Operator Theory Operator Algebras and Applications

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Preface

During the last twenty years operator theory has come of age. The subject has developed in several directions, using new and powerful methods that have led to the solution of basic problems thought to be inaccessible in the sixties. Some of these developments have made mutually enriching contact with other areas of mathematics, including algebraic topology and index theory, complex analysis in one and several variables, and probability theory.

This period has seen the full characterization of quasitriangular operators in terms of the Fredholm index, the classification of families of essentially normal operators via C^*-algebraic extensions, and the consequences of the latter subject culminating in the unification of C^*-algebraic K-homology and K-cohomology in the Kasparov KK-bifunctor. The invariant subspace problem has been solved for subnormal operators and related classes. The classical Weyl-von Neumann theorem has been vastly generalized to separable C^*-algebras. The Ringrose problem on the multiplicity of nests of subspaces has been solved, using algebraic methods that have provided striking insight into the behavior of operators under similarity transforms. The classical perturbation theory for Schrödinger operators has been transformed and simplified by the use of path integrals and the Feynman-Kac formula. A rich theory of completely positive and completely bounded maps of C^*-algebras has emerged, and this has had significant implications for operator theory, including dilation theory, the characterization of operators having annular spectral sets, and the partial solution of Sz.-Nagy's problem. The C^*-algebras of Toeplitz operators associated with a large class of domains in C^n are now clearly understood. Finally, the structure of broad classes of reflexive operator algebras has been penetrated and put into the context of "noncommutative" spectral synthesis.

These algebraic methods are diverse, and they touch upon a broad area of mathematics. In addition to the interrelations alluded to above, there have been direct applications to systems theory, complex variables, and statistical mechanics. Moreover, significant problems and motivations have arisen from the subject's traditional underpinnings for partial differential equations. While it would not be possible or perhaps even desirable to attempt to unify these results and methods, it seemed an appropriate time to summarize progress and examine the common points of view that now run through the subject. Thus, we organized the 1988 AMS Summer Institute on Operator Theory/Operator Algebras and Applications. The present two volumes contain papers representing most of the invited talks, as well as many of the seminar talks that were given there. We think that the points made above are amply borne out in these *Proceedings*.

William B. Arveson
Ronald G. Douglas

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PART 2

Proceedings of Symposia in Pure Mathematics
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The Spectral C^*-Algebra of an E_0-Semigroup

WILLIAM ARVESON

Introduction. An E_0-*semigroup* is a one-parameter semigroup $\{\alpha_t : t \geq 0\}$ of normal *-endomorphisms of the algebra $\mathscr{B}(H)$ of all bounded operators on a separable Hilbert space H, satisfying $\alpha_t(1) = 1$ for $t \geq 0$, and such that $\langle \alpha_t(A)\xi, \eta \rangle$ is continuous in t for fixed $A \in \mathscr{B}(H)$ and ξ, $\eta \in H$. E_0-semigroups were introduced by Powers [**10**], and their theory has been undergoing development by Powers [**11**], Powers and Robinson [**13**], Powers and Price [**12**], Price [**16**], and the author [**1, 2, 4, 5**].

E_0-semigroups occur naturally in a number of ways. For example, if we are given a one-parameter unitary group $\{U_t : t \in \mathbf{R}\}$ acting on a separable Hilbert space H, and a type-I subfactor M of $\mathscr{B}(H)$ which is *invariant* in the sense that $U_t M U_t^*$ is contained in M for every $t \geq 0$, then we obtain *two* semigroups α, β acting respectively on M and its commutant M' by

$$\alpha_t(A) = U_t A U_t^*, \qquad A \in M,\ t \geq 0,$$
$$\beta_t(B) = U_t^* B U_t, \qquad B \in M',\ t \geq 0.$$

If we realize the type-I factors M, M' as $\mathscr{B}(K)$, $\mathscr{B}(K')$, respectively, then of course α and β are seen to be E_0-semigroups. More specifically, if one is given a system of local observables which is acted upon by the inhomogeneous Lorentz group in such a way that the Haag-Kastler axioms are satisfied ([**9**, p. 99]), then it is a simple matter to write down nontrivial examples of pairs $\{U_t\}$, M satisfying the above conditions. Some more elementary constructions of E_0-semigroups are described in [**3**] and [**10**].

It is correct to think of E_0-semigroups as quantized versions of semigroups of isometries [**3**], but one must not push this analogy too far. For example, by the Wold decomposition, every semigroup of isometries $\{U_t : t \geq 0\}$ acting on a Hilbert space decomposes uniquely into a direct sum

$$U_t = V_t \oplus W_t,$$

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where W is a semigroup of unitary operators and V is the direct sum of a number n of copies of the natural semigroup of shifts acting on $L^2(0, \infty)$. The pair (n, W) completely determines U to unitary equivalence in the sense that if

$$U'_t = V'_t \oplus W'_t$$

is another semigroup of isometries decomposed in the above way, then U is unitarily equivalent to U' iff $n = n'$ and W is unitarily equivalent to W'.

This might lead one to suspect that the problem of classifying E_0-semigroups up to conjugacy should be similar in that (a) a given E_0-semigroup should decompose into a tensor product of a "pure" E_0-semigroup with a semigroup of *-automorphisms, and (b) the pure E_0-semigroups should have a relatively simple classification to within conjugacy (an E_0-semigroup α is said to be *pure* if the family of von Neumann algebras $M_t = \alpha_t(\mathcal{B}(H))$ decreases to $\mathbf{C}1$ as t tends to infinity). It is interesting that these naive assertions are both entirely wrong. It is known, for example, that there exist E_0-semigroups α having the property that $\bigcap\{\alpha_t(\mathcal{B}(H)) : t \geq 0\}$ is a factor of type II or III. This implies that nothing like (a) can be true. More significantly, Powers has shown that, in addition to the natural examples of pure E_0-semigroups, there is an enormous variety of others which are not conjugate (or even cocycle-conjugate) to these elementary ones [**11**].

These remarks show that, unlike the theory of one-parameter groups of automorphisms of $\mathcal{B}(H)$, the theory of semigroups of endomorphisms of $\mathcal{B}(H)$ is rather subtle. In [**3**], we summarized the results of some recent work on an index theory appropriate for E_0-semigroups. Those results relate to the problem of classifying E_0-semigroups up to cocycle-conjugacy. The purpose of this paper is to summarize progress on *spectral* invariants of E_0-semigroups, with emphasis on problems that remain open. Here, the appropriate notion of spectrum is a noncommutative topological space, namely a separable C^*-algebra which has the "correct" representation theory.

1. Preliminaries. Our approach to the theory of E_0-semigroups is based on the notion of continuous tensor product systems (product systems, for short). A *product system* is a measurable family

$$p: E \to (0, \infty)$$

of nonzero separable Hilbert spaces over the open interval $(0, \infty)$, on which there is defined an associative multiplication which *acts like tensoring*. This means that the operation is measurable and bilinear on fiber spaces, and such that for each s, $t > 0$, the fiber space $E_{s+t} = p^{p-1}(s + t)$ is spanned by all products $\{xy : x \in E_s, y \in E_t\}$, and we have

$$\langle xy, x'y' \rangle = \langle x, x' \rangle \langle y, y' \rangle,$$

for all x, x' in E_s and all y, y' in E_t. We will write $\{E_t : t > 0\}$, or simply E, for a product system $p: E \to (0, \infty)$.

Every E_0-semigroup is associated with a canonical product system. Indeed, let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $\mathcal{B}(H)$. For every $t > 0$, let E_t be the complex vector space of operators defined by

$$E_t = \{A \in \mathcal{B}(H) : \alpha_t(T)A = AT, T \in \mathcal{B}(H)\}.$$

We may define an inner product in E_t by the formula

$$B^* A = \langle A, B \rangle 1, \qquad A, B \in E_t,$$

and this defines a family of Hilbert spaces. The total space of this family is the set of ordered pairs $E = \{(t, A) : A \in E_t, t > 0\}$, and the projection

$$p : E \to (0, \infty)$$

is defined by $p(t, A) = t$. If we use operator multiplication to define a binary operation in E in the natural way by

$$(s, A)(t, B) = (s + t, AB), \qquad A \in E_s,\ B \in E_t,$$

then E becomes a product system (cf. [**1**] or [**3**] for more detail).

Notice that for this *particular* product system, we can define a natural operator-valued mapping $\phi : E \to \mathcal{B}(H)$, namely

$$\phi((t, A)) = A, \qquad (t, A) \in E.$$

The map ϕ is an *essential representation* of E, as defined below in (2.1). Thus, an E_0-semigroup acting on $\mathcal{B}(H)$ gives rise to a pair (E, ϕ) consisting of a product system E and an essential representation $\phi : E \to \mathcal{B}(H)$. Conversely, if one is given a product system E and an essential representation of E on a separable Hilbert space, then it is not hard to write down an E_0-semigroup which is associated to the pair (E, ϕ) as above (see [**1**, Proposition 2.7]).

In this way, one can obtain information about the general problem of classifying E_0-semigroups by analyzing the structure of their associated product systems, and by seeking to understand the representation theory of product systems.

2. The spectral C^*-algebra. Let E be a product system. By a *representation* of E we mean a weakly measurable operator-valued function $\phi : E \to \mathcal{B}(H)$ having the following properties:

$$\text{(2.1)} \quad \begin{aligned} &\text{(i)} \quad \phi(xy) = \phi(x)\phi(y), \quad \text{for all } x, y \text{ in } E, \\ &\text{(ii)} \quad \phi(y)^*\phi(x) = \langle x, y \rangle 1, \quad \text{for all } x, y \text{ in } E_t \text{ and every } t > 0. \end{aligned}$$

Antirepresentations are defined similarly, except that the order of the factors on the right of (2.1) part (i) is reversed. (2.1) part (ii) implies that the restriction of ϕ to every fiber E_t, $t > 0$, is a linear isometry from the Hilbert space E_t to $\mathcal{B}(H)$ ([**1**, Section 1]). For every $t > 0$, we have a subspace H_t of H defined by

$$\text{(2.2)} \qquad H_t = [\phi(x)\xi : x \in E_t,\ \xi \in H].$$

These subspaces are decreasing in t, and their union is dense in H ([**1**, Corollary of Proposition 2.4]). ϕ is called *singular* or *essential* according as $\bigcap\{H_t : t > 0\} = \{0\}$, or $H_t = H$ for every $t > 0$. This terminology differs slightly from previous usage (in [**1**] and [**3**], essential representations were called *nonsingular*), and is somewhat more convenient.

We now introduce a C^*-algebra $C^*(E)$ [**4**], which plays the role of the spectrum of the product system E in the sense that the (separable) representations of E correspond precisely to the (separable) *-representations of $C^*(E)$. Let $L^2(E)$ be the Hilbert space of all square-integrable sections of E. The inner product in $L^2(E)$ is given by

$$\langle f, g\rangle = \int_0^\infty \langle f(t), g(t)\rangle \, dt.$$

We have a direct integral decomposition of $L^2(E)$ over the measure space $((0,\infty)\, dt)$,

$$L^2(E) = \int^\oplus E_t \, dt, \tag{2.3}$$

which shows that $L^2(E)$ is a continuous analogue of the full Fock space over an infinite dimensional one-particle space [**1**]. In particular, for every v in E, we can define left and right creation operators $l(v)$, $r(v)$ on $L^2(E)$ by

$$l(v)\xi(x) = \begin{cases} v\xi(x-t), & \text{if } x > t, \\ 0, & \text{if } 0 < x \le t, \end{cases}$$
$$r(v)\xi(x) = \begin{cases} \xi(x-t)v, & \text{if } x > t, \\ 0, & \text{if } 0 < x \le t \end{cases}$$

for ξ in $L^2(E)$. l is a singular representation of E, and r is a singular antirepresentation. The two sets of operators $l(E)$ and $r(E)$ mutually commute (though of course neither $l(E)$ nor $r(E)$ is a commutative set of operators), but $l(E)^*$ does not commute with $r(E)$. Indeed, both $l(E)\bigcup l(E)^*$ and $r(E)\bigcup r(E)^*$ are irreducible sets of operators ([**4**, Theorem 5.2]). l (resp. r) is called the *regular representation* (resp. *regular antirepresentation*) of E.

More generally, if $\phi: E \to \mathscr{B}(H)$ is an arbitrary representation or antirepresentation of E, and f belongs to the Banach space $L^1(E)$ of all integrable sections of E, then the weak integral

$$\int_0^\infty \phi(f(t))\, dt$$

defines a bounded operator on H. This integral defines a linear mapping of $L^1(E)$ into $\mathscr{B}(H)$ of norm of, at most, one, which we will denote by the same letter ϕ. One may verify that for f, g in $L^1(E)$, we have

$$\phi(f)\phi(g) = \phi(f * g),$$

where $f * g$ denotes the *convolution* of f and g,

$$f * g(x) = \int_0^x f(t)g(x-t)\, dt, \qquad x > 0. \tag{2.4}$$

The multiplication defined on $L^1(E)$ by (2.4) makes $L^1(E)$ into a Banach algebra, and $\phi\colon L^1(E) \to \mathscr{B}(H)$ is a contractive homomorphism of $L^1(E)$ onto a nonselfadjoint algebra of operators, whose norm-closure in $\mathscr{B}(H)$ is a separable Banach algebra. If one starts with an antirepresentation $\phi\colon E \to \mathscr{B}(H)$, then this integration process obviously produces a contractive antihomomorphism of Banach algebras $\phi\colon L^1(E) \to \mathscr{B}(H)$.

Applying this to the left and right regular representations, we see that for f in $L^1(E)$, $l(f)$, and $r(f)$ are respectively left and right convolution by f:

$$(l(f)\xi)(x) = \int_0^x f(t)\xi(x-t)\,dt,$$
$$(r(f)\xi)(x) = \int_0^x \xi(x-t)f(t)\,dt,$$

for every ξ in $L^2(E)$. Moreover, using (2.1) part (ii), it is rather easy to show that for every pair f, g of functions in $L^1(E)$, there are functions h_1, h_2 in $L^1(E)$ such that

$$l(g)^*l(f) = l(h_1) + l(h_2)^*;$$

indeed, a straightforward computation allows one to write down explicit formulas for h_1 and h_2 in terms of f and g. It follows that the norm-closed linear span

$$C^*(E) = \operatorname{span}\{l(f)l(g)^*\colon f, g \in L^1(E)\} \tag{2.5}$$

is a separable C^*-algebra.

DEFINITION 2.6. The C^*-algebra defined by (2.5) is called the spectral C^*-algebra of E.

The reader may note that Definition 2.6 is simpler and considerably more concrete than the definition of $C^*(E)$ given in [**4**]. On the other hand, it is not clear at all that the C^*-algebra defined by 2.5 has the correct representation theory. The fact that it does follows from the results of [**4**] and [**5**]. For the reader's convenience, we indicate how the proof of this basic universal property can be dug out of those two references.

THEOREM 2.7. *Let E be a product system. For every representation ϕ of E on a separable Hilbert space H, there is a unique *-representation π of $C^*(E)$ on H satisfying*

$$\pi(l(f)l(g)^*) = \phi(f)\phi(g)^*, \tag{2.8}$$

for every f, g in $L^1(E)$. π is necessarily nondegenerate, and $\phi(E)$ and $\pi(C^*(E))$ generate the same von Neumann algebra. Conversely, every nondegenerate representation π of $C^*(E)$ on a separable Hilbert space has the form (2.8) *for a unique representation ϕ of E.*

PROOF. Let $\mathscr{A}$ denote the C^*-algebra defined in ([**4**, Definition 2.12]); and for every f, g in $L^1(E)$, let $f \otimes \bar{g}$ be the element of $\mathscr{A}$ defined in the discussion preceding ([**4**, Proposition 2.13]). Theorem 2.16 of [**4**] asserts that for every separable representation ϕ of E, there is a unique *-representation π of $\mathscr{A}$ which satisfies the analogue of (2.8):

$$\pi(f \otimes \bar{g}) = \phi(f)\phi(g)^*, \qquad f, g \in L^1(E). \tag{2.9}$$

Conversely, every separable nondegenerate representation π of $\mathscr{A}$ is related to a unique representation ϕ of E by the formula (2.9) ([**4**, Corollary 2 of Theorem 3.4]). Thus, the desired universal property holds for $\mathscr{A}$.

Applying this universal property to the left regular representation $l: E \to \mathscr{B}(L^1(E))$, we obtain a *-representation λ of $\mathscr{A}$ on $L^2(E)$ such that

$$\lambda(f \otimes \bar{g}) = l(f)l(g)^*, \qquad f, g \in L^1(E).$$

By ([**5**, Corollary 3 of Theorem 3.1]), λ is a faithful representation of $\mathscr{A}$. It follows that $C^*(E)$ inherits the required universal property of $\mathscr{A}$ through λ. □

Evans has shown that the Cuntz C^*-algebra O_∞ is isomorphic to the C^*-algebra generated by all left creation operators acting on the full Fock space $\mathscr{F}(H)$ over an infinite dimensional (separable) one-particle space H [**8**]. Thus, the spectral C^*-algebras $C^*(E)$ are properly thought of as continuous analogues of O_∞. They are unitless, separable, nuclear C^*-algebras which are, in most cases, simple ([**4**, Theorem 4.1, and Corollary 2 of Theorem 8.2]). Our proof of simplicity does not work for product systems which contain no "units"; nevertheless, we conjecture that, in general, *$C^*(E)$ is simple for every nontrivial product system E.*

Very little is known about the classification of these spectral C^*-algebras. In more detail, let $\theta: E \to F$ be an isomorphism of product systems. This means that θ is a measurable bijection which preserves multiplication and restricts to a unitary operator on each fiber space E_t, $t > 0$. The set $\operatorname{aut}(E)$ of all automorphisms of a given product system E is obviously a group. It is possible to compute $\operatorname{aut}(E)$ very explicitly for the simplest product systems E, and in those cases there is a natural topology on $\operatorname{aut}(E)$, making it into a Polish group which is often *locally compact* ([**1**, Theorem 8.8]). This group involves the canonical commutation relations in an essential way. The structure of $\operatorname{aut}(E)$ for general product systems E is unknown.

Due to the functorial nature of the construction of $C^*(E)$, every isomorphism $\theta: E \to F$ of product systems induces an isomorphism of C^*-algebras $\hat{\theta}: C^*(E) \to C^*(F)$. More explicitly, θ induces a unitary operator U_θ from $L^2(E)$ to $L^2(F)$ by way of

$$U_\theta \xi(t) = \theta(\xi(t)), \qquad \xi \in L^2(E), \quad t > 0.$$

$\hat{\theta}$ is the corresponding spatial isomorphism of $\mathscr{B}(L^2(E))$ to $\mathscr{B}(L^2(F))$,

$$\hat{\theta}(A) = U_\theta A U_\theta^*, \qquad A \in \mathscr{B}(L^2(E)).$$

Notice that $\hat{\theta}$ carries a generator $l(f_1)l(f_2)^*$ of $C^*(E)$ to the generator of $C^*(F)$ given by $l(\tilde{f}_1)l(\tilde{f}_2)^*$ where, for f in $L^1(E)$, $\tilde{f}$ is the element of $L^1(F)$