Stochastic Processes and Their Applications

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Edited by K. Itô and T. Hida

PREFACE

The Fifteenth Conference on Stochastic Processes and Their Applications was held in Nagoya, Japan, for the period July 2-6, 1985. This volume contains the invited papers presented at this conference.

The conference was attended by 360 scientists, from all over the world, and 110 among them were participants from overseas, whose attendance is greatly appreciated.

In addition to the invited paper sessions, there were contributed paper sessions on the following topics.

Gaussian processes and fields
Branching, population and biological models
Stochastic methods in physics
Stochastic differential equations
Probability distributions and limit theorems
Stable processes and measures
Random walks and i.i.d. random variables
Filtering, control and optimization
Statistical inference
Diffusion processes
Markov processes
Storage and reliability
Ergodic theorems
Martingales and processes with independent increments
Point processes and applications
Stochastic processes in random media

The organizers regret that the papers in these sessions could not be included in this volume.

We should like to express sincere thanks to Professors Ken-iti Sato and Tadahisa Funaki who have helped us in editing this volume.

September 15, 1985

Kiyosi Itô

Takeyuki Hida

The following lectures were delivered at the conference, but are not included in this volume.

G. Kallianpur

Some recent results in nonlinear filtering theory with finitely additive white noise

H. Kesten

First-passage percolation and maximal flows

S. Orey

On getting to a goal fast

Yu.A. Rozanov

On stochastic partial differential equations

A.S. Sznitman

A "propagation of chaos" result for Burger´s equation

The following special lecture was also given.

K. Itô

An outline of the development of stochastic processes.

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established that by choosing suitable versions of $\{\xi_t(x) : t \geq 0\}$ for each $x \in M$ we obtain a random process $\{\xi_t : t \geq 0\}$ with values in Diff(M), the group of smooth diffeomorphisms of M with the topology of uniform C^r convergence for all $r \geq 0$. The process $\{\xi_t : t \geq 0\}$ is called a stochastic flow (of diffeomorphisms). For results on the existence of the stochastic flow of diffeomorphisms for equation (1.1) see [6], [12], [15] or [19].

The process $\{\xi_t : t \geq 0\}$ has the following properties

(i) independent increments on the left (i.e. if $0 \leq t_0 < t_1 < \ldots < t_n$ then $\xi_{t_i}\xi_{t_{i-1}}^{-1}$, $1 \leq i \leq n$, are independent)

(ii) time homogeneous (i.e. if $t > s$ the distribution of $\xi_t\xi_s^{-1}$ depends only on $t-s$)

(iii) continuous sample paths with probability 1 (w.p.1)

(iv) $\xi_0 = I$ w.p.1.

It is shown in [2] that any process $\{\xi_t : t \geq 0\}$ in Diff(M) satisfying (i)-(iv) arises as the solution of an equation of the form (1.1). See also [23]. The law of the process $\{\xi_t : t \geq 0\}$ is determined uniquely by the drift $V_0 \in C^\infty(TM)$ and the Hilbert space H. Since H is continuously included in $C^\infty(TM)$ it is a reproducing kernel Hilbert space of sections of TM and so it is determined by a positive semi-definite reproducing kernel b. We have

$$\begin{aligned} b(x,y) &= E(W_1(x) \otimes W_1(y)) \\ &= \sum_{i\geq 1} V_i(x) \otimes V_i(y) \qquad (1.4) \\ &\in T_xM \otimes T_yM \end{aligned}$$

for $x,y \in M$. V_0 and b may be interpreted as the mean and variance of the random vector field which is driving (1.1). Notice that the individual V_i, $i \geq 1$, determine the law of the flow only so far as they contribute to the sum (1.4). It is important to distinguish between the behaviour of the one-point motion $\{\xi_t(x) : t \geq 0\}$, which is characterised by L, and the behaviour of the flow $\{\xi_t : t \geq 0\}$. The extra information given by V_0 and b which is not contained in L is the correlation between the one-point motions $\{\xi_t(x) : t \geq 0\}$ for different x but the same noise $\{W_t^i : t \geq 0\}$, $i \geq 1$.

In this paper we shall be concerned with the following questions about the stochastic flow $\{\xi_t : t \geq 0\}$.

(1) Geometrical nature of ξ_t. Does there exist a nice subset D of Diff(M) such that $\xi_t \in D$ for all $t \geq 0$ w.p.1? Since the stochastic flow has continuous sample paths we can always take D equal to the

identity component $Diff_I(M)$ of Diff(M). Of more interest is the case where G is a subgroup of $GL(d,\mathbb{R})$ and D is the group of automorphisms of some G-structure on M. See Kobayashi [18] for details on G-structures. For example D might be the group of isometries of some Riemannian structure on M or the group of diffeomorphisms which preserve some smooth volume element on M.

(2) <u>Stability of solutions</u>. For distinct $x,y \in M$ what happens to $d(\xi_t(x),\xi_t(y))$ as $t \to \infty$? (Here d denotes the metric corresponding to some Riemannian structure on M.) This is a question about the 2-point motion $\{(\xi_t(x),\xi_t(y)) : t \geq 0\}$ on M^2. We obtain a similar question, but one which involves infinitely many points, by considering $\operatorname{diam}(\xi_t(U))$ for some neighbourhood U of x in M.

(3) <u>Induced measures</u>. Let $\mathcal{P}(M)$ denote the space of Borel probability measures on M. For $\rho \in \mathcal{P}(M)$ let $\rho_t = \rho\xi_t^{-1}$. Then $\{\rho_t : t \geq 0\}$ is a Markov process in $\mathcal{P}(M)$. In [22] Le Jan obtains results on the nature of the stationary distribution for this Markov process. See also [20]. We shall say $\rho \in \mathcal{P}(M)$ is invariant under the stochastic flow if $\rho_t = \rho$ for all $t \geq 0$ w.p.1. In this case we may take D in question 1 to be the group of ρ-preserving diffeomorphisms of M. This is a much stronger property of ρ than the fact that ρ is a stationary measure for the one-point process; in fact ρ is stationary for the one-point process if and only if $E(\rho_t) = \rho$ for all $t \geq 0$. Since the one-point process is a Feller process and M is compact there exists at least one stationary probability measure. Moreover if L is elliptic then ρ is unique and the one-point process is ergodic with respect to ρ. Henceforth we shall use ρ to denote a stationary probability measure on M. In general ρ will not be invariant under the flow and we wish to describe the behaviour of ρ_t as $t \to \infty$.

We shall describe some general results in Sections 3 and 4. Before that we consider two specific examples of stochastic flows on the circle; in each case we may write down an explicit solution. In the final section we consider a family of examples of stochastic flows on the torus.

For results for similar questions applied to isotropic stochastic flows in Euclidean spaces see Baxendale and Harris [5] and Le Jan [21].

2. Two stochastic flows on the circle

Before proceeding with general results we consider the following examples of stochastic flows. In each case $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with the Riemannian structure inherited from the standard inner product on $\mathbb{R}$. We write θ for elements of both $\mathbb{R}$ and S^1 and observe that equations (2.1) and (2.3) below have period 2π.

Example 1

$$d\xi_t(\theta) = dW_t. \tag{2.1}$$

The solution is given by

$$\xi_t(\theta) = \theta + W_t \pmod{2\pi}, \tag{2.2}$$

that is, ξ_t is rotation of S^1 through the angle W_t.

Example 2

$$d\xi_t(\theta) = \sin(\xi_t(\theta)) \circ dW_t^1 + \cos(\xi_t(\theta)) \circ dW_t^2. \tag{2.3}$$

This is the equation for the gradient stochastic flow obtained by taking the standard embedding of S^1 as the unit circle in $\mathbb{R}^2$. See Carverhill, Chappell and Elworthy [10] (which also contains a computer simulation of the 10-point motion due to P. Townsend and D. Williams). The solution is given by

$$\tan(\tfrac{1}{2}\xi_t(\theta)) = \frac{\tan(\frac{1}{2}U_t^2) + a_t(\tan(\frac{1}{2}\theta) - b_t)}{1 - a_t(\tan(\frac{1}{2}\theta) - b_t)\tan(\frac{1}{2}U_t^2)} \tag{2.4}$$

where

$$a_t = \exp(-U_t^1 + \tfrac{1}{2}t) \tag{2.5}$$

$$b_t = \int_0^t a_s^{-1}\, dU_s^2 \tag{2.6}$$

and $\{U_t^i : t \geq 0\}$, $i = 1,2$, are Brownian motions given by

$$\begin{bmatrix} dU_t^1 \\ dU_t^2 \end{bmatrix} = R_t \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}$$

where R_t denotes rotation in $\mathbb{R}^2$ through the angle $\xi_t(\pi)$.

In both examples the one-point motion has generator given by $L = \frac{1}{2}\frac{d^2}{d\theta^2}$, so that the one-point motion is Brownian motion on S^1. The stationary probability measure ρ is normalised Lebesgue measure. In example 1 the stochastic flow consists of rotations, whereas in example 2 the stochastic flow lies in a 3-dimensional Lie subgroup (isomorphic to $SL(2,\mathbb{R})$) of $\mathrm{Diff}(S^1)$. In example 1 distances between points are preserved and the measure ρ is invariant under the flow.

In example 2 the fact that $a_t \to \infty$ as $t \to \infty$ w.p.1 implies that distances and the measure ρ become more and more distorted by ξ_t as $t \to \infty$. More precisely b_t converges to some random value b_∞, say, as $t \to \infty$ w.p.1, and if θ_1 and θ_2 are distinct points of S^1 with $\tan(\frac{1}{2}\theta_i) \neq b_\infty$, $i = 1,2$, then

$$\lim_{t\to\infty} \frac{1}{t} \log d(\xi_t(\theta_1), \xi_t(\theta_2)) = \lim_{t\to\infty} \frac{1}{t} \log \left(\frac{1}{a_t}\right)$$

$$= -\frac{1}{2}.$$

Also the measure ρ_t becomes more and more concentrated but does not converge as $t \to \infty$. Instead it looks more and more like a unit mass attached to a Brownian particle. In fact for any $\theta \in S^1$ such that $\tan(\frac{1}{2}\theta) \neq b_\infty$ then $\rho_t - \delta(\xi_t(\theta)) \to 0$ weakly as $t \to \infty$ (where $\delta(\theta)$ denotes the unit mass concentrated at θ). For a generalisation of example 2 to gradient stochastic flows on spheres see [3].

3. The support theorem for stochastic flows

Let C denote the space of continuous functions $f : [0,\infty) \to \mathrm{Diff}(M)$ such that $f(0) = I$, with the compact-open topology. By properties (iii) and (iv) of stochastic flows, $\xi = \{\xi_t : t \geq 0\}$ is a C-valued random variable. In this section we study ν, the distribution of ξ. The following theorem is due to Ikeda and Watanabe [16].

Theorem 3.1. Let $\mathcal{U}$ denote the space of piecewise constant functions $u : [0,\infty) \to H$. Let $\xi^u \in C$ denote the solution of

$$\frac{d}{dt}(\xi^u_t(x)) = (V_0 + u(t))(\xi^u_t(x))$$

$$\xi^u_0(x) = x.$$

Then the support of ν is the closure in C of $\{\xi^u : u \in \mathcal{U}\}$.

Theorem 3.1 can sometimes provide an answer to question 1. Let LA(H) denote the closed Lie subalgebra of $C^\infty(TM)$ generated by H (or, equivalently, generated by $\{V_i : i \geq 1\}$). Notice that LA(H) depends only on H as a set and not on its inner-product. Theorem 3.1 remains valid if we replace H by LA(H) in the definition of $\mathcal{U}$. It follows that if $LA(H) = C^\infty(TM)$ then the support of ν is C itself, and so the only closed subset D of Diff(M) satisfying $\xi_t \in D$ for all $t \geq 0$ w.p.1 is $D = \mathrm{Diff}_I(M)$. On the other hand if the (deterministic) flows along the individual vector fields V_i, $i \geq 0$, all lie in some closed subgroup D of Diff(M) then $\xi_t \in D$ for all $t \geq 0$ w.p.1. The stochastic flows studied in the previous section provide examples of this phenomenon. For an example where D is the group of conformal diffeomorphisms of a sphere see [3]. In general, even if the noise in (1.1) is only finite-dimensional the resulting stochastic flow is infinite-dimensional.

When considering the behaviour of ξ_t as $t \to \infty$ the support theorem is often of little use. This is because the topology on C is that of uniform convergence on compact subsets of $[0,\infty)$ and so, for example, the set

$$\{f \in C : d(f(t)(x), f(t)(y)) \to 0 \text{ as } t \to \infty\},$$

for fixed x and y in M, is not closed in C. The failure of the support of ν to provide information about ν itself is shown up in the following result.

<u>Theorem 3.2</u>. Let ν and ν' denote the distributions in C of the stochastic flows corresponding to pairs (V_0, H) and (V_0', H').

(i) If $LA(H) = LA(H')$ and $V_0' - V_0 \in LA(H)$ then ν and ν' have the same support.

(ii) Either ν and ν' are singular or $\nu = \nu'$. The latter case occurs if and only if $V_0 = V_0'$ and $H = H'$ (as Hilbert spaces).

<u>Proof</u>. (i) This follows directly from Theorem 3.1.

(ii) For $T > 0$ let $C_T = \{f|_{[0,T]} : f \in C\}$ and let ν_T denote the distribution of $\{\xi_t : 0 \leq t \leq T\}$ in C_T. Consider the isomorphism of C with $(C_T)^\infty$ given by

$$f_n(t) = f(t+nT)(f(nT))^{-1}, \quad n \geq 0, \ t \in [0,T], \ f \in C.$$

Under this isomorphism ν corresponds to the infinite product of copies of ν_T. The first part of (ii) now follows from a theorem of Kakutani [17] on infinite product measures. The second part is contained in [2]. □

4. Lyapunov exponents for stochastic flows

Consider the derivative flow $\{D\xi_t : t \geq 0\}$ on the tangent bundle TM of M. For each $x \in M$ and $t \geq 0$, $D\xi_t(x)$ is a random linear mapping from T_xM to $T_{\xi_t(x)}M$. In this section we shall consider the limiting rate of growth of $D\xi_t(x)$ as $t \to \infty$ for $x \in M$, and its effect on the nature of the stochastic flow as $t \to \infty$. If we impose a Riemannian structure we may consider $(D\xi_t(x)^* D\xi_t(x))^{1/2} : T_xM \to T_xM$, the positive part of $D\xi_t(x)$. Here $D\xi_t(x)^* : T_{\xi_t(x)} \to T_xM$ denotes the adjoint of $D\xi_t(x)$. The following theorem goes back to results of Furstenberg

[13] on products of independent identically distributed random matrices and Oseledec [24] on products of a stationary ergodic sequence of matrices. It was adapted to apply to deterministic flows by Ruelle [25] and its present formulation for stochastic flows is due to Carverhill [7].

Theorem 4.1. Assume ρ is a stationary ergodic probability measure for the one-point process of a stochastic flow $(\xi_t : t \geqslant 0)$ defined on a probability space $(\Omega, \mathfrak{F}, P)$. For $P \times \rho$-almost all $(\omega, x) \in \Omega \times M$,

$$(D\xi_t(x)^* D\xi_t(x))^{1/2t} \to \Lambda_{(\omega,x)} \quad \text{as } t \to \infty \tag{4.1}$$

where $\Lambda_{(\omega,x)}$ is a random linear map on T_xM with non-random eigenvalues

$$e^{\lambda_1} \geqslant e^{\lambda_2} \geqslant \dots \geqslant e^{\lambda_d} > 0. \tag{4.2}$$

The values $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_d$ are called the Lyapunov exponents for the stochastic flow. Since M is compact, any two Riemannian structures on M are uniformly equivalent and so the Lyapunov exponents are independent of the choice of Riemannian structure. They are non-random because they depend only on the remote future of the stochastic flow, whereas $\Lambda_{(\omega,x)}$ is in general random because the eigenspaces corresponding to distinct eigenvalues depend on the entire evolution of the stochastic flow. Roughly speaking, the theorem implies that the positive part of $D\xi_t(x)$ has eigenvalues growing like $e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_d t}$ as $t \to \infty$. More precisely we have:

Corollary 4.2.

(i) For ρ-almost all $x \in M$ and Lebesgue-almost all $v \in T_xM \setminus \{0\}$

$$P\{\tfrac{1}{t} \log \|D\xi_t(x)(v)\| \to \lambda_1 \text{ as } t \to \infty\} = 1. \tag{4.3}$$

(ii) For ρ-almost all $x \in M$

$$P\{\tfrac{1}{t} \log \det(D\xi_t(x)) \to \lambda_1 + \lambda_2 + \dots + \lambda_d \text{ as } t \to \infty\} = 1. \tag{4.4}$$

We comment that if the generator L for the one-point process is elliptic then we may remove the condition on x in Corollary 4.2. If the generator for the induced one-point process $\{D\xi_t(x)(v) : t \geqslant 0\}$ in TM is elliptic on $\bigcup_{y \in M} (T_yM \setminus \{0\})$ then we may remove the condition on $v \in T_xM \setminus \{0\}$ also.

Corollary 4.2(i) provides an answer to the infinitesimal version of question 2. Carverhill [7], following Ruelle [25], has a local stable manifold theorem which enables us to obtain answers to question 2 as originally posed. We state a special case of the theorem.

Theorem 4.3. Suppose $\lambda_1 < 0$. If $0 > \mu > \lambda_1$ then for $P \times \rho$-almost all (ω,x) in $\Omega \times M$ there exist (measurable) $r(\omega,x) > 0$ and $\gamma(\omega,x) > 0$ such that $d(x,y_i) < r(\omega,x)$ for $i = 1,2$ implies

$$d(\xi_t(y_1),\xi_t(y_2)) < \gamma(\omega,x)d(y_1,y_2)e^{\mu t} \tag{4.5}$$

for all $t \geq 0$.

An immediate consequence of Theorem 4.3 is that if $\lambda_1 < 0$ then for ρ-almost all $x \in M$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$P\{\text{diam } \xi_t(B(x,\delta)) \to 0 \text{ as } t \to \infty\} > 1 - \varepsilon \tag{4.6}$$

where $B(x,\delta)$ denotes the open ball with centre x and radius δ.

Let us review example 2. Let $\overline{\theta}(\omega)$ be the random point of S^1 given by $\tan(\frac{1}{2}\overline{\theta}(\omega)) = b_\infty$. Since the distribution of $\{\xi_t : t \geq 0\}$ is rotation invariant it follows that $\overline{\theta}(\omega)$ is uniformly distributed on S^1. The set of full $P \times \rho$ measure on which (4.1) is valid is the set $\{(\omega,\theta) : \theta \neq \overline{\theta}(\omega)\}$, and $\lambda_1 = -\frac{1}{2}$. In Theorem 4.3 we need $r(\omega,\theta) < d(\theta,\overline{\theta}(\omega))$. In (4.6) notice that diam $\xi_t(B(\theta,\delta)) \to 0$ as $t \to \infty$ whenever $\overline{\theta}(\omega) \in S^1 - B(\theta,\delta)$, and this happens with probability $1 - \frac{\delta}{\pi}$. So corresponding to $\varepsilon > 0$ we need $\delta < \pi\varepsilon$.

The following result provides a global version of Corollary 4.2(ii), so long as ρ is closely related to the Riemannian measure m, say, on M. Let $\lambda_\Sigma = \lambda_1 + \lambda_2 + \ldots + \lambda_d$.

<u>Theorem 4.4</u>. Suppose ρ has a positive C^2 density with respect to m. Then

$$\lim_{t\to\infty} \frac{1}{t} I(\rho_t|\rho) = -\lambda_\Sigma \qquad \text{w.p.1}$$

where $I(\rho_t|\rho)$ denotes the relative entropy of ρ_t with respect to ρ. In particular $\lambda_\Sigma \leq 0$, and $\lambda_\Sigma = 0$ if and only if ρ is invariant under the stochastic flow $\{\xi_t : t \geq 0\}$.

<u>Proof</u>. See Baxendale [4], Theorem 5.2.

The highest Lyapunov exponent λ_1 may be evaluated using a formula due to Carverhill [8], following the method of Has'minskii [14] for linear stochastic differential equations. We shall obtain a special case of the formula in the next section. The formula may be generalised so as to obtain the sum of the k highest Lyapunov exponents for $1 \leq k \leq d$. See Baxendale [4] for details. The sum λ_Σ has been studied by Chappell [11].

For a recent survey article on Lyapunov exponents for stochastic flows see Carverhill [9].

5. A family of stochastic flows on the torus

Let $M = (\mathbb{R}/2\pi\mathbb{Z})^2 = T^2$, the two-dimensional torus, with coordinates $x = (x^1, x^2) \in M$ and Riemannian structure inherited from the standard inner product on $\mathbb{R}^2$. Consider the one-parameter family of stochastic flows on M, parametrised by $\alpha \in [0,\pi/2]$, given by

$$d\xi_t(x) = \sum_{i=1}^{4} V_i(\xi_t(x)) \circ dW_t^i \tag{5.1}$$

where the vector fields are as follows

$$V_1(x) = \sin x^1 (\cos\alpha \frac{\partial}{\partial x^1} + \sin\alpha \frac{\partial}{\partial x^2})$$

$$V_2(x) = \cos x^1 (\cos\alpha \frac{\partial}{\partial x^1} + \sin\alpha \frac{\partial}{\partial x^2})$$

$$V_3(x) = \sin x^2 (-\sin\alpha \frac{\partial}{\partial x^1} + \cos\alpha \frac{\partial}{\partial x^2})$$

$$V_4(x) = \cos x^2 (-\sin\alpha \frac{\partial}{\partial x^1} + \cos\alpha \frac{\partial}{\partial x^2}).$$

For each α the infinitesimal generator for the one-point motion is one half of the Laplace-Beltrami operator on M, so that the one-point motion is Brownian motion on M. The stationary probability measure ρ for the one-point motion is normalized Lebesgue measure. It is easily checked by calculating the covariance $b(x,y)$ that different values of $\alpha \in [0,\pi/2]$ give rise to stochastic flows with different distributions.

We may simulate the k-point motion $\{(\xi_t(x_1),\ldots,\xi_t(x_k)) : t \geq 0\}$ in M^k for any $k \geq 1$, $(x_1,\ldots,x_k) \in M^k$ and α using the scheme

$$\xi_{nh+h}(x_r) = \xi_{nh}(x_r) + \sqrt{h} \sum_{i=1}^{4} X_i^{(n)} V_i(\xi_{nh}(x_r))$$

for $1 \leq r \leq k$ and $n \geq 0$, where $\{X_i^{(n)} : 1 \leq i \leq 4,\ n \geq 0\}$ are independent $N(0,1)$ random variables. Figures 1 to 4 show the result of simulating with $h = 0{\cdot}01$ the 225-point motion started from the regular 15×15 lattice J in M given by $J = \{(\frac{(2i-1)\pi}{15}, \frac{(2j-1)\pi}{15}) : 1 \leq i,j \leq 15\}$. We show $[0,2\pi] \times [0,2\pi]$ squares whose edges are to be identified to give M. The pictures correspond to different values of α but they are all at the same time $t = 0{\cdot}5$ and are all generated by the same sequence of $X_i^{(n)}$. In order to appreciate better the distortion of M caused by ξ_t we have added in figures 2, 3 and 4 the straight line segments joining the current positions of nearest neighbour pairs in the original lattice J.

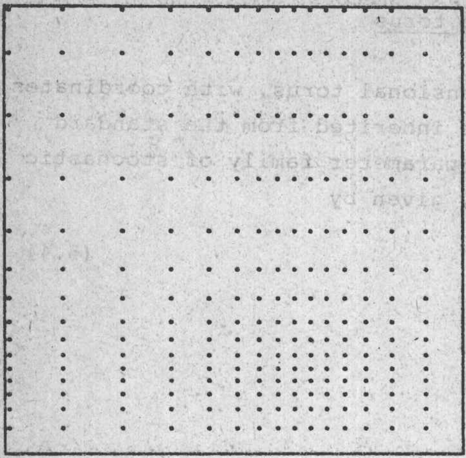

Figure 1 : $\alpha = 0$, $t = 0\cdot5$

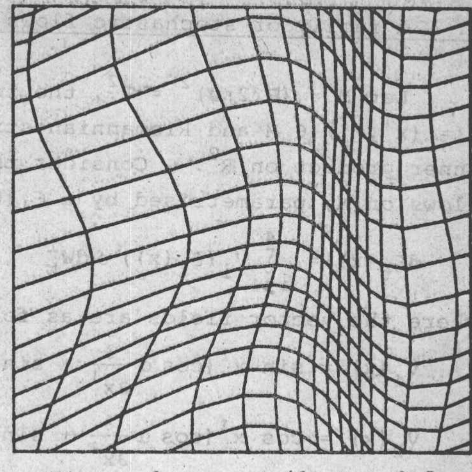

Figure 2 : $\alpha = \pi/6$, $t = 0\cdot5$

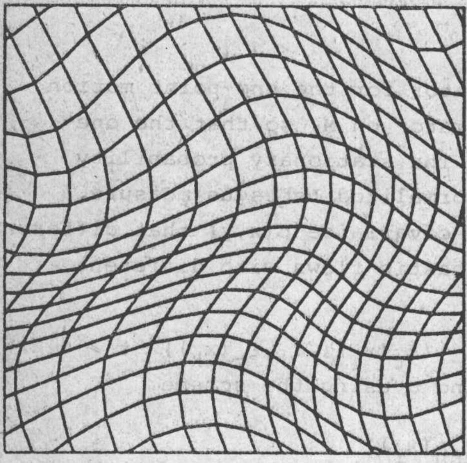

Figure 3 : $\alpha = \pi/3$, $t = 0\cdot5$

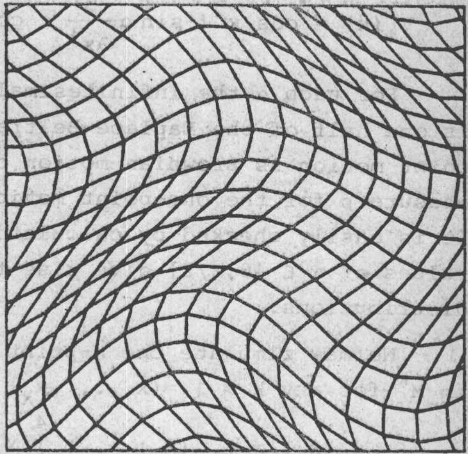

Figure 4 : $\alpha = \pi/2$, $t = 0\cdot5$

Consider the two extreme values of α. If $\alpha = 0$ then we obtain

$$\xi_t(x) = (\xi_t^1(x^1), \xi_t^2(x^2))$$

where $\{\xi_t^1 : t \geqslant 0\}$ and $\{\xi_t^2 : t \geqslant 0\}$ are independent copies of the stochastic flow on S^1 studied in example 2. This decomposition shows up in Figure 1 in the way in which horizontal lines are mapped to horizontal lines and vertical lines are mapped to vertical lines. The stochastic flow takes values in a 6-dimensional Lie subgroup (isomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$) of $\mathrm{Diff}(M)$. This is the only value of α for which ξ_t takes values almost-surely in a finite-dimensional subgroup of Diff(M) because in all other cases the Lie algebra generated by $\{V_1, V_2, V_3, V_4\}$ is infinite-dimensional. From the calculations in example 2 we see that for $x, y \in M$ with $x \neq y$

$$P\{\lim_{t\to\infty} \frac{1}{t} \log d(\xi_t(x), \xi_t(y)) = -\frac{1}{2}\} = 1$$

and that for $x \in M$

$$P\{\text{weak-}\lim_{t\to\infty} (\rho_t - \delta(\xi_t(x))) = 0\} = 1.$$

If $\alpha = \pi/2$ then we obtain the stochastic flow studied by Ikeda and Watanabe in [16]. In this case div $V_i = 0$ for $1 \leqslant i \leqslant 4$ so that ρ is invariant under the stochastic flow. See [16] for an exact characterisation of the support of the stochastic flow. For $x \neq y$ the two-point process $\{(\xi_t(x), \xi_t(y)) : t \geqslant 0\}$ on $M^2 \setminus D$ (where D denotes the diagonal $\{(u,v) \in M^2 : u = v\}$) has finite stationary probability measure $\rho \times \rho$. Hence the process $\{(\xi_t(x), \xi_t(y)) : t \geqslant 0\}$ on $M^2 \setminus D$ is recurrent on $M^2 \setminus D$ and so

$$P\{d(\xi_t(x)), \xi_t(y)) \to 0 \text{ as } t \to \infty\} = 0.$$

From the above discussion we have $\lambda_1 = \lambda_2 = -\frac{1}{2}$ when $\alpha = 0$, and $\lambda_1 + \lambda_2 = 0$ (so that $\lambda_1 = -\lambda_2 \geqslant 0$) when $\alpha = \pi/2$. We proceed to calculate λ_1 and λ_2 for general α. We shall write $\lambda_i(\alpha)$ to denote the dependence of λ_i on α. In view of Theorems 4.3 and 4.4 we wish to discover when $\lambda_1(\alpha) < 0$ and when $\lambda_1(\alpha) + \lambda_2(\alpha) = 0$.

We may treat (5.1) as an equation in $\mathbb{R}^2$. Converting to Itô form we obtain

$$d\xi_t(x) = \sum_{i=1}^{4} V_i(\xi_t(x))dW_t^i$$

since in this case the Itô correction term is identically zero. Differentiating with respect to x and writing $D\xi_t(x) = A_t$, we obtain

$$dA_t = \sum_{i=1}^{4} DV_i(\xi_t(x))A_t dW_t^i. \tag{5.2}$$

Calculating $DV_i(y)$ for $1 \leqslant i \leqslant 4$ and substituting in (5.2) we obtain

$$dA_t = \begin{bmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \end{bmatrix} A_t dU_t^1 + \begin{bmatrix} 0 & -\sin\alpha \\ 0 & \cos\alpha \end{bmatrix} A_t dU_t^2 \tag{5.3}$$

where

$$\left.\begin{aligned} dU_t^1 &= (\cos\xi_t^1(x))dW_t^1 - (\sin\xi_t^1(x))dW_t^2 \\ dU_t^2 &= (\cos\xi_t^2(x))dW_t^3 - (\sin\xi_t^2(x))dW_t^4 \end{aligned}\right\} \tag{5.4}$$

and $\xi_t(x) = (\xi_t^1(x), \xi_t^2(x))$. From (5.4) we see by Lévy's theorem that if $U_0^1 = U_0^2 = 0$ then $\{U_t^1 : t \geqslant 0\}$ and $\{U_t^2 : t \geqslant 0\}$ are Brownian motion processes. It follows from (5.3) that for this family of examples the law of $\{D\xi_t(x) : t \geqslant 0\}$ for fixed $x \in M$ is independent of x. Fix $v \in T_xM \cong \mathbb{R}^2$ with $v \neq 0$ and let $v_t = D\xi_t(x)(v) = A_t v$. Then

$$dv_t = \begin{bmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \end{bmatrix} v_t dU_t^1 + \begin{bmatrix} 0 & -\sin\alpha \\ 0 & \cos\alpha \end{bmatrix} v_t dU_t^2. \tag{5.5}$$

Write $v_t = r_t(\cos\theta_t, \sin\theta_t)$. Then by Itô's lemma we obtain from (5.5)

$$d\theta_t = -\cos\theta_t \sin(\theta_t - \alpha)dU_t^1 + \sin\theta_t \cos(\theta_t - \alpha)dU_t^2 + \tfrac{1}{2}(\sin(2\theta_t - \alpha)\cos(2\theta_t - \alpha) - \sin\alpha\cos\alpha)\,dt$$

and

$$d(\log r_t) = \cos\theta_t \cos(\theta_t - \alpha)dU_t^1 + \sin\theta_t \sin(\theta_t - \alpha)dU_t^2 + q_\alpha(\theta_t)dt$$

where

$$q_\alpha(\theta) = \tfrac{1}{2}\{-1 + \sin^2\alpha + \sin^2(2\theta - \alpha)\}. \tag{5.6}$$

Therefore $\{\theta_t : t \geqslant 0\}$ is a diffusion on S^1 with infinitesimal generator given by

$$L_1 = \tfrac{1}{4}(\sin^2\alpha + \sin^2(2\theta - \alpha))\frac{d^2}{d\theta^2} + \tfrac{1}{2}(\sin(2\theta - \alpha)\cos(2\theta - \alpha) - \sin\alpha\cos\alpha)\frac{d}{d\theta}$$

and so for $0 < \alpha \leqslant \frac{\pi}{2}$ it has a unique stationary probability measure μ_α, say, with smooth positive density g_α, say, with respect to Lebesgue measure. Also

$$\log r_t = \log r_0 + M_t + \int_0^t q_\alpha(\theta_s)ds$$

where M_t is a martingale with $\frac{d}{dt}\langle M\rangle_t \leqslant 1$. Therefore

$$\lim_{t\to\infty} \frac{1}{t}\log r_t = \lim_{t\to\infty} \frac{1}{t}\int_0^t q_\alpha(\theta_s)ds \quad \text{w.p.1}$$

$$= \int_{S^1} q_\alpha(\theta)d\mu_\alpha(\theta) \quad \text{w.p.1}. \tag{5.7}$$

Since (5.7) is valid for all $x \in M$ and $v \in T_xM \setminus \{0\}$ we have

$$\lambda_1(\alpha) = \int_{S^1} q_\alpha(\theta)d\mu_\alpha(\theta). \tag{5.8}$$

Formula (5.8) is a special case of Carverhill's formula. In general the integral is taken over SM, the unit sphere bundle of M. In our case $SM = M \times S^1$ and by the remark after (5.3) and (5.4) it turns out that both the integrand and the density of the measure depend only upon the second factor in the product $M \times S^1$.

When $\alpha = \frac{\pi}{2}$ then $g_\alpha(\theta) = k(\sin^2\alpha + \sin^2(2\theta - \alpha))^{-1/2}$ where k is chosen to ensure total mass 1. Then

$$\begin{aligned}\lambda_1(\tfrac{\pi}{2}) &= -\tfrac{1}{2} + E(2^{-1/2})/K(2^{-1/2})\\ &= 4\pi^2(\Gamma(\tfrac{1}{4}))^{-4}\\ &\sim 0\cdot 228\end{aligned}$$

where K and E denote complete elliptic integrals of the first and second kinds (see Whittaker and Watson [26]). For $0 < \alpha < \frac{\pi}{2}$ we do not have such a nice formula for $g_\alpha(\theta)$. Notice that for $0 < \alpha \leqslant \frac{\pi}{2}$ both $q_\alpha(\theta)$ and $g_\alpha(\theta)$, and hence also $\lambda_1(\alpha)$, depend analytically upon α. However as $\alpha \to 0$ the generator L_1 becomes singular and we need to investigate $g_\alpha(\theta)$ in more detail in order to describe the behaviour of $\lambda_1(\alpha)$ as $\alpha \to 0$.

Since the generator L_1 is invariant under the transformation $\theta \mapsto \theta + \frac{\pi}{2}$ it follows that the density g_α has period $\frac{\pi}{2}$. We shall give a formula for $g_\alpha(\theta)$ for $\frac{\alpha}{2} - \frac{\pi}{4} \leqslant \theta \leqslant \frac{\alpha}{2} + \frac{\pi}{4}$. Define

$$f_\alpha(\phi) = (\sin^2\alpha + \sin^2\phi)^{-1/2}\exp\left(\frac{\cos\alpha}{\sqrt{(1+\sin^2\alpha)}}\arctan\left(\frac{\sqrt{(1+\sin^2\alpha)}}{\sin\alpha}\tan\phi\right)\right) \tag{5.9}$$

for $-\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2}$. Then

$$g_\alpha(\theta) = k_\alpha(\sin^2\alpha + \sin^2(2\theta-\alpha))^{-1}(f_\alpha(2\theta-\alpha))^{-1}\left(1 + \ell_\alpha\int_{-\pi/2}^{2\theta-\alpha} f_\alpha(\phi)\,d\phi\right) \tag{5.10}$$

for $\frac{\alpha}{2} - \frac{\pi}{4} \leqslant \theta \leqslant \frac{\alpha}{2} + \frac{\pi}{4}$, where ℓ_α is chosen so that $g_\alpha(\frac{\alpha}{2} - \frac{\pi}{4}) = g_\alpha(\frac{\alpha}{2} + \frac{\pi}{4})$ and then k_α is chosen to ensure total mass 1. Since

$$\begin{aligned}1 \leqslant 1 + \ell_\alpha\int_{-\pi/2}^{2\theta-\alpha} f_\alpha(\phi)\,d\phi &\leqslant 1 + \ell_\alpha\int_{-\pi/2}^{\pi/2} f_\alpha(\phi)\,d\phi\\ &= \exp\left(\frac{\pi\cos\alpha}{\sqrt{(1+\sin^2\alpha)}}\right) \leqslant e^{\pi}\end{aligned}$$

we obtain from (5.9), (5.10)

$$\frac{k_\alpha e^{-\pi/2}}{\sqrt{(\sin^2\alpha + \sin^2(2\theta-\alpha))}} \leqslant g_\alpha(\theta) \leqslant \frac{k_\alpha e^{3\pi/2}}{\sqrt{(\sin^2\alpha + \sin^2(2\theta-\alpha))}}\ .$$

Therefore

$$\tfrac{1}{2}\, e^{-2\pi}\, \frac{E(\gamma)}{\gamma^2 K(\gamma)} \leqslant \lambda_1(\alpha) + \tfrac{1}{2} \leqslant \tfrac{1}{2}\, e^{2\pi}\, \frac{E(\gamma)}{\gamma^2 K(\gamma)}$$

where $\gamma = (1 + \sin^2\alpha)^{-1/2}$. Now as $\alpha \to 0$ then $\sqrt{1-\gamma^2} \sim \alpha$ and so

$\frac{E(\gamma)}{\gamma^2 K(\gamma)} \sim 1/\log(1/\alpha)$ (see [25], p.521). It follows that $\lambda_1 : [0,\frac{\pi}{2}] \to \mathbb{R}$ is continuous but not differentiable at $\alpha = 0$.

Returning to (5.3) we obtain by Itô's lemma

$$d(\log\det A_t) = (\cos\alpha)dU_t^1 + (\cos\alpha)dU_t^2 - (\cos^2\alpha)dt$$

so that

$$\lambda_1 + \lambda_2 = \lim_{t\to\infty} \frac{1}{t}\log\det A_t = -\cos^2\alpha. \tag{5.11}$$

Combining (5.6), (5.8) and (5.11) we have

$$\lambda_i(\alpha) = \frac{1}{2}\left(-\cos^2\alpha + (-1)^{i-1}\int_{S^1} \sin^2(2\theta-\alpha)d\mu_\alpha(\theta)\right). \tag{5.12}$$

The table gives values of $\lambda_i(\alpha)$ for α at intervals of 0·1 radians. Values of $\int_{S^1} \sin^2(2\theta-\alpha)d\mu_\alpha(\theta)$ were obtained by numerical integration and substituted into (5.12). We obtain $\lambda_1(\alpha) < 0$ for

α	$\lambda_1(\alpha)$	$\lambda_2(\alpha)$
0	-0·5	-0·5
0·1	-0·346	-0·644
0·2	-0·307	-0·653
0·3	-0·268	-0·641
0·4	-0·226	-0·622
0·5	-0·179	-0·591
0·6	-0·129	-0·552
0·7	-0·077	-0·508
0·8	-0·024	-0·462
0·9	0·029	-0·415
1·0	0·078	-0·370
1·1	0·123	-0·328
1·2	0·161	-0·292
1·3	0·192	-0·263
1·4	0·214	-0·243
1·5	0·226	-0·231
$\pi/2$	0·228	-0·228

$\alpha < 0{\cdot}845 = 0{\cdot}269\pi$ radians. From (5.11) we have $\lambda_1(\alpha) + \lambda_2(\alpha) < 0$ for $\alpha < \frac{\pi}{2}$, so that $\alpha = \frac{\pi}{2}$ is the only case in which the measure ρ is invariant