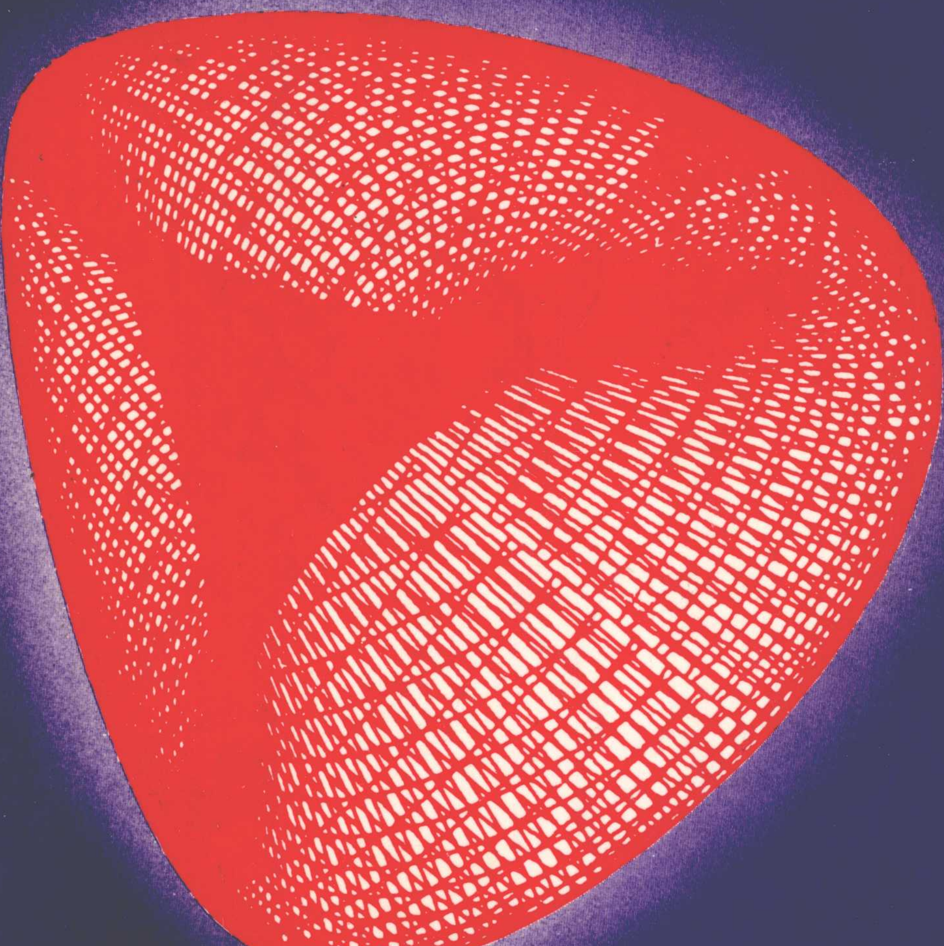
Ellis Horwood Series MATHEMATICS AND ITS APPLICATIONS

HARMONIC AND MINIMAL MAPS with applications in geometry and physics

Gábor Tóth



HARMONIC AND MINIMAL MAPS: With Applications in Geometry and Physics

GÁBOR TÓTH Ph.D.

Mathematical Institute of the
Hungarian Academy of Sciences
Budapest, Hungary.

Translation Editor:
T. J. WILLMORE, M.Sc.,Ph.D.,D.Sc.,FIMA
Professor of Pure Mathematics
University of Durham

ELLIS HORWOOD LIMITED
Publishers · Chichester

Halsted Press: a division of
HN WILEY & SONS
ne · Chichester · Toronto

First published in 1984 by

ELLIS HORWOOD LIMITED
Market Cross House, Cooper Street, Chichester, West Sussex,
PO19 1EB, England

The publisher's colophon is reproduced from James Gillison's drawing of the ancient Market Cross, Chichester.

Distributors:
Australia, New Zealand, South-east Asia:
Jacaranda-Wiley Ltd., Jacaranda Press,
JOHN WILEY & SONS INC.,
G.P.O. Box 859, Brisbane, Queensland 4001, Australia.

Canada:
JOHN WILEY & SONS CANADA LIMITED
22 Worcester Road, Rexdale, Ontario, Canada.

Europe, Africa:
JOHN WILEY & SONS LIMITED
Baffins Lane, Chichester, West Sussex, England.

North and South America and the rest of the world:
Halsted Press: a division of
JOHN WILEY & SONS
605 Third Avenue, New York, N.Y. 10158, U.S.A.

British Library Cataloguing in Publication Data

British Library Cataloguing in Publication Data
Toth, Gabor
Harmonic and Minimal Maps:
With Applications in Geometry and Physics.—
Ellis Horwood Series in mathematics and its applications
1. Physics 2. Geometry 3. Harmonic maps
4. Minimal maps
I. Title
530'.01'51474 QC20.7.H3

ISBN 0-85312-738-7 (Ellis Horwood Limited)
ISBN 0-470-20127-4(Halsted Press)

Printed in Great Britain by Unwin Brothers of Woking.

PREFACE

Ever since the year 1964, when the fundamental work of Eells-Sampson, "*Harmonic mappings of Riemannian manifolds*" appeared, the theory of harmonic maps has far outgrown its original scope to develop the first general properties of harmonic maps and to study the existence of harmonic maps in given homotopy classes of maps between Riemannian manifolds (existence theory). To mention only a few related branches of mathematics in which harmonic maps occur not only on the level of illustrating examples but on the level of rethinking the basic ideas, I refer to the recent developments of complex analysis, Morse theory with calculus of variations and the theory of minimal maps.
The rapid expansion of the theory of harmonic maps, as mentioned above, is to be thanked for the advantage due to its universal features but this fact, as far as writing a book on the subject is concerned, plays the role of certain difficulty as well.
The present book is therefore, by no means, intended to be a comprehensive introduction to harmonic maps but, rather, an expository which studies the geometry of harmonic and minimal maps into spaces of constant curvature. Though I have tried to keep the prerequisites to a minimum, and to make the book accessible for students of mathematics, mathematicians and physicists, the presentation can only be considered to be almost self-contained as basic Riemannian geometry and representation theory are applied rather than developed here; nevertheless, only those proofs are omitted which would be incompatible with the general scope. The background materials are given in their right places without bogging the reader down with a long introductory chapter. Each chapter begins with a concise introduction and ends with a set of problems to help the reader's comprehension of the material. The exposition has a gradually increasing speed which, I hope, would keep pace with the increasing interest.

The first (introductory) chapter displays a basic material on the general theory of differential operators on vector bundles. The stress being on applicability, I give here a detailed account on first and second order classical differential operators of Riemannian geometry. For the convenience of the reader, two specific sections summarize the basic facts of the spectrum and the (full) isometry group of Riemannian manifolds.

In Chapter II particular emphasis is placed on the generalized Hodge-de Rham theory by developing a differential calculus on twisted tensor bundles which is then specified leading to the concept of harmonicity of maps.The great majority of this chapter is devoted to the works of Eells-Sampson and Lichnerowicz on harmonic maps into flat codomains; a theory which has also proved to be fruitful in describing the geometric structure of the nonnegatively Ricci curved manifolds. Basic fibre bundle theory is used here.

The principal part of the book is essentially contained in ChaptersIII-IV, where the variational theories of harmonic and minimal maps are developed in a rather similar way offered by the calculus of variations. The basic existence theory of harmonic maps between spheres, due to R.T. Smith, is presented in Chapter III in full details. Based on different ideas, the counterpart of this theory for minimal maps, the Do Carmo-Wallach classification and rigidity of minimal immersions between spheres, is treated in Chapter IV with the necessary representation theoretical background material summarized in a specific section.

In view of Chapter IV, a particularly interesting problem in the theory of harmonic maps is their application to the study of their rigidity properties. In Chapter V, introducing the concept of infinitesimal rigidity, I deal with infinitesimal deformations and classification of harmonic maps of constant energy density into spheres. A variety of examples is also given here illustrating the fundamental concepts.

The differential geometry developed in Chapters I-II can also be served to give global formulation of various physical theories. In Chapter VI, as a theory of specific interest, I describe some basic concepts of Yang-Mills fields. As a detailed exposition on the subject could fill a whole monograph, I restrict myself to point out the close relationship between energy, volume and Yang-Mills functionals by giving a calculus of variations for Yang-Mills fields and proving Simons' instability theorem.

I owe a great deal of help and encouragement to Professor James Eells who, with a deep insight in the theory combined with his kind personality, has guided me towards the problems that determined my research activity over the last 5 years. I would like to express my sincere gratitude to him. I am especially grateful to Professor T.J. Willmore for his hard and dedicated work who has acted as a referee and translation editor. Their comments on the preliminary draft of the manuscript were invaluable aids for preparing the final version.
I had many long and informative discussions with Á. Elbert on differential equations used in the book; I am also indebted to him. Thanks are due to numerous mathematicians and physicists who, by their lectures, private talks and papers, taught me a lot of aspects of differential geometry; I am particularly grateful to J.C. Wood and L. Lemaire for their useful suggestions to improve the original text and to G. D'Ambra, P. Forgács, A. Lichnerowicz, S. Rallis, H.J.C. Sealey, J.H. Sampson, J. Szenthe and N. Wallach. Last (but not least) I would like to express my gratitude to my wife for her boundless patience and excellent work of typing the whole manuscript.

Gábor Tóth
The Ohio State University
231 West 18th Avenue
Columbus, Ohio 43210
U.S.A.
January 1984

Table of Contents

Preface iii

Chapter I – DIFFERENTIAL OPERATORS ON VECTOR BUNDLES

1. Differential operators and symbols 1
2. Lie differentiation 13
3. Exterior differentiation 15
4. Covariant differentiation 24
5. The Exterior codifferentiation 40
6. The Laplacian 45
7. The spectrum of a Riemannian manifold 51
8. Isometries 56

Problems for Chapter I 63

Chapter II – HARMONIC MAPS INTO FLAT CODOMAINS AND THE STRUCTURE OF COMPACT NONNEGATIVELY RICCI CURVED RIEMANNIAN MANIFOLDS

1. Calculus on twisted tensor bundles 66
2. Generalities on harmonic maps 76
3. Existence and uniqueness of harmonic maps into flat codomains 96
4. The Albanese map 108
5. Riemannian manifolds with positive semidefinite Ricci tensor field 132

Problems for Chapter II 143

Chapter III – HARMONIC MAPS INTO SPHERES

1. First and second variation formulas 146
2. Instability 156
3. Existence of harmonic maps between spheres 160
4. Study of the pendulum equation 166
5. Regularity 175
6. Examples of harmonically representable homotopy classes 186

Problems for Chapter III 187

Chapter IV – MINIMAL SUBMANIFOLDS IN SPHERES
1. First and second variation formulas 189
2. Instability 202
3. Existence of minimal immersions of a homogeneous manifold into spheres 208
4. Rigidity 232
5. Higher fundamental forms 240
6. Rigidity of minimal immersions between spheres 248
7. Representations of the special orthogonal group 260

Problems for Chapter IV 268

Chapter V – INFINITESIMAL DEFORMATIONS OF HARMONIC MAPS INTO SPHERES
1. Conformal diffeomorphisms of 2-manifolds 270
2. Infinitesimal rigidity 278
3. Harmonic maps of a homogeneous manifold into spheres 289
4. Infinitesimal rigidity of minimal immersions between spheres 300

Problems for Chapter V 304

Chapter VI – APPLICATIONS IN THEORETICAL PHYSICS
1. First and second variation formulas for the Yang-Mills functional 309
2. Instability 320

Specific Bibliography for Chapter VI 324

Summary of Basic Notations 328

Bibliography 333

Index 339

CHAPTER I

Differential Operators on Vector Bundles

In this chapter we present some preliminary material on differential operators acting on (real) vector bundles. In Section 1 we review, in some detail, the elementary theory of differential operators. The proof of the finiteness theorem and the Hodge-de Rham decomposition for elliptic self-adjoint differential operators is omitted, since proving it in details would take us too far afield. The general framework is specified in Sections 2-6 to present a rapid elementary course on first and second order classical differential operators occurring in Riemannian geometry whose inevitable importance is reflected throughout this book as well. For later reference, to close the chapter, we attach two additional sections dealing with the spectrum and the (full) isometry group of Riemannian manifolds. Some of the well-known proofs will only be sketched or even omitted since all these are readily available in standard books on differential geometry, such as Berger, Gauduchon and Mazet (1971), Helgason (1978), Kobayashi and Nomizu (1963) and (1969), Lichnerowicz (1955), de Rham (1955), Wells (1973).

1. DIFFERENTIAL OPERATORS AND SYMBOLS

Let V and W be (real, finite dimensional) vector spaces and denote by ε_U^V and ε_U^W the trivial vector bundles over an open set $U \subset \mathbf{R}^m$ with typical fibres V and W, respectively. *A differential operator of order* r is a linear map

$$D_o : C^\infty(\varepsilon_U^V) \to C^\infty(\varepsilon_U^W)$$

(sending a section of ε_U^V to a section of ε_U^W) of the form

$$D_0 = \sum_{|\rho| \leq r} A_\rho D^\rho \,, \tag{1.1}$$

where the summation runs over all multi-indices $\rho=(\rho^1,\ldots,\rho^m)\in \mathbf{Z}_+^m$ with $|\rho| = \sum_{i=1}^m \rho^i \leq r$, $D^\rho = (\frac{\partial}{\partial x^1})^{\rho_1} \ldots (\frac{\partial}{\partial x^m})^{\rho_m}$ and

$$A_\rho : U \to \mathrm{Hom}(V,W)$$

is a (smooth) map.† The *symbol* $\sigma_r(D_0)$ of the differential operator D_0 is the map

$$\sigma_r(D_0) : U\times\mathbf{R}^m \to \mathrm{Hom}(V,W)$$

defined by

$$\sigma_r(D_0)(x,y) = \sum_{|\rho|=r} y^\rho A_\rho(x),\ x\in U\,, \qquad y = (y^1,\ldots,y^m)\in\mathbf{R}^m\,, \tag{1.2}$$

where y^ρ stands for $(y^1)^{\rho_1}\ldots(y^m)^{\rho_m}\in\mathbf{R}$. For fixed $x\in U$, the symbol $\sigma_r(D_0)(x,.)$ is then a homogeneous polynomial of degree r with coefficients in $\mathrm{Hom}(V,W)$. The differential operator D_0 is said to be *elliptic* if $\sigma_r(D_0)(x,y)\in\mathrm{Hom}(V,W)$ is an isomorphism for all $x\in U$ and $0\neq y\in\mathbf{R}^m$.

Example 1.3. Set $V=W=\mathbf{R}$ and $r=2$. Then (1.1) can be rewritten as

$$D_0 = \sum_{i,j=1}^m C_{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \text{linear part,}$$

with $C_{ij} = C_{ji} : U\to\mathbf{R}$, $i,j=1,\ldots,m$, and its symbol, for fixed $x\in U$, is the quadratic form

$$\sigma_2(D_0)(x,y) = \sum_{i,j=1}^m C_{ij}(x)y^i y^j\,,\ y = (y^1,\ldots,y^m)\in\mathbf{R}^m\,.$$

†Throughout this book, unless stated otherwise, all manifolds, maps, bundles, etc. will be smooth, i.e. of class C^∞. For the notations explained briefly in the text, see Summary of Basic Notations.

Thus, D_0 is elliptic if and only if the matrix $C(x) = (C_{ij}(x))_{i,j=1}^{m} \in M(m,m)$ is nonsingular for all $x \in U$.

Let ξ and η be (real) vector bundles over an m-dimensional manifold M † with typical fibres V and W, respectively. A linear map

$$D : C^{\infty}(\xi) \to C^{\infty}(\eta)$$

is said to be a *local operator* if for any section $v \in C^{\infty}(\xi)$ vanishing on an open set $U \subset M$ we have $(Dv)|U=0$. Then, for each open set $U \subset M$, there is induced a local operator

$$D : C^{\infty}(\xi|U) \to C^{\infty}(\eta|U) .$$

In fact, given $x_0 \in U$, choose a scalar $\mu \in C^{\infty}(M)$ on M with compact support in U such that μ equals 1 in a neighbourhood of x_0. Then, for $v \in C^{\infty}(\xi|U)$, we set

$$(Dv)(x_0) = (D(\mu v))(x_0) ,$$

where

$$(\mu v)(x) = \begin{cases} \mu(x)v(x) & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \notin U . \end{cases}$$

Further, if (U,ϕ) is a local coordinate neighbourhood on M such that the restrictions $\xi|U$ and $\eta|U$ are trivial then, choosing trivializations

$$h_U : \xi|U \to \varepsilon_U^V \quad \text{and} \quad k_U : \eta|U \to \varepsilon_U^W , \tag{1.4}$$

the local operator D gives rise to a linear map

$$D_0 : C^{\infty}(\varepsilon_{\phi(U)}^V) \to C^{\infty}(\varepsilon_{\phi(U)}^W)$$

by setting

† Unless stated otherwise, all manifolds considered in this book are assumed to be connected.

$$D_0(v) = \{k_U(D(h_U^{-1}(v\circ\phi)))\}\circ\phi^{-1} \;, \quad v\in C^\infty(\varepsilon^V_{\phi(U)}) \;. \qquad (1.5)$$

The local operator $D:C^\infty(\xi) \to C^\infty(\eta)$ is said to be a *differential operator of order* r if for every local coordinate neighbourhood (U,ϕ) and trivializations h_U, k_U (in (1.4)) the corresponding local operator D_0 is a differential operator of order r, i.e. it has the form given in (1.1). D_0 is then called a *local coordinate representation of* D. The set of all differential operators $D:C^\infty(\xi) \to C^\infty(\eta)$ of (a given) order $r\in\mathbf{Z}_+$ is denoted by $\mathrm{Diff}_r(\xi,\eta)$; it is, in fact, a module over the algebra $C^\infty(M)$ of scalars on M. Clearly, $\mathrm{Diff}_0(\xi,\eta) = \mathrm{Hom}(\xi,\eta)$ and, for each $r\in\mathbf{Z}_+$, we have the canonical inclusion $\mathrm{Diff}_r(\xi,\eta)\subset\mathrm{Diff}_{r+1}(\xi,\eta)$.

To define the symbol of a differential operator $D\in\mathrm{Diff}_r(\xi,\eta)$ of order $r\in\mathbf{Z}_+$, we consider the fibre bundle $T'(M)$ over M with typical fibre $\mathbf{R}^m\setminus\{0\}$ obtained from the *cotangent bundle* $T^*(M)$ of M by removing the zero section. The *symbol* $\sigma_r(D)$ of D will then associate to each cotangent vector $\alpha_x\in T'_x(M)$ $(=T^*_x(M)\setminus\{0\})$, $x\in M$, a linear map

$$\sigma_r(D)(\alpha_x) \;:\; \xi_x \to \eta_x$$

between the fibres of ξ and η at x , respectively, in the following way. First, choose a scalar $\mu\in C^\infty(M)$ with $\mu(x)=0$ such that α_x equals the total differential $d\mu$ at $x\in M$. Then, for each $v_x\in\xi_x$, we set

$$\sigma_r(D)(\alpha_x)v_x = (D(\tfrac{1}{r!}\,\mu^r\cdot v))(x)\in\eta_x \;, \qquad (1.6)$$

where $v\in C^\infty(\xi)$ is an extension of v_x . We claim that the symbol $\sigma_r(D)$ is well-defined (i.e. the value $\sigma_r(D)(\alpha_x)v_x$ does not depend on the choice of μ and v) and that, for the trivial bundles $\xi=\varepsilon^V_U$ and $\eta=\varepsilon^W_U$, with $U\subset\mathbf{R}^m$ open, we recover our earlier agreement (1.2). Assume that $D\in\mathrm{Diff}_r(\xi,\eta)$ has local coordinate representation (1.1) with respect to a local coordinate neighbourhood (U,ϕ) and trivializations h_U , k_U in (1.4). Then, setting $w = \frac{1}{r!}\,\mu^r\cdot v\in C^\infty(\xi)$, by (1.5), we have

$$k_U\{(D(w))(x)\} = (k_U D\{h_U^{-1}(h_U w)\})(x) = D_0((h_U w)\circ\phi^{-1}))(\phi(x)) =$$

$$= \sum_{|\rho|\le r} A_\rho(\phi(x))(D^\rho((h_U w)\circ\phi^{-1}))(\phi(x)) \ .$$

As the scalar $\mu^r \circ \phi^{-1}$ vanishes in r-th order at $\phi(x)$,

$$D^\rho((h_U w)\circ\phi^{-1}) = \{D^\rho(\frac{1}{r!}\,\mu^r\circ\phi^{-1}\cdot(h_U v)\circ\phi^{-1})\}(\phi(x)) =$$

$$= D^\rho(\frac{1}{r!}\,\mu^r\circ\phi^{-1})(\phi(x))\cdot h_U(v_x),$$

for every multi-index $\rho\in\mathbf{Z}_+^m$ with $|\rho|\le r$. Further, by $d(\mu\circ\phi^{-1})_{\phi(x)} = (\phi^{-1})^* d\mu_x = (\phi^{-1})^*\alpha_x \in T^*_{\phi(x)}(\mathbf{R}^m)$, an elementary calculation yields

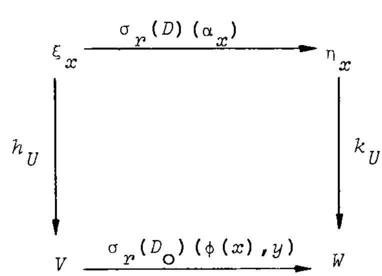
$$D^\rho(\frac{1}{r!}\,\mu^r\circ\phi^{-1})(\phi(x)) = \begin{cases} (((\phi^{-1})^*\alpha_x)^\vee)^\rho & \text{if} \quad |\rho|=r \\ 0 & \text{if} \quad |\rho|<r \end{cases}$$

where $^\vee : T^*(\mathbf{R}^m)\to\mathbf{R}^m$ stands for the canonical identification defined by parallel transport of cotangent vectors in $T^*(\mathbf{R}^m)$ to the origin. Summarizing, we obtain

$$k_U\{(D(\frac{1}{r!}\,\mu^r v))(x)\} = \sum_{|\rho|=r} y^\rho A_\rho(\phi(x))h_U(v_x) =$$

$$= \sigma_r(D_0)(\phi(x),\ y)h_U(v_x) \ ,$$

where $y = ((\phi^{-1})^*\alpha_x)^\vee$. In other words, the diagram



commutes and the claim follows.

Extending the earlier definition, a differential operator $D \in \mathrm{Diff}_r(\xi,\eta)$ is said to be *elliptic* if the linear map $\sigma_r(D)(\alpha_x):\xi_x \to \eta_x$ is an isomorphism for all $\alpha_x \in T'(M)$.

Remark. A result of Peetre asserts that any local operator $D \in C^\infty(\xi) \to C^\infty(\eta)$ has local coordinate representation (1.1) over suitably chosen coordinate neighbourhoods (U,ϕ) of M . (For a proof, see Narasimhan (1968) pages 175-176.)
Differential operators compose well as stated in the following.

Proposition 1.7. *Let* ξ,η *and* ζ *be vector bundles over a manifold* M . *If* $D \in \mathrm{Diff}_r(\xi,\eta)$ *and* $D' \in \mathrm{Diff}_s(\eta,\zeta)$ *then the linear map*

$$D' \circ D : C^\infty(\xi) \to C^\infty(\zeta)$$

is a differential operator of order $r+s$ *and its symbol is given by*

$$\sigma_{r+s}(D' \circ D) = \sigma_s(D') \circ \sigma_r(D) \ .$$

Proof. The composition $D' \circ D$ being a local operator, we work out a local coordinate representation of $D' \circ D$ over a coordinate neighbourhood (U,ϕ) . Assume that D and D' have local coordinate representations

$$D = \sum_{|\rho| \le r} A_\rho D^\rho$$

and

$$D' = \sum_{|\sigma| \le s} B_\sigma D^\sigma$$

over U , respectively. By making use of the Leibniz rule, we have

$$D' \circ D = \sum_{|\sigma| \le s} B_\sigma D^\sigma \{ \sum_{|\rho| \le r} A_\rho D^\rho \} =$$

$$= \sum_{|\rho| \le r} \sum_{|\sigma| \le s} B_\sigma \sum_{\gamma \le \sigma} \binom{\sigma}{\gamma} (D^{\sigma-\gamma} A_\rho) D^{\rho+\gamma} =$$

$$= \sum_{|\rho|\leq r} \sum_{|\sigma|\leq s} B_\sigma \sum_{\tau\leq\rho+\sigma} \binom{\sigma}{\tau-\rho} (D^{\rho+\sigma-\tau} A_\rho) D^\tau = \sum_{|\tau|\leq r+s} C_\tau D^\tau ,$$

where

$$C_\tau = \sum_{\substack{|\rho|\leq r, |\sigma|\leq s \\ \rho\leq\tau\leq\rho+\sigma}} \binom{\sigma}{\tau-\rho} B_\sigma D^{\rho+\sigma-\tau} A_\rho . \tag{1.8}$$

Hence, $D'\circ D$ is a differential operator of order $r+s$. To compute its symbol, we first note that, for $|\tau|=r+s$, (1.8) reduces to the form

$$C_\tau(x) = \sum_{\substack{|\rho|=r \\ \rho+\sigma=\tau}} B_\sigma(x) A_\rho(x) , \quad x\in\phi(U) .$$

Thus, by the identity $y^{\rho+\sigma} = y^\rho y^\sigma$, $y\in\mathbf{R}^m$, (1.2) entails

$$\sigma_{r+s}(D'\circ D)(x,y) = \sum_{|\tau|=r+s} y^\tau C_\tau(x) =$$

$$= \sum_{|\tau|=r+s} y^\tau \sum_{\substack{|\rho|=r \\ \rho+\sigma=\tau}} B_\sigma(x) A_\rho(x) = \sum_{|\rho|=r} \sum_{|\sigma|=s} y^\rho y^\sigma B_\sigma(x) A_\rho(x) =$$

$$= \{\sum_{|\sigma|=s} y^\sigma B_\sigma(x)\}\{\sum_{|\rho|=r} y^\rho A_\rho(x)\} = \sigma_s(D')(x,y)\circ\sigma_r(D)(x,y) . \quad \checkmark$$

To introduce the notion of adjoint for differential operators we have to endow the linear subspace $C_0^\infty(\xi)\subset C^\infty(\xi)$ of sections of compact support with a suitable prehilbert structure. For this, we first denote by $\mathcal{D}^r(M)(=C^\infty(\Lambda^r(T^*(M))))$, $r\in\mathbf{Z}_+$, the *vector space of forms of degree* r *on* M . Clearly, $\mathcal{D}^r(M)=\{0\}$ for $r>m$. The direct sum $\mathcal{D}(M) = \bigoplus_{r=0}^{m} \mathcal{D}^r(M)$ with the *exterior multiplication* Λ becomes an algebra over $\mathbf{R}$ (usually called the *exterior algebra of* M) . Taking M to be a Riemannian manifold, the metric tensor $g = (,)\in C^\infty(S^2(T^*(M)))$, being nonsingular (in fact, positive definite) on each of the fibres of

$T(M)$, induces a canonical isomorphism $\gamma : T(M) \to T^*(M)$ by setting

$$(\gamma(X_x))(Y_x) = g(X_x, Y_x) = (X_x, Y_x) \ , \qquad X_x, Y_x \in T_x(M) \ , \quad x \in M \ . \tag{1.9}$$

(The induced isomorphism between the $C^\infty(M)$-modules of vector fields $V(M)(=C^\infty(T(M)))$ and 1-forms $\mathcal{D}^1(M)$ on M is also denoted by γ .) Furthermore, assuming that M is oriented, the *volume form* $\mathrm{vol}(M,g) \in \mathcal{D}^m(M)$ can be locally given by

$$\mathrm{vol}(M,g) = \gamma(E^1) \wedge \ldots \wedge \gamma(E^m) \ , \tag{1.10}$$

where $\{E^i\}_{i=1}^m$ is an oriented orthonormal moving frame of $T(M)$ over a local coordinate neighbourhood (U,ϕ) . (More generally, if $\{E^i\}_{i=1}^m \subset V(U)$ is *any* local oriented moving frame with

$$g = \sum_{i,j=1}^m g_{ij} \gamma(E^i) \otimes \gamma(E^j) \ , \quad g_{ij} \in C^\infty(U) \ , \quad i,j=1,\ldots,m \ ,$$

then

$$\mathrm{vol}(M,g) = \sqrt{\det(g_{ij})_{i,j=1}^m} \ \gamma(E^1) \wedge \ldots \wedge \gamma(E^m) \ .$$

For example, classically, if (U,ϕ) is oriented with associated local base $\{dx^i\}_{i=1}^m \subset \mathcal{D}^1(U)$ then the metric tensor and volume form can be written as

$$g = \sum_{i,j=1}^m g_{ij} dx^i \otimes dx^j \ , \ g_{ij} \in C^\infty(U) \ , \qquad i,j=1,\ldots,m \ , \tag{1.11}$$

and

$$\mathrm{vol}(M,g) = \sqrt{g} \cdot dx^1 \wedge \ldots \wedge dx^m \ , \qquad g = \det(g_{ij})_{i,j=1}^m \ , \tag{1.12}$$

respectively.) By standard Riemannian geometry, the existence of the volume form enables to integrate scalars of compact support on M and, as usual,

$$\int_M \mu \operatorname{vol}(M,g) \ , \quad \mu \in C_0^\infty(M) \ ,$$

stands for the integral of μ over M.

A fibre metric on a (real) vector bundle ξ over M is a section $(,)_\xi \in C^\infty(S^2(\xi^*))$ whose restriction to each of the fibres of ξ is positive definite. The vector bundle ξ with $(,)_\xi$ (or rather, the pair $(\xi, (,)_\xi)$) is said to be a *Euclidean vector bundle* over M. Then, for $v,v' \in C^\infty(\xi)$, the *scalar product* $(v,v')_\xi$ is a scalar on M and, assuming supp $v \cap$ $\cap$ supp $v' \subset M$ to be compact, the *global scalar product of* v and v' is defined by

$$((v,v'))_\xi = \int_M (v,v')_\xi \operatorname{vol}(M,g) \ . \tag{1.13}$$

The linear space $C_0^\infty(\xi)$ endowed with the global scalar product $((,))_\xi$ becomes a prehilbert space.

Proposition 1.14. *Let ξ and η be Euclidean vector bundles with fibre metrics $(,)_\xi$ and $(,)_\eta$, respectively, over an m-dimensional oriented Riemannian manifold M. Then, for any differential operator $D \in \operatorname{Diff}_r(\xi,\eta)$ there exists a unique differential operator $D^* \in \operatorname{Diff}_r(\eta,\xi)$ such that*

$$((Dv,w))_\eta = ((v,D^*w))_\xi \tag{1.15}$$

for all $v \in C_0^\infty(\xi)$ and $w \in C_0^\infty(\eta)$. Moreover, the symbols are related by the formula

$$\sigma_r(D^*) = (-1)^r \cdot {}^t\sigma_r(D) \ , \tag{1.16}$$

where ^{t} denotes the transpose.

Proof. As $C_0^\infty(\xi)$ is a prehilbert space, uniqueness of the adjoint *on* $C_0^\infty(\xi)$ is obvious. Also, by standard argument, D^* is unique on the whole of $C^\infty(\xi)$. The question of existence, by uniqueness, can easily be reduced to local existence of D^*, i.e. we may set $M=\mathbf{R}^m$, $\xi=\varepsilon^V_{\mathbf{R}^m}$, $\eta=\varepsilon^W_{\mathbf{R}^m}$ and the integration on $\mathbf{R}^m$