MATHEMATICS FOR ELEMENTARY TEACHERS

Eugene F. Krause

FOR ELEMENTARY TEACHERS

Eugene F. Krause

University of Michigan
Ann Arbor

PRENTICE-HALL, INC., ENGLEWOOD CLIFFS, NEW JERSEY 07632

Library of Congress Cataloging in Publication Data

Krause, Eugene F (date)
Mathematics for elementary teachers.

Includes index.
1. Mathematics—1961- I. Title.
QA39.2.K69 372.7 77-7188
ISBN 0-13-562702-8

MATHEMATICS FOR ELEMENTARY TEACHERS

Eugene F. Krause

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Prentice-Hall International, Inc., *London*
Prentice-Hall of Australia Pty. Limited, *Sydney*
Prentice-Hall of Canada, Ltd., *Toronto*
Prentice-Hall of India Private Limited, *New Delhi*
Prentice-Hall of Japan, Inc., *Tokyo*
Prentice-Hall of Southeast Asia Pte. Ltd., *Singapore*
Whitehall Books Limited, *Wellington, New Zealand*

PREFACE

This book is my answer to the question, "What do you do in a first mathematics course for elementary school teachers?" It evolved during ten years of teaching such a course and trying to mold it to the students' and my own satisfaction.

The choice of content was pragmatic and straightforward. The arithmetic of the elementary grades, from whole numbers through integers and rational numbers, is covered thoroughly. The treatment of the real numbers is sketchier, but adequate for the purposes of elementary teachers. The basic topics from geometry that appear in elementary texts are investigated. These include simple geometric figures and relations; the common tools of geometry; and the fundamental notions of congruence, measurement, symmetry, and similarity. The minimal algebra and number theory of the elementary grades are also covered, along with accessible and appropriate ideas from such areas as sets and functions, proof, graph theory, flow charting, and mathematical systems. Finally, the steady march of probability and statistics down through the junior high school grades makes it desirable that elementary teachers have some understanding of those topics. The basic ideas are covered in one self-contained chapter.

The approach to this content was influenced greatly by the course structure and students at the University of Michigan. The fact that we have but one required mathematics course for elementary teachers virtually dictated that essential ideas from the several mathematical areas mentioned above be integrated with the main topic of arithmetic—lest they not get covered at all. Such integration may or may not be an absolute pedagogical virtue. In our situation, however, it has worked out satisfactorily.

Students in our classes always range from very weak to very strong. For the latter people, there are quite a few challenging starred problems in this book. There are also quite a few simple problems, not unlike those found in elementary school texts. Differential assignments are easy to make.

Although our students take a separate "methods" course in the school of education, there is no point in trying to create a pure "content" course. It is important that connections be noted between the mathematics that these future teachers study and the mathematics they will teach. Consequently, methods creep in throughout the book and at times seem to take over almost completely.

Most of our students have had two or more years of high school mathematics. They may be a bit rusty and mathematics may be rather low on their lists of favorite things, but still they deserve a college-level course. It is demeaning and a great disservice to such students to give them a dull remedial course that ranges over some list of minimal survival skills. It is far better to give them an interesting, intellectually stimulating course that provides some perspective on mathematics. More and more mathematics educators are beginning to agree that the future teacher should be able to look at arithmetic from three viewpoints: (1) external meanings, where all the symbols have concrete references in the real world (This is where mathematics is born and where it returns in applications.); (2) algorithms, where the symbols are manipulated in mechanical, well-rehearsed ways; and (3) mathematical systems, where the internal harmony and basic simplicity of each mathematical structure is perceived. In our course we have used these three viewpoints as a framework on which to build mathematical understandings.

One personal pedagogical principle guided the writing of this text: There is no royal road to anywhere. To understand mathematics one must do mathematics. The lifeblood of any text is its exercise lists. Even the most idiosyncratic and bizarre exposition can be tolerated if the problems are good. Perhaps that is why most of the exercises in this book were written three or four years before the text itself.

A few words about the size of the book and the Contents seem to be in order. First, although the book is based on lectures from a one-semester course, it would be an extraordinary class that could cover everything in such a short time. In translating lectures into print some form of Parkinson's Law seems to come into effect. More thorough explanations, additional illustrative material, and topics that one would love to explore "if only there were time" find their way in and never find their way out again. My own experience is that Chapters 1 through 7 by themselves fill up one semester.

By omitting much of Chapter 5 I have been able to steal an extra week in which to preview sketchily a few of the most basic ideas from Chapters 8 and 9, but I have never touched Chapter 10 in the first course. Alternatively one could buy additional time for Chapters 8 and 9 by de-emphasizing the theoretical material of Chapter 4 and sections 6.3, 6.4, 6.5, 7.4, and 7.5. If one had the luxury of a two-term course, then the entire book could be covered thoroughly.

Second, the Contents is somewhat deceptive because of the integrated approach. Chapter titles tend to be milestones along the main arithmetic path, but in every chapter there is much more than just arithmetic. In particular there is a lot of geometry; it just is not segregated in the canonical two chapters at the end of the text. Sometimes there is a bloc of geometry that is big enough to warrant a full section. In one instance geometry dominates an entire chapter (Chapter 9). Often, however, the geometric thread is carried along in sections bearing titles that give no hint of its presence. Many instructors, particularly those who offer a follow-up course in geometry, prefer to minimize the geometric content of their first course. This can be accomplished by omitting sections 1.2, 3.2, 3.6, 6.6, and 6.7, Chapter 9, and the obviously geometric exercises that are sprinkled throughout the book.

I would like to thank the Harper & Row Publishing Company for their permission to use certain material from my book *A Second Mathematics Course for Elementary Teachers* (Harper and Row, 1974). I would also like to acknowledge three groups that influenced the writing of this book: my colleagues in mathematics-education at the University of Michigan, particularly Professor Charles Brumfiel who has worked most closely with me; the Comprehensive School Mathematics Project, St. Louis, Missouri; and finally, the many students and teaching assistants in Math 385, 1968–1977.

EUGENE F. KRAUSE

Ann Arbor, Michigan

TO THE STUDENT

To be a good teacher of elementary school mathematics it is necessary to know a good deal of mathematics. Willingly or not the elementary teacher in a self-contained classroom is the mathematician in residence. As such she or he must be able to interpret and supplement the text, answer offbeat mathematical questions, guide students into productive patterns of thought by *asking* appropriate questions, offer insights and illuminating examples, exhibit a genuine enthusiasm for the subject, challenge the strong, and devise alternative strategies for teaching the weak. To do any of these things requires an understanding of mathematics that goes well beyond grade level.

Mathematical competence, however, is not enough. We have all met people who "knew their mathematics," but couldn't teach it because they couldn't get down to the students' level. An illustration of this difficulty is furnished by two solutions to the following word problem. Before looking at the solutions you might find it interesting to work the problem yourself.

Brumfiel's Problem. In a class of 60 children there are $\frac{2}{3}$ as many boys as girls. How many boys are there?

Solution 1. Begin by introducing two unknowns: let b be the number of boys and g be the number of girls. Next set up a pair of simultaneous equations.

$$b + g = 60$$
$$b = \tfrac{2}{3}g$$

Now solve the pair of equations by substituting from the second into the first

$$\tfrac{2}{3}g + g = 60$$

simplifying

$$\tfrac{5}{3}g = 60$$

multiplying both sides by $\frac{3}{5}$

$$g = \tfrac{3}{5} \cdot 60 = 36$$

and substituting again, this time into the second equation

$$b = \tfrac{2}{3}g = \tfrac{2}{3} \cdot 36 = 24$$

This is a very nice, systematic solution; just the sort one hopes for in an Algebra I class. The problem, however, could be posed to fourth graders. After all, it involves only a small whole number and a common fraction. For fourth graders, solution (1) would be incomprehensible. For them, the following solution would make more sense.

Solution 2. The phrase *there are* $\frac{2}{3}$ *as many boys as girls* means that for every 2 boys in the class there are 3 girls. Thus, one can begin to draw a picture of the class as follows:

b	*b*	*b*	*b*. . .
b	*b*	*b*	*b*. . .
g	*g*	*g*	*g*. . .
g	*g*	*g*	*g*. . .
g	*g*	*g*	*g*. . .

From this picture, one reasons as follows: There are 60 children in the class and 5 rows of children, so there are 12 children in each row. The boys in the class make up 2 of these rows. Therefore the number of boys is

$$2 \times 12 = 24$$

As an elementary school teacher it will, of course, be important that you be able to solve problems, but as the foregoing example shows, *how* you solve them also will be important.

In this text the number systems of the elementary grades—whole numbers, integers, rational numbers—will be studied from three different points of view. The first will be that of the child in which there are concrete *external meanings* attached to the numbers and the arithmetic operations. This point of view will lead to simple graphical techniques, like solution (2) just completed, for solving word problems. The second point of view will emphasize computational techniques, or *algorithms*, for calculating answers to purely mathematical problems. This is the viewpoint that seems to dominate most of the elementary years. Finally, the numbers and the arithmetic operations will be viewed as *mathematical systems*—abstract structures that have their own internal harmony and logic. This third point of view will be of less practical use to you in the classroom than the first two, but it should help you to organize things in your own mind, give you a clearer perspective of the whole field of arithmetic, and indicate what is down the road for your students in their future schooling.

While the study of number systems from the three viewpoints just mentioned provides the framework for the course, many other topics from elementary mathematics will appear as well. It is pointless to try to present a "pure" arithmetic course. There are times when it is illuminating to relate the arithmetic to some simple ideas of geometry, or algebra, or measurement. And, of course, these are practical topics to study for their own sake since they too occupy a place in the elementary school curriculum.

Our analysis of elementary mathematics will begin with Chapter 2 after we have sharpened some tools in Chapter 1.

CONTENTS

2

THE WHOLE NUMBERS: EXTERNAL MEANINGS 54

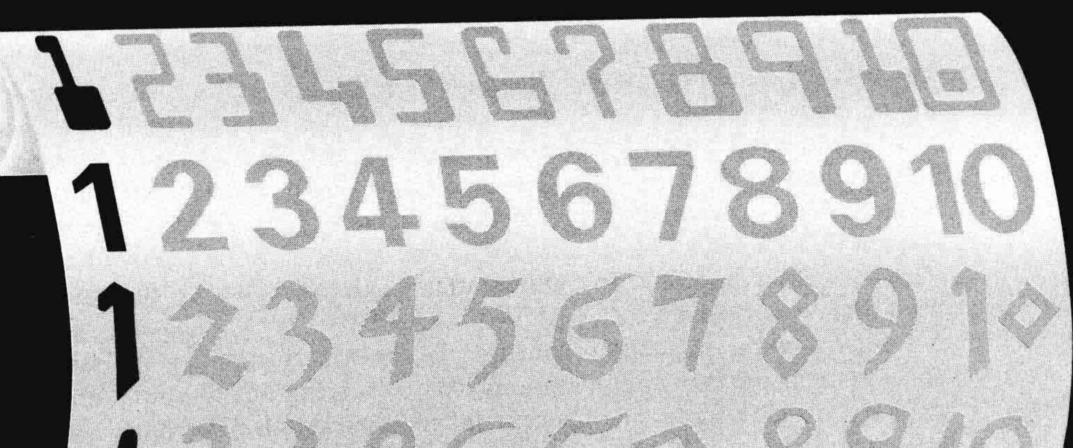
3

THE WHOLE NUMBERS: ALGORITHMS 99

4

THE WHOLE NUMBERS: MATHEMATICAL SYSTEMS 147

SETS AND FUNCTIONS



APOLOGY

Two major mathematical concepts that give unity and coherence to elementary mathematics are those of set and function. Most high school graduates have had considerable contact with both of these ideas, so much of this chapter might be review. There will be some new ideas, though, and it will be important in future chapters that we all agree on a standard terminology and symbolism. Furthermore, woven into this introductory material on sets and functions are some basic notions from geometry and algebra on which subsequent work will build. Thus it would be dangerous to skip this chapter. If indeed most of the ideas are familiar, you might endure the review by looking at each topic and asking yourself, "How would I teach that to kids?" This is a practical question to ponder because the ideas of set and function have secured themselves a niche in most elementary school mathematics texts.

1.1 SETS: THE BASIC CONCEPTS

The concept of **set** is a very general and a very simple one.

Any collection of things is a set.

The things can be tangible or intangible: a set of golf clubs, a set of dishes, a set of rules for how to play checkers. In certain contexts more colorful words than *set* are used to refer to collections: a *pod* of whales, a *sounder* of hogs, a *muster* of peacocks. On an election night a few years ago, when Rockefellers were leading in elections in New York, West Virginia, and Arkansas, Eric Severeid remarked that we might soon be confronted by "a *wealth* of Rockefellers." In mathematical contexts, however, the word *set* is used almost exclusively.

Surrounding the basic concept of set are a number of associated concepts and notational conventions. We can bring out the most important of these by looking at one specific set in some detail. We have chosen to investigate the set of U.S. Presidents, 1940–1970, which we shall denote by *P*.

Membership

The things that make up the set *P* are: FDR, Truman, Ike, JFK, LBJ, and Nixon. Rather than call them *things* we will refer to them as **members**, or **elements**, of *P*. The conventional shorthand way of reporting that "Ike is a member of P" is to write

$$\text{Ike} \in P$$

That is, the symbol for membership is $\in$, which can be read "is a member of" or "belongs to."

> The symbolic expression $a \in A$ indicates that a is a member of the set A.

A slash is used to indicate nonmembership. For example,

$$\text{Ford} \notin P$$

means that Ford is not a member of P.

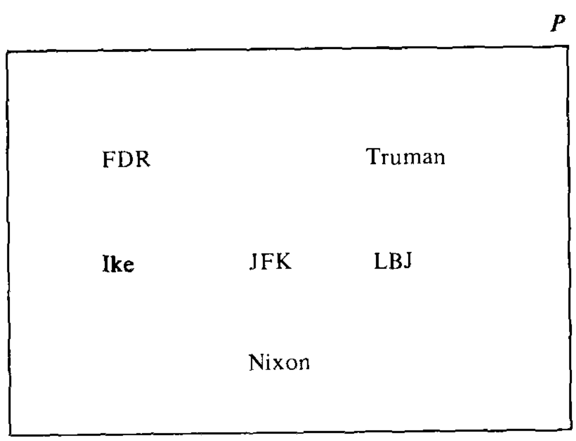
Notation for Sets

The most common notation for sets is **roster** notation. To describe a set in this notation one simply lists the elements of the set within a pair of braces. For our example,

$$P = \{\text{FDR, Truman, Ike, JFK, LBJ, Nixon}\}$$

The braces are pronounced "the set consisting of," so that the symbols above are read "P equals the set consisting of FDR, Truman, Ike, JFK, LBJ, and Nixon." It may seem fastidious to insist on braces when listing a set, but this is the notation that is universally used and understood. Ordinary round parentheses will carry other meanings as we shall soon see.

Another way of describing a set is by means of a **Venn diagram**. In its simplest form a Venn diagram for a set is just a rectangle or circle or other convenient figure with the elements of the set listed inside:



Important variations on the basic Venn diagram will be discussed shortly.