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Edited by A. Dold and B. Eckmann

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Gauss Sums and p-adic Division Algebras

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INTRODUCTION

Let F be a finite field extension of the p-adic rational field $\mathbb{Q}_p$, and let D be a central F-division algebra of finite dimension n^2. These notes have two principal aims. The first of these is to develop the theory of a congruence Gauss sum $\tau(\pi)$ attached to an irreducible admissible representation π of the multiplicative group $D^\times$ of D. For the second, we assume

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that the index n of the division algebra D is not divisible by the residual characteristic p of F. In this situation, Corwin and Howe [3] have constructed a bijective correspondence between these representations π and continuous irreducible representations σ, of degree dividing n, of the Weil group $\mathcal{W}_F$ of F. Koch and Zink then took up the subject again, and gave a more complete account in [10]. Our second aim is then to derive a precise comparison between the constant $W(\pi)$ in the Godement-Jacquet functional equation attached to π and the Langlands local constant $W(\sigma)$ of the corresponding representation σ of $\mathcal{W}_F$. We shall see that the root numbers $W(\pi)$ and $W(\sigma)^{n/\dim(\sigma)}$ differ at most by a 4-th root of unity factor which can be written down explicitly. This enables us to introduce certain (not entirely canonical) "twists" of the Corwin-Howe-Koch-Zink correspondence to obtain another correspondence which satisfies in full the postulates of Langlands' philosophy (see Theorem (11.3.4) below).

The crucial link between these two aims is the fact that the constants $W(\pi)$ can be computed in terms of the Gauss sums $\tau(\pi)$, just as in the classical situation of local fields treated in Tate's thesis [15]. (This holds in full generality: we do not need here the hypothesis that n is relatively prime to p.)

In describing the correspondence between representations π of $D^\times$ and representations σ of $\mathcal{W}_F$ when p does not divide n, we mainly follow the paper of Koch and Zink [10]. However, we have to analyse the separate stages in more detail. This, together with the necessary representation-theoretic background, takes up a major portion of the notes. (It should be noted that, in the main body of the notes, we actually work with a correspondence between "finite" representations of $D^\times$ and representations of the absolute Galois group Ω_F of F, rather than $\mathcal{W}_F$. This is for convenience only, and does not really affect the results. For the transition to Weil groups, see (5.5) below.)

The new congruence Gauss sums arise as a non-abelian generalisation of the classical Gauss sums for quasicharacters of local fields. Via class field theory, the latter can be considered as attached to abelian representations of $\mathcal{W}_F$. This point of view has led to another non-abelian generalisation, the so-called Galois Gauss sums (see [8], [12], [5], [17], [4], [6]). Just as our congruence Gauss sums provide a formula for $W(\pi)$, so the Galois Gauss sums determine the Langlands constant $W(\sigma)$. In both cases, the Gauss sum also determines the conductor. This parallel between two kinds of Gauss sum, which in the abelian case reduces to a consequence of class field theory, is quite analogous to, but in a sense independent of (and simpler than), the parallel between two kinds of L-functions which is central to Langlands' philosophy.

Apart from the connections between the two kinds of Gauss sum which will be established here via the correspondence between representations, there are - in full generality - some startling similarities in their properties, e.g. as regards congruence behaviour and Galois action: compare our § 2 with [12] or [6]. Nevertheless, in some ways, they are objects of two essentially different sorts. A congruence Gauss sum is given quite explicitly as the eigenvalue of a certain scalar operator attached to a representation which is necessarily irreducible. On the other hand, the Galois Gauss sums should be viewed as the values of a certain homomorphism of the additive group of virtual characters, which possesses important Frobenius induction properties. Moreover, one of the deepest results on Galois Gauss sums asserts that, at least in the tame case, this homomorphism is essentially a group determinant (see [17]).

In another direction, congruence Gauss sums can be defined in the context of arbitrary p-adic simple algebras (of finite dimension). Here, however, a number of new phenomena come into play, and these demand a separate treatment. It is clear to us that many of the results and techniques of this paper apply in the broader set-up, but many of the details remain to be worked out.

In connection with this topic, the work of Lamprecht should be acknowledged. In 1957, he introduced Gauss sum matrices associated with representations of the multiplicative group of a finite ring. Doubtless, our congruence Gauss sums could be defined as arising from a specialisation of Lamprecht's matrices. Of course, such a specialisation was necessary in order to get objects with the strong arithmetic properties we need. In particular, one must remember that the Godement-Jacquet functional equation was unknown at the time of Lamprecht's papers [11].

We now give a brief survey of the contents of the individual sections. §1 is introductory: it recalls the basic concepts and fixes some notation. In §2, the theory of congruence Gauss sums is developed. Only a part of this section is needed in the remainder of the present notes. The Gauss sum form of the functional equation is established in §3. In §4, we show, without as yet any restriction on n, that for abelian representations the Godement-Jacquet constant and the classical local constant for the centre essentially coincide. Much of the material of §4 has already appeared in the different context of [1]. It was originally written for this work, and we retain it for the sake of completeness.

From §5 on, we assume that n is not divisible by p. §5 itself is a survey of [10], and elaborates the division of the Corwin-Howe correspondence into separate stages. §6 and §7 give more detail, and contain the basic Gauss sum computations. An irreducible representation π of $D^{\times}$ is determined by an abelian representation χ of $A^{\times}$, for a certain subalgebra A of D. At the end of §7, we have enough information to compare $W(\pi)$ and $W(\chi)$, except, in one case, for an undetermined sign. To dispose of this, we need a detour into abstract representation theory (in §8), followed by a laborious calculation in §9. The character χ is given by a character ϕ of the centre, K say, of A. The relation between $W(\chi)$ and $W(\phi)$ has already been worked out in §4. The representation σ of $\mathcal{W}_F$ corresponding to π is then induced from ϕ (viewed as a character of W_K via class field theory). One knows that $W(\phi)$ and $W(\sigma)$ differ by a certain "induction constant" which is a 4-th root of unity depending only on the field extension K/F. We work out these constants explicitly in §10, except, in one case, for a sign, which the reader can look up in [6]. In §11, we assemble all the partial results to get the relation between $W(\sigma)$ and $W(\pi)$.

It was already known that the correspondence between representations of $D^{\times}$ and $\mathcal{W}_F$ given in [3] and [10] was inconsistent with Langlands' philosophy in one important respect concerned with the relation between the restriction ω_{π} of π to the centre $F^{\times}$ and the determinant character $\det(\sigma)$ of σ. Our root number calculations give another inconsistency. The explicit formulas of §11 make it clear that these two problems are aspects of the same phenomenon, which again arises when one compares the field-theoretic properties of the corresponding Gauss sums. One is then led to try to modify the correspondence by "twisting" the various steps of the construction by tame characters of the fields involved. We pursue this idea systematically in §12. We exhibit one modified correspondence, which seems to be the simplest available, but which cannot in the present context be called canonical. However, it does at least provide an example of a bijection between representations of $D^{\times}$ and of $\mathcal{W}_F$ which satisfies in full the postulates of Langlands' philosophy.

Finally, we would like to thank Mrs. Joan Bunn for retyping this manuscript for publication with her usual efficiency and style.

CONTENTS

§1 Arithmetic of local division algebras

(1.1) We start by fixing some of our basic notations, and recalling a few elementary facts about division algebras over local fields. Everything in this section is both well-known and easily verified: [13] and [18] are convenient general references.

Let F be a non-Archimedean local field of characteristic zero and residual characteristic p. Thus F is a finite field extension of the p-adic rational field $\mathbb{Q}_p$. We let

(1.1.1)
$$\begin{aligned} \mathfrak{o}_F &= \text{the valuation ring in } F,\\ \mathfrak{p}_F &= \text{the maximal ideal of } \mathfrak{o}_F,\\ \bar{F} &= \mathfrak{o}_F/\mathfrak{p}_F, \text{ the residue class field of } F,\\ q = q_F &= N\mathfrak{p}_F = |\bar{F}|, \text{ the cardinality of } \bar{F}. \end{aligned}$$

We write $R^\times$ for the group of invertible elements of a ring R. The unit group $\mathfrak{o}_F^\times$ has a filtration

$$\mathfrak{o}_F^\times \supset 1 + \mathfrak{p}_F \supset 1 + \mathfrak{p}_F^2 \supset \dots ,$$

and we denote these subgroups by

(1.1.2) $$U_0(F) = \mathfrak{o}_F^\times, \quad U_i(F) = 1 + \mathfrak{p}_F^i, \ i \geq 1.$$

We also write

(1.1.3) $$\nu_F\colon F^\times \to \mathbb{Z}$$

for the canonical (surjective) valuation of F, and

(1.1.4) $$\|x\|_F = q_F^{-\nu_F(x)}, \quad x \in F^\times.$$

If we have a finite field extension E/F, we use analogous notations, and write $N_{E/F}$, $Tr_{E/F}$ for the relative norm and trace respectively.

The field F has a canonical continuous additive character ψ_F defined by

(1.1.5) $$\psi_F = \psi_{\mathbf{Q}_p} \cdot Tr_{F/\mathbf{Q}_p},$$

where $\psi_{\mathbf{Q}_p}$ is the composition of canonical maps

$$\mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p \to \mathbf{Q}/\mathbf{Z} \to \mathbf{C}^\times,$$

the last of these being $x \mapsto e^{2\pi i x}$. The pairing $(x,y) \mapsto \psi_F(xy)$, $x,y \in F$, in nondegenerate, and may be used to identify the locally compact abelian group F with its Pontrjagin dual $\hat{F}$.

It is useful to note that the largest $\underline{\underline{o}}_F$-lattice contained in the kernel of ψ_F is $\underline{\underline{D}}_F^{-1} = \underline{\underline{D}}_{F/\mathbf{Q}_p}^{-1}$, the inverse of the absolute different of F.

(1.2) Now let D be a finite-dimensional central F-division algebra, with, say, $n^2 = \dim_F(D)$. The homomorphism $n\nu_F : F^\times \to \mathbf{Z}$ extends to a surjective homomorphism

(1.2.1) $$\nu_D : D^\times \to \mathbf{Z}$$

which is indeed a valuation. The set

(1.2.2) $$\underline{\underline{O}}_D = \{x \in D : \nu_D(x) \geq 0\}$$

is a ring, with the usual convention $\nu_D(0) = \infty$, and it is the unique maximal order in D. This ring has a unique maximal ideal

(1.2.3) $$\underline{\underline{P}}_D = \{x \in D : \nu_D(x) \geq 1\},$$

and moreover any left (or right) $\underline{\underline{O}}_D$-lattice spanning D over F is of the form

$$\underline{\underline{P}}_D^i = \{x \in D : \nu_D(x) \geq i\},$$

for some uniquely determined $i \in \mathbb{Z}$. In particular, it is a 2-sided fractional ideal of $\underline{\underline{O}}_D$.

The residue class ring

(1.2.4) $$\bar{D} = \underline{\underline{O}}_D/\underline{\underline{P}}_D$$

is a field, and indeed an extension of $\bar{F}$ of degree n.

We again have a chain of subgroups

$$\underline{\underline{O}}_D^\times \supset 1 + \underline{\underline{P}}_D \supset 1 + \underline{\underline{P}}_D^2 \supset \dots ,$$

each of them compact, open, and normal in $D^\times$, and we denote them by

(1.2.5) $$U_0(D) = \underline{\underline{O}}_D^{\times}, \quad U_i(D) = 1 + \underline{\underline{P}}_D^i, \quad i \geq 1.$$

We have canonical isomorphisms

$$U_0(D)/U_1(D) \cong \bar{D}^{\times},$$

$$U_i(D)/U_{i+1}(D) \cong \underline{\underline{P}}_D^i/\underline{\underline{P}}_D^{i+1}, \quad i \geq 1,$$

and therefore, for $i \geq 1$, $U_i(D)/U_{i+1}(D)$ is an elementary abelian p-group of order q_F^n.

We write

(1.2.6) $$\mathrm{Nrd}_D : D^{\times} \to F^{\times}, \quad \mathrm{Trd}_D : D \to F$$

for the reduced norm and trace respectively. It is also convenient to have the notation

(1.2.7) $$N_D\underline{\underline{A}} = |\underline{\underline{O}}_D/\underline{\underline{A}}|,$$

for an $\underline{\underline{O}}_D$-ideal $\underline{\underline{A}}$. Thus

$$N_D\underline{\underline{P}}_D^i = q_F^{ni}, \quad i \geq 0.$$

We frequently omit the D's from these notations when there is no danger of confusion.

The definition

(1.2.8) $$\psi_D = \psi_F \cdot \mathrm{Trd}_D$$

provides a canonical continuous additive character of D, and the pairing $D \times D \to \mathbb{C}^\times$ given by

(1.2.9) $$(x,y) \mapsto \psi_D(xy)$$

is nondegenerate. It may be used to identify D with its Pontrjagin dual $\hat{D}$.

The largest $\underline{\underline{O}}_D$-lattice contained in the kernel of ψ_D is the inverse $\underline{\underline{D}}_D^{-1}$ of the absolute different of D. One verifies easily that

(1.2.10) $$\underline{\underline{D}}_D = \underline{\underline{P}}_D^{n-1} \underline{\underline{D}}_F.$$

Now let i,j be integers, with $1 \leq i \leq j \leq 2i$. Then, for $x,y \in \underline{\underline{P}}_D^i$, we have

$$(1 + x)(1 + y) \equiv 1 + x + y \pmod{\underline{\underline{P}}_D^j}.$$

It follows that the group $U_i(D)/U_j(D)$ is abelian, and canonically isomorphic to $\underline{\underline{P}}_D^i/\underline{\underline{P}}_D^j$.

We can also use the character ψ_D to obtain a very useful description of the Pontrjagin dual $(U_i/U_j)^\wedge$ of the finite abelian group U_i/U_j. For, let $\gamma \in \underline{\underline{D}}_D^{-1}\underline{\underline{P}}_D^{-j}$, and consider the map

$$\theta_\gamma : 1 + x \mapsto \psi_D(\gamma x), \ x \in \underline{\underline{P}}_D^i.$$

This is a homomorphism $U_i \to \mathbb{C}^\times$ which is trivial on U_j. Moreover, θ_γ is the