in Equation 19 is, of course, quite arbitrary. For instance, if one is willing to tolerate a change of 5% at a given axial location as signifying negligible change, this factor would become approximately 1/30. In any case, it is clear that as time increases, the solution becomes progressively worse for any finite slab. Incidentally, the solutions of this and other closely related problems are given in the book by Carslaw and Jaeger (3).

NOMENCLATURE

C = constant
erf = error function
erfc = complementary error function
L = length of the slab
t = time
T = temperature field in the slab
T_o = initial temperature in the slab
T_1 = temperature $x=0$ in the slab
x = axial coordinate in the slab
α = thermal diffusivity
δ = thermal boundary layer thickness
η = similarity coordinate defined in Equation 5
ϕ = function of η satisfying Equations 7, 10, and 11

LITERATURE CITED

1. Abramowitz, M., and I. A. Stegun, "Handbook of Mathematical Functions," Chapter 7, Dover, New York (1968).
2. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," Chapter 17, John Wiley, New York (1960).
3. Carslaw, H. S., and J. C. Jaeger, "Conduction of Heat in Solids," Chapter II, Oxford, London, (1959).
4. Knudsen, J. G., and D. L. Katz, "Fluid Dynamics and Heat Transfer," Chapter 13, McGraw-Hill, New York (1958).
5. Ruckenstein, E., *Chemical Engng. Sci., 23, 363* (1968).

STUDY PROBLEMS

1. Solve Equation 8 for the boundary layer thickness $\delta(t)$, apply the initial condition on δ and, therefore, derive the result in Equation 15.

2. Solve Equation 9 for $\phi(\eta)$. Apply the boundary conditions on ϕ to derive the result in Equation 16.

3. We went through some discussion to establish that the system can be considered semi-infinite; that is, x ranges from 0 to ∞ in the model. Is this a necessary condition for our mathematical technique to work? Why?

4. Is it necessary for the condition at $t=0$ in our problem to merge with the condition as $x\to\infty$ for the similarity method to work? Can you think of a typical physical situation (in heat or mass transfer) where the two conditions may not merge?

HOMEWORK PROBLEMS

1. Using values of the error function given in tables by Abramowitz and Stegun (1), prepare a plot of the dimensionless temperature in the slab (from the left-hand side of Equation 18) against the similarity coordinate η in the range $0 \leqq \eta \leqq 2$ (assume $C=2$).

2. In this problem, we shall model the steady heat transfer to a fluid in laminar flow in a circular tube in the thermal entrance region. This problem was originally set up and solved by Leveque in 1928. After making appropriate assumptions, the following partial differential equation can be derived for the temperature in the fluid.

$$\beta y \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{H-1}$$