Lecture Notes in Mathematics

1569

Vilmos Totik

Weighted Approximation with Varying Weight

$W^n P_n$

Springer-Verlag

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Weighted Approximation with Varying Weight

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

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Mathematics Subject Classification (1991): 41A10, 41A17, 41A25, 26Cxx, 31A10, 31A99, 41A21, 41A44, 42C05, 45E05

ISBN 3-540-57705-X Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-57705-X Springer-Verlag New York Berlin Heidelberg

Library of Congress Cataloging-in-Publication Data. Totik, V. Weighted approximation with varying weights / Vilmos Totik. p. cm. – (Lecture notes in mathematics; 1569) Includes bibliographical references and index. ISBN 3-540-57705-X (Berlin: softcover: acid-free). – ISBN 0-387-57705-X (New York: acid-free)
1. Approximation theory. 2. Polynomials. I. Title. II. Series: Lecture notes in mathematics (Springer-Verlag); 1569. QA3.L28 no. 1569 [QA221] 510 s–dc20 [511'.42] 93-49416

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Printed in Germany

SPIN: 10078788 46/3140-543210 - Printed on acid-free paper

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Lecture Notes in Mathematics 1569

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1 Introduction

In this work we are going to discuss polynomial approximation with weighted polynomials of the form $w^n P_n$, where w is some fixed weight and the degree of P_n is at most n. We emphasize that the exponent of the weight w^n changes with n, so this is a different (and in some sense more difficult) type of approximation than what is usually called weighted approximation. In fact, in the present case the polynomial P_n must balance exponential oscillations in w^n. To have a basis for discussion let us consider first an important special case.

Let $w(x) = \exp(-c|x|^\alpha)$, $c > 0$ be a so called Freud weight. H. N. Mhaskar and E. B. Saff [34] considered weighted polynomials of the form $w^n P_n$, where the degree of P_n is at most n. They found that the norm of these weighted polynomials live on a compact set $\mathcal{S}_w$, i.e. for every such weighted polynomial we have

$$\|w^n P_n\|_{\mathbf{R}} = \|w^n P_n\|_{\mathcal{S}_w},$$

furthermore, $w^n P_n / \|w^n P_n\|_{\mathcal{S}_w}$ tends to zero outside $\mathcal{S}_w$. They also explicitly determined $\mathcal{S}_w$:

(1.1) $$\mathcal{S}_w = [-\gamma_\alpha^{1/\alpha} c^{-1/\alpha}, \gamma_\alpha^{1/\alpha} c^{-1/\alpha}]$$

where

$$\gamma_\alpha := \int_0^1 \frac{v^{\alpha-1}}{\sqrt{1-v^2}}\, dv = \Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})/(2\Gamma(\frac{\alpha}{2}+\frac{1}{2})),$$

(see Section 3 below).

One of the most challenging problems of the eighties in the theory of orthogonal polynomials was Freud's conjecture (see Section 3) about the asymptotic behavior of the recurrence coefficients for orthogonal polynomials with respect to the weights w. The solution came in three papers [16], [29] and [27] by D. S. Lubinsky, A. Knopfmacher, P. Nevai, S. N. Mhaskar and E. B. Saff. The most difficult part of the proof was the following approximation theorem ([29]).

Theorem 1.1 *If $w_\alpha(t) = \exp(-\gamma_\alpha |t|^\alpha)$, $\alpha > 1$ is a Freud weight normalized so that $\mathcal{S}_{w_\alpha} = [-1, 1]$, then for every continuous f which vanishes outside $(-1, 1)$ there are polynomials P_n of degree at most n, $n = 1, 2, \ldots$ such that $w_\alpha^n P_n$ uniformly tends to f on the whole real line.*

Let us mention that it follows from what we have said about $w_\alpha^n P_n$ tending to zero outside $[-1, 1]$, if f can be uniformly approximated by $w_\alpha^n P_n$, then it must vanish outside $[-1, 1]$.

In the next section we shall present a rather elementary and direct proof for Theorem 1.1. Then, in Section 3, we shall derive a short proof for the strong asymptotic result of Lubinsky and Saff for an extremal problem associated with Freud weights. With this we will provide a self contained and short proof for the most important result of the monograph [28].

In Section 4.1 we shall considerably generalize Theorem 1.1 and solve the analogous approximation problem for a large family of weights. In earlier works the approximation problem was mostly considered for concrete weights such as

Freud, Jacobi or Laguerre weights. The generalization given in Theorem 4.2 is the first general result in the subject and is far stronger than the presently existing results (e.g. it allows S_w to lie on different intervals). It also solves several open conjectures. However, the new and relatively simple *method* is perhaps the most important contribution of the present paper (Lubinsky and Saff themselves generalized Theorem 1.1 in a different direction, see e.g. [28] and Section 12). We shall first restrict our attention to the important special case given in Theorem 1.1 in order to get a simple proof for the above mentioned asymptotics (and hence for the so called Freud conjecture) and in order not to complicate our method with the technical details that are needed in the proof of Theorem 4.2 (see Section 5).

In the third part of this work we shall present a modification of the method. This will allow us to consider varying weights in the stronger sense, that we shall allow even w_n to vary with n. Recently a lot of attention has been paid to such varying weights which are connected to some interesting applications to be discussed in Chapter IV.

In essence our approximation problem can be reformulated as follows: how well can we discretize logarithmic potentials, i.e. replace them by a potential of a discrete measure which are the sums of n ($n = 1, 2, \ldots$) equal point masses (see the discussion below for the relevant concepts). The usual procedure is the following: divide the support into $n+1$ equal parts with respect to the measure and place masses $1/n$ at these division points. This approach has proven to be sufficient and useful in many problem. However, the process introduces singularities on the support which has to be avoided in finer problems. Our method in its simplest form is a modification of the previous idea. We also divide the support into n equal parts with respect to the measure, but we use the *weight points* of these parts instead of their endpoints for placing the mass points to, then we *vertically shift* this discrete measure by an amount L_n/n where $L_n \to \infty$ is appropriately chosen. This modification will result in a dramatic increase in the speed of approximation.

* * * * * * * * * * * * *

In the rest of this introduction we shall briefly outline the results from the theory of weighted potentials that we will need in the paper.

We shall use logarithmic potentials of Borel measures. If μ is a finite Borel measure with compact support, then its logarithmic potential is defined as its convolution with the logarithmic kernel:

$$U^\mu(z) = \int \log \frac{1}{|z-t|} \, d\mu(t).$$

Let Σ be a closed subset of the real line. For simplicity we shall assume that "Σ is regular with respect to the Dirichlet problem in $\mathbf{C} \setminus \mathbf{R}$", by which we mean that every point x_0 of Σ satisfies Wiener's condition: if

$$E_n := \{x \in \Sigma \,|\, 2^{-n-1} \le |x - x_0| \le 2^{-n}\},$$

then

$$\sum_{n=1}^{\infty} \frac{n}{\log 1/\mathrm{cap}(E_n)} = \infty \tag{1.2}$$

(see the following discussion for the definition of the logarithmic capacity). In particular, this is true if Σ consists of finitely many (finite or infinite) intervals. This regularity condition is not too essential in our considerations, but it simplifies some of our proofs.

A weight function w on Σ is said to be *admissible* if it satisfies the following three conditions

(1.3)
(i) w is continuous;
(ii) $\Sigma_0 := \{x \in \Sigma \,|\, w(x) > 0\}$ has positive capacity;
(iii) if Σ is unbounded, then $|x|w(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$.

We are interested in approximation of continuous functions by weighted polynomials of the form $w^n P_n$. To understand the behavior of such polynomials we have to recall a few facts from [34] and [35] about the solution of an extremal problem in the presence of a weight (often called external field).

We define $Q = Q_w$ by

$$w(x) =: \exp(-Q(x)). \tag{1.4}$$

Then $Q : \Sigma \to (-\infty, \infty]$ is continuous everywhere where w is positive, that is where Q is finite.

Let $\mathcal{M}(\Sigma)$ be the set of all positive unit Borel measures μ with $\mathrm{supp}(\mu) \subseteq \Sigma$, and define the weighted energy integral

$$\begin{aligned} I_w(\mu) &:= \iint \log[|z-t|w(z)w(t)]^{-1} d\mu(z)d\mu(t) \\ &= \iint \left[\log\frac{1}{|z-t|} + Q(z) + Q(t)\right] d\mu(z)d\mu(t). \end{aligned} \tag{1.5}$$

The classical case corresponds to choosing Σ to be compact and $w \equiv 1$ on Σ: If μ is a Borel measures with compact support on $\mathbf{R}$, then its *logarithmic energy* is defined as

$$I(\mu) := \int U^{\mu}(z)d\mu(z) = \int\int \log\frac{1}{|z-t|} d\mu(t)d\mu(z).$$

If K is a compact set, then its logarithmic *capacity* $\mathrm{cap}(K)$ is defined by the formula

$$\log\frac{1}{\mathrm{cap}(K)} := \inf\{I(\mu) \,|\, \mu \in \mathcal{M}(K)\}. \tag{1.6}$$

Now the capacity of an arbitrary Borel set B is defined as the supremum of the capacities of compact subsets of B, and a property is said to hold *quasi–everywhere* on a set A if it holds at every point of A with the exception of points of a set of capacity zero.

The equilibrium measure (see [51] or [17]) ω_K of K is the unique probability measure ω_K minimizing the energy integrals in (1.6). Its potential has the following properties:

$$U^{\omega_K}(z) \le \log \frac{1}{\mathrm{cap}(K)} \quad \text{for} \quad z \in \mathbf{C}, \tag{1.7}$$

$$U^{\omega_K}(z) = \log \frac{1}{\mathrm{cap}(K)} \quad \text{for quasi-every} \quad z \in K. \tag{1.8}$$

If K is regular (which means that its complement $\mathbf{C} \setminus K$ is regular with respect to the Dirichlet problem), then we have equality for every z in (1.8).

Returning to the general case of weighted energies, the next theorem was essentially proved in [34] and [35].

Theorem A *Let w be an admissible weight on the set Σ, and let*

$$V_w := \inf\{I_w(\mu) \mid \mu \in \mathcal{M}(\Sigma)\}. \tag{1.9}$$

Then the following properties are true.

(a) *V_w is finite.*

(b) *There exists a unique $\mu_w \in \mathcal{M}(\Sigma)$ such that*

$$I_w(\mu_w) = V_w.$$

Moreover, μ_w has finite logarithmic energy.

(c) *$\mathcal{S}_w := \mathrm{supp}(\mu_w)$ is compact, is contained in Σ_0 (c.f. property* (ii) *above), and has positive capacity.*

(d) *The inequality*

$$U^{\mu_w}(z) \ge -Q(z) + V_w - \int Q d\mu_w =: -Q(z) + F_w$$

holds on Σ.

(e) *The inequality*

$$U^{\mu_w}(z) \le -Q(z) + F_w$$

holds for all $z \in \mathcal{S}_w$.

(f) *In particular, for every $z \in \mathcal{S}_w$,*

$$U^{\mu_w}(z) = -Q(z) + F_w.$$

The proof is an adaptation of the classical Frostman method. In fact, in [34] and [35] property (d) was proved to hold for quasi-every $z \in \Sigma$. But the regularity of Σ implies that then the set of points where

$$\int \log \frac{1}{|z-t|} d\mu_w(t) + Q(z) \ge V_w - \int Q d\mu_w =: F_w$$

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holds is dense at every point of Σ in the fine topology (see [12, Chapter 10] or [17, Chapter III]), hence the inequality in question is true at every $z \in \Sigma$ by the continuity of Q (where it is finite) and the continuity of logarithmic potentials in the fine topology.

The measure μ_w is called the *equilibrium* or *extremal measure* associated with w.

Above we have used the abbreviation

$$F_w := V_w - \int Q d\mu_w$$

for this important quantity.

We cite another theorem of H. N. Mhaskar and E. B. Saff [34, Theorem 2.1], which says that the supremum norm of weighted polynomials $w^n P_n$ lives on $\mathcal{S}_w$. Let us agree that whenever we write P_n, then it is understood that the degree of P_n is at most n.

Theorem B *Let w be an admissible weight on $\Sigma \subseteq \mathbf{R}$. If P_n is a polynomial of degree at most n and*

$$(1.10) \qquad |w(z)^n P_n(z)| \le M \quad \text{for} \quad z \in \mathcal{S}_w,$$

then for all $z \in \mathbf{C}$

$$(1.11) \qquad |P_n(z)| \le M \exp\left(n(-U^{\mu_w}(z) + F_w)\right).$$

Furthermore, (1.10) *implies*

$$(1.12) \qquad |w(z)^n P_n(z)| \le M \quad \text{for} \quad z \in \Sigma.$$

This theorem asserts that every weighted polynomial must assume its maximum modulus on $\mathcal{S}_w$. Soon we shall see that $\mathcal{S}_w$ is the smallest set with this property.

Theorem B is an immediate consequence of the principle of domination (see the proof of Lemma 5.1 in Section 5).

Part I
Freud weights

In the first part of the paper we shall consider exponential type (also called Freud) weights. We shall illustrate our method on them. The other purpose of this part is to give a self-contained and relatively short proof for the strong asymptotic results of Lubinsky and Saff [28].

2 Short proof for the approximation problem for Freud weights

In this section we give a short and simple proof for Theorem 1.1.

Let $Q(x) = \gamma_\alpha |x|^\alpha$, so that $w_\alpha(x) = w(x) = \exp(-Q(x))$. First we simplify the problem.

I. Obviously, it is enough to consider f's that are positive in $(-1, 1)$ and less than, say, 1. Furthermore, we know that it is sufficient to approximate on, say, $[-2, 2]$, because $w^n P_n$ tends to zero outside $[-3/2, 3/2]$ (see Theorems A and B from the introduction and the formula (3.7) in Section 3).

II. It is enough to approximate by the absolute values of weighted polynomials. In fact, if $w^n|P_n|$ uniformly tends to $\sqrt{f}$, then $w^{2n}|P_n|^2$ uniformly tends to f, and here $|P_n|^2$ is already a real polynomial. This shows our claim when the degree n is even. For odd degree one can get the statement by approximating f/w with even degree polynomials and then by multiplying through by w.

III. It is enough to show the following: for every $\epsilon > 0$ and $L > 0$ there is a continuous function g_L and for every large n polynomials Q_n of degree at most n such that with $J_\epsilon := [-1 + \epsilon, 1 - \epsilon]$

$$w^n(x)|Q_n(x)| = \exp(g_L(x) + R_L(x)), \qquad x \in J_\epsilon, \tag{2.1}$$

where the remainder term $R_L(x)$ satisfies $|R_L(x)| \le C_\epsilon/L$ uniformly for $x \in J_\epsilon$ with some $C_\epsilon \ge 1$ independent of L, and for every $x \in [-3, 3]$

$$w^n(x)|Q_n(x)| \le Dn^3, \tag{2.2}$$

where $D = D_{L,\epsilon}$ is a constant independent of n.

In fact, suppose this is true, and apply it to w^λ instead of w with some $\lambda > 1$. The corresponding extremal support is $[-\theta_\lambda, \theta_\lambda]$ with $\theta_\lambda = \lambda^{-1/\alpha}$ tending to 1 together with λ, hence, by choosing $\lambda > 1$ close to 1 and then applying the statement above to a smaller ϵ if necessary, we can see that there are polynomials $Q_{[n/\lambda]}$ of degree at most $[n/\lambda]$ such that with some g_L and R_L as above

$$w^n(x)|Q_{[n/\lambda]}(x)| = \exp(g_L(x) - (n - \lambda[n/\lambda])Q(x) + R_L(x)), \qquad x \in J_\epsilon,$$

and

$$w^n(x)|Q_{[n/\lambda]}(x)| \le D_1 n^3, \qquad x \in [-2,2].$$

Since $0 \le n - \lambda[n/\lambda] \le \lambda$, and the family of function $\{g_L - sQ \mid 0 \le s \le 1\}$ (considered on $[-1+\epsilon, 1-\epsilon]$) is compact, for every large n there are polynomials $S_{n-[n/\lambda]}$ of degree at most $n - [n/\lambda]$ such that

$$|S_{n-[n/\lambda]}(x) - f(x)\exp(-g_L(x) + (n-\lambda[n/\lambda])Q(x))|$$

$$\le \exp(-g_L(x) + (n-\lambda[n/\lambda])Q(x))/L, \qquad x \in J_{2\epsilon},$$

$$|S_{n-[n/\lambda]}(x)| \le f(x)\exp(-g_L(x) + (n-\lambda[n/\lambda])Q(x)), \qquad x \in J_{2\epsilon} \setminus J_\epsilon,$$

and

(2.3) $$|S_{n-[n/\lambda]}(x)| \le n^{-4}, \qquad x \in [-2,2] \setminus J_\epsilon.$$

Now we set $P_n = Q_{[n/\lambda]} S_{n-[n/\lambda]}$, which has degree at most n. If $\eta > 0$ is given, then choose first $\epsilon > 0$ so that the maximum of f outside $J_{2\epsilon}$ is smaller than η, then chooose $\lambda > 1$ as above, and finally choose L large enough to have $C_\epsilon/L < \eta$. Then our estimates show that for sufficiently large n the difference $|w^n|P_n| - f|$ is at most 3η on $[-2,2]$, and this is what we need to prove.

IV. Thus, we only have to verify (2.1) and (2.2).

Let us consider the so called Ullman distribution μ_w given by its density function

(2.4) $$v(t) = \frac{\alpha}{\pi}\int_{|t|}^{1} \frac{u^{\alpha-1}}{\sqrt{u^2 - t^2}}\, du.$$

It is well-known (see the computation in Section 3, especially (3.6) and (3.7)) that $w(x)$ and $\exp(U^{\mu_w}(x))$ differ on $[-1,1]$ only in a multiplicative constant, and elsewehere the weight $w(x)$ is smaller than $\exp(U^{\mu_w}(x))$ times this constant. Hence it is enough to show (2.1) and (2.2) with $w = w_\alpha$ replaced by $\exp(U^{\mu_w})$. In doing so we are going to use the standard discretization technique for logarithmic potentials (c.f. [42] and [28]) with some modifications, but exactly these modifications permit good approximation.

Let v be the density of the Ullman distribution μ_w (see (2.4)), and let us divide $[-1,1]$ by the points $-1 = t_0 < t_1 < \ldots < t_n = 1$ into n intervals I_j, $j = 0, 1, \ldots, n-1$ with $\mu_w(I_j) = 1/n$. Since v is continuous and positive in $(-1,1)$, there are two constants c, C (depending on ϵ) such that if $I_j \cap J_{\epsilon^2} \ne \emptyset$, then $c/n \le |I_j| \le C/n$.

Let

$$\xi_j := \frac{1}{\mu(I_j)}\int_{I_j} t\, d\mu(t) = n\int_{I_j} t\, d\mu(t)$$

be the weight point of the restriction of μ_w to I_j, and set

$$Q_n(t) = \prod_j (t - iL/n - \xi_j).$$

We claim that this choice will satisfy (2.1) and (2.2) (with w replaced by $\exp(U^{\mu_w})$).

First of all let us consider the partial derivative of $U^{\mu_w}(z)$ at $z = x + iy$ with respect to y:

$$\frac{\partial U^{\mu_w}(z)}{\partial y} = -\int_{-1}^{1} \frac{y}{(x-t)^2 + y^2} v(t)dt \to \pi v(x) \tag{2.5}$$

as $y \to 0 - 0$ uniformly for $x \in J_\epsilon$ by the properties of the Poisson kernel. This, and the mean value theorem implies that

$$U^{\mu_w}(x) - U^{\mu_w}(x - iL/n) = \frac{\pi L v(x)}{n} + o\left(\frac{L}{n}\right) \tag{2.6}$$

uniformly in $x \in J_\epsilon$. The same argument shows that

$$|U^{\mu_w}(x) - U^{\mu_w}(x - iL/n)| = O(\frac{L}{n}) \tag{2.7}$$

uniformly for $x \in \mathbf{R}$.

Actually, (2.5) and (2.6) uniformly hold on $\mathbf{R}$ because v is continuous (even at ± 1) and vanishes outside $[-1, 1]$. We shall use this fact in Section 3, but for the present purposes we keep the above formulation because in Section 4.1 we shall consider weights the density of which is not necessarily continuous around the endpoints, and it will be easier to point out the necessary changes if we work with (2.6) and (2.7).

Let $\mu_n(t) = \mu_w(t - iL/n)$, i.e. we are defining μ_n on the interval $[-1, 1] + iL/n$, which is obtained by shifting $[-1, 1]$ upwards on the plane by the amount L/n. Then the preceding two estimates tell us how far apart the two potentials U^{μ_w} and U^{μ_n} can be on $[-1, 1]$ and on $\mathbf{R}$. Next we estimate for $x \in J_\epsilon$, $x \in I_{j_0}$

$$|\log|Q_n(x)| + nU^{\mu_n}(x)| \tag{2.8}$$

$$= \left|\sum_{j=0}^{n-1} n \int_{I_j} (\log|x - iL/n - t| - \log|x - iL/n - \xi_j|)\, d\mu_w(t)\right|.$$

Here the integrand is

$$\log\left|1 + \frac{\xi_j - t}{x - iL/n - \xi_j}\right| = \Re \log\left(1 + \frac{\xi_j - t}{x - iL/n - \xi_j}\right).$$

Since the absolute value of

$$\frac{\xi_j - t}{x - iL/n - \xi_j}, \qquad t \in I_j$$

is at most $1/2$ for large L (check this separately for $|\xi_j - t| \le C/n$ and for the opposite case which can only occur if $I_j \cap J_{\epsilon^2} = 0$ and hence $|x - \xi_j| > \epsilon/2$ while $|I_j| < \epsilon^2$), it easily follows that then the last expression can be written in the form

$$= (\xi_j - t)\Re\frac{1}{x - iL/n - \xi_j} + O\left(\frac{|\xi_j - t|^2}{|x - iL/n - \xi_j|^2}\right),$$

and since the integral of the first term on I_j against $d\mu_w(t)$ is zero because of the choice of ξ_j, we have to deal only with the second term. For it we have the upper estimate

$$O\left(\frac{(C/n)^2}{(L/n)^2+(c(j-j_0)/n)^2}\right)$$

if $I_j \cap J_{\epsilon^2} \neq \emptyset$ and

$$O\left(\frac{|I_j|^2}{\epsilon^2}\right)$$

otherwise (recall that $x \in J_\epsilon$), hence we can continue (2.8) as

$$\leq C_1 \sum_{k=0}^{\infty} \frac{C^2}{L^2+c^2k^2} + C_1 \max_j |I_j| \sum_{I_j \cap J_{\epsilon^2} = \emptyset} |I_j|\epsilon^{-2} \leq \frac{C_\epsilon}{L}$$

if n is sufficiently large.

Now

$$\log|Q_n(x)| + nU^{\mu_w}(x) = (\log|Q_n(x)| + nU^{\mu_n}(x)) + (nU^{\mu_w}(x) - nU^{\mu_n}(x)),$$

and here, by the preceding estimate, the first term is at most C_ϵ/L in absolute value, while by (2.6) the second term is $\pi v(x)L + o(L)$ uniformly in $x \in J_\epsilon$ as $n \to \infty$. This gives (2.1) (recall that we are working with $\exp(U^{\mu_w})$ instead of w).

The proof of (2.2) is standard: using the monotonicity of the logarithmic function we have for example for $x \in I_{j_0}$, $j_0 < j < n-1$ the inequality

$$\log|x - iL/n - \xi_j| \leq n \int_{I_{j+1}} \log|x - iL/n - t|d\mu_w(t),$$

and adding these and the analogous inequalities for $j < j_0$ together one can easily deduce the estimate

$$\log|Q_n(x)| + nU^{\mu_n}(x) \leq 3\log 6 + \tag{2.9}$$

$$+ \sum_{j=j_0-1}^{j_0+1} n \int \log \frac{1}{|x - iL/n - t|} d\mu_w(t) \leq 3\log 6n/L$$

for every $x \in [-3, 3]$. This and (2.7) prove (2.2). ∎

3 Strong asymptotics

The theorems of this section are not new, they can be found in the monograph [28] by D. S. Lubinsky and E. B. Saff. We closely follow many steps from [28], but we substitute the approximation part of the proof with the simple method of Section 2 which allows us to make shortcuts and simplifications, thereby

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