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strict convexity and complex strict convexity

theory and applications

Vasile I. Istrăţescu

STRICT CONVEXITY AND COMPLEX STRICT CONVEXITY

Theory and Applications

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Preface

Interest in the geometric properties of Banach spaces is due, to a great extent, to the fact that the linear topological properties, which are extremely useful in applications, are inseparably linked with a fixed geometrical object, namely, the closed unit ball of the space [i.e., the set $B_1(0) = \{x : \|x\| \le 1\}$]. Thus we are led naturally to consider linear topological properties within the framework of a given norm on the space. The purpose of this book is to present a comprehensive survey of those properties of a Banach space related to strict convexity, together with some applications.

The book contains three chapters. The first chapter is devoted to some of the basic results of linear functional analysis; readers who have had a one-year course in functional analysis may omit this chapter. Our treatment of the subject of the book begins in Chap. 2. The class of strictly convex spaces was first defined and investigated by J. Clarkson and M. G. Krein. We present several characterizations of these spaces, using real extreme points and certain classes of spaces related to the Banach space, such as the $\ell^p(X)$ spaces. An interesting and important characterization of strictly convex spaces mentioned in the text is the one involving duality mappings (introduced by A. Beurling and A. Livingston). This was studied by many mathematicians, among whom we mention only a few: F. Browder, W. L. Bynum, and S. Gudder. Some relations between strict convexity (or some of its generalizations) and the extension of continuous linear functionals are also mentioned. Further, the problem of the strict convexity of subspaces, products, and quotient subspaces is discussed.

Since in applications it is convenient to have some concrete classes of Banach spaces which share the property of strict convexity, we present some of these. Among the most important and interesting is the class of uniformly convex spaces, introduced by J. A. Clarkson in 1936 and since then investigated and generalized in a great number of papers. Among the contributors to the study of uniformly convex spaces and related classes of Banach spaces we note V. Smulian, D. Milman, B. Pettis, R. C. James, I. Glicksberg, J. Lindenstrauss, M. I. Kadets (M. I. Kadec), P. Enflo, V. Klee, A. Lovaglia, E. Asplund, G. Nordlander, V. D. Milman, S. L. Troiansky, and V. Zizler.

Next we consider the modulus of convexity and the modulus of smoothness of a Banach space and related classes of Banach spaces. Here the basic results of J. Lindenstrauss, M. I. Kadets, and V. Milman are presented. We mention also a function, introduced by V. Milman, which facilitates the description of some classes of spaces, e.g., the uniformly convex spaces and the uniformly nonsquare spaces of James. Further, we study relations between the differentiability properties of the norm and some geometric properties of the Banach space. The problem of deciding when a Banach space has an equivalent norm with given properties is treated in the last part of this chapter, for the properties of strict convexity and uniform convexity. Here we present the theorems of V. Klee on strict convexity and the remarkable result of R. C. James and P. Enflo on uniformly convex spaces.

The chapter continues with some applications. First we give applications to the theory of approximation (the proximity operator, or the Chebyshev operator) and further to fixed point theory. Here we note that the first result in which uniform convexity plays a fundamental role was a theorem of F. Browder that a nonexpansive mapping defined on a bounded, closed, and convex subset in an uniformly convex space has a fixed point. We present results related to this theorem as well as for a class of mappings called T-maps, which were first studied by the Italian mathematician Francesco Tricomi. Also, the normal structure and geometric properties of Banach spaces and fixed point theory are discussed. Special attention is given to a new class of spaces, the so-called normed probabilistic metric spaces introduced by A. Šerstnev, which are a particular case of probabilistic metric spaces, introduced by K. Menger in 1942. A property is considered which seems to be a good candidate for strict convexity in this setting.

In their interesting study of the space of all continuous (real-valued) functions on a compact Hausdorff space, R. Arens and J. L. Kelley have determined explicitly the form of extreme points of the closed unit ball of the conjugate space. It was shown that these coincide with the set of all nontrivial multiplicative functionals. The case of linear operators between two spaces of continuous functions was first considered by C. I. Tulcea and A. I. Tulcea. The problem has attracted the attention of many people and further theorems were obtained by R. Blumenthal, F. Bonsall, J. Lindenstrauss, R. Phelps, and M. Sharir. The second chapter ends by presenting results related to those of Arens-Kelley and Tulcea.

One of the most important properties of the space of complex analytic functions is the validity of the maximum principle for elements in the space. It is quite natural to ask if this fundamental property holds for Banach-valued analytic functions. A well-known example (mentioned, for example, in the book of Hille et al.) shows that this is not the case for all Banach spaces. In the third chapter we solve the problem of characterizing those Banach spaces for which the maximum modulus holds for analytic functions with values in the space. This is achieved by defining the so-called complex extreme points. By the use of this class of points in the unit ball of a Banach space we study the class of complex strictly convex spaces (i.e., those spaces for which all elements of norm 1 are complex extreme points). Further, we present some classes of spaces which are, in some sense, analogous to some classes of Banach spaces considered in Chap. 2.

Almost all the results we include have appeared in the literature. We make no claim of encyclopedic coverage, for we have concentrated on those aspects which seem to us most interesting and significant. Also, we have attempted to provide the reader with a bibliography, which does not pretend to be complete, but which, we hope, will serve as an adequate guide to the history and the current status of the topics presented in this book.

We wish to acknowledge with thanks conversations and correspondence in functional analysis (geometric theory of Banach spaces) from which we have benefited. Out appreciation goes also to the editors of the Marcel Dekker, Inc. for their attention to this volume.

Vasile I. Istrăţescu

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1
Banach Spaces

1.1 LINEAR SPACES

A fundamental notion in linear and nonlinear analysis, that of linear space over a field $\mathbb{K}$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, the set of all real numbers or the set of all complex numbers, respectively) is considered in the following definition.

DEFINITION 1.1.1 The nonempty set V is a linear space over $\mathbb{K}$ if there exist two functions defined on $V \times V$ and $\mathbb{K} \times V$, denoted by + and $\cdot$, respectively, which satisfy the following properties:

1. $x + (y + z) = (x + y) + z$ (associativity)
2. $x + y = y + x$ (commutativity)
3. There exists an element 0 (zero) of V such that $x + 0 = x$
4. $0 \cdot x = 0$
5. $a \cdot (x + y) = a \cdot x + a \cdot y$
6. $a \cdot (b \cdot x) = ab \cdot x$
7. $(a + b) \cdot x = a \cdot x + b \cdot x$
8. $1 \cdot x = x$

for all x,y,z in V and a,b in $\mathbb{K}$.

Sometimes $a \cdot x$ is denoted simply as ax, and we adopt this convention in what follows. The function + is called addition and the function $\cdot$ is said to be multiplication by scalars (of $\mathbb{K}$).

We now give some examples of linear spaces.

EXAMPLE 1.1.2 We consider an arbitrary set S, with V the set of all functions defined on S with values in $\mathbb{K}$. If f,g are two elements of V, then $h = f + g$ is defined by

$$h(s) = f(s) + g(s)$$

for all s in S. For any a in $\mathbb{K}$ and f in V, the element af is the function defined as follows:

$$(af)(s) = af(s)$$

for all s in S. With the addition and multiplication by scalars defined above it is clear that V is a linear space over $\mathbb{K}$.

As a matter of terminology, we mention that a linear space is called a real linear space if $\mathbb{K} = \mathbb{R}$ and a complex space if $\mathbb{K} = \mathbb{C}$.

EXAMPLE 1.1.3 Let n be a finite integer or ∞ and consider the set V of all sequences of the form $x = (x_1, x_2, \ldots, x_n)$ if n is finite and $x = (x_1, x_2, x_3, \ldots)$ otherwise. Here x_i are elements of $\mathbb{K}$. We show that V can be structured as a linear space over $\mathbb{K}$. Indeed, if $y = (y_1, y_2, \ldots, y_n)$ is another element of V then we set as $x + y$ the element

$$(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3,\ \ldots,\ x_n + y_n)$$

and if a is an arbitrary element of $\mathbb{K}$ the element $a \cdot x$ is defined as

$$(ax_1, ax_2, ax_3, \ldots, ax_n)$$

It is obvious that V with + and $\cdot$ defined as above is a linear space over $\mathbb{K}$. We can proceed in a similar way for ∞.

REMARK 1.1.4 It is clear that Example 1.1.3 is a particular case of the space considered in 1.1.2, namely, when the set S is countable.

EXAMPLE 1.1.5 Let X be a compact Hausdorff space and C(X) be the set of all continuous complex-valued functions on X. With the natural functions + and $\cdot$, C(X) is a complex linear space over $\mathbb{C}$. If we consider only the real-valued continuous functions, then it is clear that we get a real linear space.

DEFINITION 1.1.6 If V is a linear space over $\mathbb{K}$ then a subset V_1 of V is said to be a linear subspace if V_1 is a linear space with the functions + and $\cdot$ defined exactly as in V.

EXAMPLE 1.1.7 If V is the linear space of Example 1.1.2 and s_0 is a fixed point of S then

$$V_{s_0} = \{f : f \in V,\ f(s_0) = 0\}$$

is obviously a linear subspace of V.

This is valid also for the space of Example 1.1.5 as well as for the space of Example 1.1.3. Other interesting examples of subspaces will be given later.

1.2 SETS IN LINEAR SPACES

In what follows we define various classes of subsets in linear spaces. First we note that the family of linear subspaces of a linear space is an important class of objects associated with the space. One of the most important classes of subsets of a linear space is that consisting of the so-called *convex sets.*

DEFINITION 1.2.1 Let V be a linear space over $\mathbb{K}$. A subset C of V is called convex if whenever x,y are in C, then for all t in (0,1) the element

$$z_t = tx + (1 - t)y$$

is in C.

DEFINITION 1.2.2 If E is a subset of a linear space V then conv E, the convex hull of E, is the smallest convex set containing E.

It is not difficult to see, using Zorn's lemma, that for each subset E, conv E exists.

The following proposition gives another description of conv E.

PROPOSITION 1.2.3 If E is a subset of a linear space V then

$$\text{conv } E = \left\{ z : z = \sum_{i=1}^{n} a_i x_i,\ a_i \geq 0,\ \sum_{i=1}^{n} a_i = 1,\ x_i \in E \right\}$$

Proof: Let us denote by E_1 the set

$$\left\{ z : z = \sum_{i=1}^{n} a_i x_i,\ a_i \geq 0,\ \sum_{i=1}^{n} a_i = 1,\ x_i \in E \right\}$$

We remark that this set is convex. Indeed, if z_1 and z_2 are two elements of E_1,

$$z_1 = a_1x_1 + \cdots + a_nx_n \qquad z_2 = b_1y_1 + \cdots + b_my_m$$

then for all t in (0,1) the element tx + (1 - t)y is in E_1, since we can write

$$tx + (1 - t)y = \sum_{i=1}^{n+m} a_i' z_i$$

with

$$a_i' = \begin{cases} ta_i & i \leq n \\ (1 - t)b_{n+m-i} & i > n \end{cases}$$

$$z_i = \begin{cases} x_i & i \le n \\ y_{i-n} & i > n \end{cases}$$

Thus the convexity of E_1 is proved. To prove that E_1 is the smallest convex set containing E, we must show that any convex set E_2 containing E contains E_1. For this it is sufficient to show that E_2 contains all the elements z of the form

$$z = a_1 x_1 + \cdots + a_n x_n \qquad a_i \in [0,1],\ \Sigma\, a_i = 1,\ x_i \in E$$

For n = 2 this is obvious. Suppose that the assertion is valid for $n \le k$. We prove it for k + 1.

Let us consider $z = a_1 x_1 + \cdots + a_k x_k + a_{k+1} x_{k+1}$ where a_k are positive numbers and $\Sigma_{i=1}^{n} a_i = 1$ and x_i are in E. We must show that this is in E_2. We can write

$$\begin{aligned} z &= a_1 x_1 + \cdots + a_k x_k + a_{k+1} x_{k+1} \\ &= a_{k+1} x_{k+1} + (1 - a_{k+1}) \left(\sum_{i=1}^{k} \frac{a_i}{1 - a_{k+1}} \right) x_i \\ &= a_{k+1} x_{k+1} + (1 - a_{k+1}) z' \end{aligned}$$

where z' is in E_2 by induction. This implies clearly that z is in E_2. Thus we get that E_1 is the smallest convex set containing E, and the proposition is proved.

We give now examples of convex sets.

EXAMPLE 1.2.4 Let V be the linear space of Example 1.1.2. Then

$$C = \{f : f \in V,\ f(s) \in (0,1),\ s \in S\}$$

is obviously a convex set.

EXAMPLE 1.2.5 Let X be a compact Hausdorff space and C(X) be the space of all complex-valued continuous functions on X. The set C defined by

$$\{f : f \in C(X),\ \sup_{s \in X} |f(s)| \le 1\}$$

is clearly a convex set.

We consider now other classes of sets in linear spaces.

DEFINITION 1.2.5 Let V be a linear space over $\mathbb{K}$ and E be a subset of V. Then E is called

1. Symmetric if $E = -E = \{-x : x \in E\}$

2. Absorbing if for each x in V there exists $t_x > 0$ such that for all $|t| \leq t_x$, $tx \in E$
3. Balanced if $tE = \{tx : x \in E, |t| \leq 1\} \subseteq E\}$
4. Affine if $tE = (1 - t)E = \{tx + (1 - t)y : x,y \in E, t \in [0,1] \subseteq E\}$
5. A line through x and y if $E = \{tx + (1 - t)y : t \in \mathbb{R}\}$
6. The segment joining x and y if $E = \{tx + (1 - t)y : t \in [0,1]\}$

We now give examples of sets having some properties stated in Definition 1.2.6.

EXAMPLE 1.2.7 Let us consider V as $\mathbb{R}^2 = \{(x_1,x_2) : x_i \in \mathbb{R}\}$ ($\mathbb{R}$ the set of all real numbers) and E as the following set:

$$E = \{(x_1,x_2) : x_1^2 + x_2^2 = 1\}$$

It is easy to see that this set is symmetric and absorbing. Obviously E is not convex.

EXAMPLE 1.2.9 Let V be the space in Example 1.2.5 and E be the set defined as follows:

$$E = \{f : f \in V, f \text{ is real valued}\}$$

Then clearly E is convex and balanced.

1.3 SEMINORMS AND NORMS ON LINEAR SPACES. LOCALLY CONVEX SPACES

Let V be a linear space over $\mathbb{K}$. An important class of functions on V is considered in the following definition.

DEFINITION 1.3.1 The function

$$p : V \to \mathbb{R}$$

is called a seminorm if the following properties hold:

$$p(x + y) \leq p(x) + p(y)$$
$$p(ax) = |a|p(x)$$

for all x,y in V and all a in $\mathbb{K}$. The seminorm is said to be norm if $p(x) = 0$ if and only if $x = 0$.

We remark that the values of a seminorm are in fact in $\mathbb{R}^+$. Indeed, if we take $x = -y$ then we get

$$p(0) = 0 \leq p(x) + p(-x) = 2p(x)$$

and the assertion is proved.

Some examples of seminorms follow.

EXAMPLE 1.3.2 If V is the linear space of Example 1.1.2 and s_0 is an arbitrary but fixed point of S, the function

$$p_{s_0}(f) = |f(s_0)|$$

is obviously a seminorm on V.

EXAMPLE 1.3.3 If C(X) is the space in Example 1.1.5 then the function

$$f \to p(f) = \sup_{s \in X} |f(s)|$$

is a norm on C(X).

As is now standard, we use the following notation for a norm:

$$p(x) = \|x\|$$

and different norms will be denoted by $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_s$, ...

If V is a linear space over $\mathbb{K}$ and p is a seminorm then the set

$$B_p(0,1) = \{x : p(x) < 1\}$$

is clearly convex, symmetric, balanced, and absorbing. The following proposition gives the connection between convex sets which are balanced and absorbing and the seminorms. This connection is expressed using the so-called *gauge* or *Minkowski functional*.

PROPOSITION 1.3.4 Let V be a linear space over $\mathbb{K}$ and E be a subset in V with the following properties:

a. E is convex.

b. E is balanced and absorbing.

Then the function p_E defined on V by the formula

$$p_E(x) = \inf\{t > 0 : x \in tE\}$$

has the following properties:

1. $p_E(x) \geq 0$.
2. $p_E(x + y) \leq p_E(x) + p_E(y)$.
3. $p_E(ax) = |a| p_E(x)$.
4. $\{x : p_E(x) < 1\} \subseteq E \subseteq \{x : p_E(x) \leq 1\}$.

Proof: Properties 1 and 3 are obvious. To prove 2, let $\varepsilon > 0$, then

$$x \in [p_E(x) + \varepsilon]E$$
$$y \in [p_E(y) + \varepsilon]E$$

and thus

$$x + y \in [p_E(x) + p_E(y) + 2\varepsilon]E$$

which follows from the convexity of E. Since ε is arbitrary we get that 2 holds. The last relations between the sets follow from the definition of p_E.

To define the very important notion of locally convex space, first we recall the notion of topological space.

DEFINITION 1.3.5 If S is an arbitrary nonempty set, then a topology on S is any collection τ of subsets of S satisfying the following properties:

1. $S \in \tau$, $\emptyset \in \tau$ (the empty set is denoted by $\emptyset$).
2. The union of an arbitrary family of elements of τ is in τ.
3. If $G_1, \ldots, G_m$ are in τ (m is finite), then $G_1 \cap G_2 \cap \cdots \cap G_m$ is in τ.

The pair (S,τ) is called a topological space. For short we say that S itself is a topological space.

DEFINITION 1.3.6 If $s \in S$, then a neighborhood of s is any subset in S containing a set V in τ such that $s \in V$.

DEFINITION 1.3.7 If S_1 and S_2 are two topological spaces and $f : S_1 \to S_2$ is a mapping defined on S_1 with values in S_2, then we say that f is continuous at the point s_1 of S_1 if for any neighborhood V^2 of $f(x_1)$ there exists a neighborhood V^1 of s_1 such that $f(s) \in V^2$ if $s \in V^1$. The function f is said to be continuous on S_1 if it is continuous at each point of S_1.

The notion of topological linear space is considered in the following definition.

DEFINITION 1.3.8 A linear space V over $\mathbb{K}$ is said to be a topological linear space if on V there exists a topology such that $V \times V$ and $\mathbb{K} \times V$ with the product topology have the property that + and $\cdot$ are continuous functions.

In this case τ is called a linear topology on V. For a detailed account of linear topological space theory we refer to the excellent texts of N. Bourbaki (1955) and A. Grothendieck (1954).

The subclass of locally convex spaces is considered in the following definition.

DEFINITION 1.3.9 A linear topology on a linear topological space V is said to be a locally convex topology if every neighborhood of 0 (the zero of V) includes a convex neighborhood of 0.

Then we have the following result.

PROPOSITION 1.3.10 If V is a locally convex space over $\mathbb{K}$ then the topology of V is determined by a family of seminorms $(p_i)_{i \in I}$.

For the proof of this assertion see the above-mentioned references.

1.4 BANACH SPACES. EXAMPLES

An important class of locally convex spaces is the class of Banach spaces, in which the family of seminorms reduces to a single norm having a special property considered below.

If

$$x \to \|x\|$$

is a norm on the linear space V then the function on V × V defined by the relation

$$(x,y) \to \|x - y\| = d(x,y)$$

defines a metric on V, i.e., d satisfies the following properties:

1. $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$
2. $d(x,y) \leq d(x,z) + d(y,z)$
3. $d(x,y) = d(y,x)$

for all x,y,z in V.

Since the proof that d has the above properties is very simple we omit it.

DEFINITION 1.4.1 If (x_n) is a sequence of elements of V then we say that it converges to $x \in V$ if $\lim \|x_n - x\| = 0$.

The notion of Cauchy sequence is defined as follows.

DEFINITION 1.4.2 A sequence (y_n) of elements of V is called a Cauchy sequence if for any $\varepsilon > 0$ there exists an integer N_ε such that for all $n,m \geq N_\varepsilon$,

$$\|x_n - x_m\| \leq \varepsilon$$

Now the notion of Banach space is considered in the following definition.

DEFINITION 1.4.3 A linear space over $\mathbb{K}$ is called a Banach space if on V there exists a norm

$$x \to \|x\|$$

such that every Cauchy sequence of elements of V is a convergent sequence. (In other words, using the terminology of metric spaces, V with the metric d defined as above is a complete metric space.)

We give now some examples of Banach spaces. First, a linear space on which there exists a norm is called a normed space, and we have real normed spaces and complex normed spaces, respectively, according as the field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Also, a Banach space is a real Banach space if the field if $\mathbb{K} = \mathbb{R}$ and a complex Banach space if $\mathbb{K} = \mathbb{C}$.

EXAMPLE 1.4.5 Suppose $X = [0,1]$ and consider the following function defined on $C(X)$, the space of all continuous $\mathbb{K}$-valued functions on X:

$$f \to \int_0^1 |f(s)|\, ds$$

which is obviously a norm on $C(X)$. In this case $C(X)$ is a normed linear space.

EXAMPLE 1.4.6 Let us consider $p \in (1,\infty)$ and let ℓ^p be the set of all sequences $x = (x_i)$, $x_i \in \mathbb{K}$, with the property that

$$x \to \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} < \infty$$

Then, using Minkowski's inequality, we get that this is a norm on ℓ^p. (For the structure of ℓ^p as a linear space see Example 1.1.2.) Also, it is not very difficult to prove that in fact ℓ^p with the above norm is a Banach space.

EXAMPLE 1.4.7 Let us denote by ℓ^∞ the set of all bounded sequences $x = (x_i)$, $x_i \in \mathbb{K}$. With the norm

$$x \to \|x\|_\infty = \sup_i |x_i|$$

ℓ^∞ is a Banach space.

We note now some very interesting and important subspaces of ℓ^∞ (ℓ^∞ is denoted sometimes by m).