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Edited by A. Dold and B. Eckmann

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Jean Moulin Ollagnier

Ergodic Theory and Statistical Mechanics

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INTRODUCTION

It can be said that for the study of dynamical systems amenability is the crucial property of the acting group. This generalization of the classical point of view is not only natural, but is also related to the applications in statistical mechanics where the acting group consists of isometries of the lattice.

This text, which grew out of a "third cycle" course in Ergodic Theory and Statistical Mechanics which I gave together with Didier Pinchon at the University of Paris VI in 1980, deals with both topological and measure-theoretic dynamical systems, and in particular the symbolic dynamical systems of statistical mechanics.

The existence of the ameaning filter for an amenable group shall be proved with the use of strongly subadditive set functions and the special dynamical system of total orders. Several ergodic theorems shall be given.

The entropy theory of measure-theoretic dynamical systems shall be completely described; and the Shannon-McMillan theorem is given as a corollary of a new ergodic theorem, the "almost subadditive ergodic theorem."

A link between topological and measure-theoretic dynamical systems shall be made by way of the variational principle for the pressure of a continuous function on a compact Hausdorff space under the action of an amenable group.

A careful study of subadditivity of set functions allows us to avoid the use of tiling methods in proving several important theorems. However, tiling is essential when proving the equivalence of a countable amenable group with Z. This proof is given in the last chapter along with Rokhlin's lemma and the proof of the hyperfiniteness of countable amenable groups (using the tower extension argument of Connes, Feldman and Weiss).

I would like to express my indebtedness to Didier Pinchon, to whom a significant portion of the material contained in this text is due. I also would like to thank Jean-Paul Thouvenot for many helpful discussions which stimulated the work on this text, and Tony Frank Paschall for his assistance with the English manuscript.

Jean Moulin Ollagnier
Villetaneuse, September 1984

CONTENTS

1. PRELIMINARY ANALYSIS

1.1. SUBLINEAR FUNCTIONS AND THE HAHN-BANACH THEOREM

1.1.1. Definition. Let E be a real vector space. A real function p on E is said to be sublinear if it is both subadditive and positively homogeneous, i.e. if the two following conditions hold:

i) for every pair (x,y) of vectors in E $\quad p(x+y) \leqq p(x) + p(y)$

ii) for every x and every positive number λ $\quad p(\lambda x) = \lambda\, p(x)$

1.1.2. Remark. It is quite clear that a linear functional is a sublinear function and that the least upper bound of a set of sublinear functions, if one exists, is still sublinear. Therefore, a least upper bound of linear functionals is a sublinear function. We shall state that this property is characteristic.

1.1.3. Example. Consider the vector space C(X) of all real continuous functions on the compact set X. Function s, defined on this space by the formula

$$s(f) = \sup_{x \in X} f(x)$$

is sublinear.

1.1.4. Extension theorem (Hahn-Banach). Let E be a real vector space and p be a sublinear function on E. Let F be a subspace of E and m be a linear functional bounded above by p on F, i.e. for every x in F, $m(x) \leqq p(x)$. Then, there exists a linear functional n on the whole space E, still bounded above by p, which is an extension of m.

The proof of the theorem essentially depends on an extension lemma, and, when the dimension is infinite, on Zorn's lemma as well.

1.1.5. Extension lemma. Let E be a real vector space, p a sublinear function on E, G a vector subspace of E non-equal to E, and f a linear functional on G bounded by p.
Let a be an element of E\G so that the vector space $G \oplus Ra$ is strictly greater than G. It is then possible to find an extension of f which is a linear functional on $G \oplus Ra$ and is still bounded by p.

Proof of the lemma. We look for a real number α such that, for every x in G and every real number λ, the following inequality holds:

$$f(x) + \lambda\alpha \leqq p(x+\lambda a)$$

Using homogeneity, we have only to verify that, for every x in G,

$$f(x) + \alpha \leqq p(x+a) \quad \text{and} \quad f(x) - \alpha \leqq p(x-a)$$

The real number α must then be chosen greater than or equal to

$$\operatorname*{Sup}_{y \in G} (f(y)-p(y-a))$$

and less than or equal to

$$\operatorname*{Inf}_{x \in G} (p(x+a)-f(x))$$

Such a choice is possible if, for every pair (x,y) of vectors in G,

$$f(y) - p(y-a) \leqq p(x+a) - f(x)$$

which is equivalent to $f(x) + f(y) \leqq p(x+a) + p(y-a)$.
Because $f(x) + f(y) = f(x+y) \leqq p(x+y) \leqq p(x+a) + p(y-a)$, the proof is achieved.

1.1.6. Proof of the theorem. Consider the set I of all pairs (G,f) where G is a subspace of E which contains F, and where f is a linear functional on G, is bounded by p, and extends m.
This set I can be ordered in the following way:

$$(G,f) \leqq (G',f') \iff G \quad G' \text{ and } f' \text{ is an extension of } f$$

Set I is non-empty because (F,m) belongs to it and it is inductive for the order.

According to Zorn's lemma, we can then find a maximal element (G^+,n) for this order. For such a maximal element, G^+ is equal to E (using lemma 1.1.5) and the proof is thereby obtained.

1.1.7. Corollary. A sublinear function is the least upper bound of all linear functionals less than or equal to it.

Proof. We want to show that, for a given sublinear function p on E and for every vector x in a vector space E, the real number p(x) is the least upper bound of all numbers f(x), where f is a linear functional on E bounded above by p.
Then, the inequality $f(x) \leqq p(x)$ holds.
On the other hand, the linear functional on the subspace generated by x, given by $f(x) = p(x)$, can be extended according to theorem 1.1.4 to the whole space.

1.1.8. Example. Continuing now example 1.1.3 of function s on C(X) and describing the linear functionals on C(X) bounded by s, we find that, for every continuous function f on X, a linear functional m bounded above by s verifies $m(f) \leqq \|f\|$, where $\| \ \|$ is the uniform norm.
We then have

$$m(f) \leqq s(f) \leqq \|f\| \quad \text{and} \quad -m(f) = m(-f) \leqq s(-f) \leqq \|f\|$$

Therefore m is a Radon measure on X.
Moreover, if f is everywhere non-negative,

$$m(-f) \leqq p(-f) \leqq 0 \quad \text{and} \quad m(f) \geqq 0$$

On the other hand, if f is equal to constant 1, both inequalities $m(1) \leqq 1$ and $m(-1) \leqq -1$ are true and therefore $m(1) = 1$.
The linear functionals on C(X) bounded by s are then positive Radon measures on X with a total mass equal to 1, i.e. Radon probability measures on X.
Conversely, all Radon probability measures are bounded by s.

1.1.9. Remark. A pseudo-norm is a sublinear function which moreover verifies the identity $p(x) = p(-x)$.
One of the more common forms of the Hahn-Banach theorem deals with the extension of linear functionals whose absolute value is bounded by a given pseudo-norm p.

If p is a pseudo-norm, the two conditions $m \leqq p$ and $|m| \leqq p$ are equivalent and theorem 1.1.4 enables us to complete the proof.

1.1.10. Remark. It is possible to consider a similar problem with complex rather than real vector spaces.
In this case, a pseudo-norm p is a subadditive positive function on complex space E that verifies, for every vector x and every complex number λ,

$$p(\lambda x) = |\lambda| p(x)$$

Then let m be a complex linear functional bounded by p on subspace F of E. According to remark 1.1.9, the real linear functional on F, Re(m), has an extension ν to the whole space, such that the absolute value of ν is bounded by p.
The complex function n on E, given by $n(x) = \nu(x) - i.\nu(ix)$, is a complex linear functional with an absolute value bounded by p, and is an extension of m.

1.2. COMPACT CONVEX SETS

1.2.1. Definition. Let E be a real vector space and E* its algebraic dual space. Every x in E defines a pseudo-norm p_x on E* in the following way:

$$p_x(f) = f(x)$$

The family of all these pseudo-norms endows E* with a topology and it becomes a locally convex topological vector space.
This topology is the restriction to E* of the product topology on R^E.
It is the coarsest topology for which the coordinate applications from E* to R, $f \longrightarrow f(x)$, are continuous.
It is called the weak* topology on E*.

1.2.2. Proposition. Let K(E,p) be the subspace of E* that consists of all linear functionals on E that are bounded above by the sublinear function p.
K(E,p) is a convex subset of E*, and it is compact for the weak* topology.

Proof. The convexity property is evident, while the compactness of $K(E,p)$ results from Tychonoff's theorem.
The product space of all segments $[-p(-x),p(x)]$ is a compact subset of R^E; and the conditions defining $K(E,p)$ in this product space make it a closed subspace.

1.2.3. Examples.
The unit ball of the strong dual of a normed space is weakly compact.
The unit ball of the space of Radon measures on a compact space X is weakly compact. The subset of this ball of all probabilities is convex and weakly compact.

The following lemma is useful in proving the converse of proposition 1.2.2.

1.2.4. Lemma. Let F be a finite dimensional real vector space. Let F' be its dual space, and K a convex compact subset of F'.
Every linear functional f on F belongs to $K(E,p)$ when, for every x in F,

$$f(x) \leqq \sup_{g\in K} (g(x))$$

Proof. Consider a Euclidian structure $\langle\ ,\ \rangle$ on F and the dual Euclidian structure on F'.
Call f' the orthogonal projection of f on K, i.e. the unique element of K such that

$$\inf_{g\in K} \langle f-g,f-g\rangle = \langle f-f',f-f'\rangle$$

Let x be the vector such that $f - f' = \langle x,.\rangle$.
For every g in K and every real number ε between 0 and 1, $(f'+\varepsilon.(g-f'))$ belongs to K and

$$(f-f')(x) \leqq \| f-f'-\varepsilon.(g-f') \|^2$$

$$= (f-f')(x) + 2\varepsilon.(f'-g)(x) + \varepsilon^2 \|f'-g\|^2$$

For every strictly positive ε

$$0 \leqq 2(f'-g)(x) + \varepsilon \|f'-g\|^2$$

When ε tends to 0, this formula reduces to $(g-f')(x) \leqq 0$, from which we derive

$$f'(x) = \operatorname*{Sup}_{g \in K} g(x)$$

Therefore f(x) is less than or equal to f'(x), and the scalar square of f-f' is non-positive, which means that f belongs to K.

1.2.5. Proposition. Let K be a convex part of the dual space E* of a real vector space E, and let it be compact in the weak* topology.
Let x be a vector in E, and consider the least upper bound p(x) of all numbers f(x), where f belongs to K. For every x in E, p(x) is finite.
So defined, function p is sublinear, and the two compact convex sets K and K(E,p) are equal.

Proof. For every x, the function $f \longrightarrow f(x)$ is continuous on K and therefore bounded. Function p is then defined on the whole space E and it is a supremum of linear functionals. According to remark 1.1.2, p is sublinear.
By definition, K is contained in K(E,p).
To prove the converse inclusion, consider an element f of K(E,p).
For every finite dimensional vector subspace of E, there exists according to lemma 1.2.4 a convex compact non-empty subset K_F of K whose elements give the same value as f to all elements of F.
The family of all compact sets K_F has the non-empty finite intersection property. Therefore, there exists a convex compact subset K_E of K, whose elements give the same value as f to all elements of E. There is only one element in K, which is indeed f, and the converse inclusion is proven.

1.2.6. Remark. It is possible to deduce the result of proposition 1.2.5 from a geometrical form of the Hahn-Banach theorem.

1.3. RADON MEASURES

1.3.1. Definition. Consider the vector space C(X) of all real continuous functions on a given compact Hausdorff space X. A Radon measure on X is a linear functional on C(X), continuous for the supremum norm given by the formula $\|f\| = \sup_{x \in X} |f(x)|$.

1.3.2. Definition. Let $(X,\mathfrak{a})$ be a measurable space, where X is a topological space and $\mathfrak{a}$ a σ-algebra containing the Borel σ-algebra. A positive real measure on this space is said to be inner regular if the measure of every measurable subset of X is the least upper bound of the measures of all its compact subsets. A positive real measure on this space is said to be outer regular if the measure of every measurable set is the greatest lower bound of the measures of all open sets containing it.

1.3.3. The Riesz representation theorem. Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$, the space of all continuous real functions on X with a compact support. Then there exists a σ-algebra $\mathcal{M}$ in X which contains all Borel sets, and a unique positive measure μ on $\mathcal{M}$ with the following properties: first, μ represents Λ

$$f \in C_c(X) \Longrightarrow \mu(f) = \int_X f \, d\mu$$

second, μ is inner and outer regular and gives a finite measure to compact sets; and, third, the σ-algebra $\mathcal{M}$ is complete for μ.

One proof of this can be found in Rudin (1).

1.3.4. Remark. Because of the above one-to-one correspondence, we shall call a Radon measure both the measure derived from a linear functional on C(X) by Riesz's theorem, and also, when necessary, the restriction to the Borel σ-algebra of this measure.

1.3.5. Corollary. Let μ be a Radon measure on a compact Hausdorff space X, and f be an upper semi-continuous function on X. Then the integral $\mu(f)$ is the supremum of all $\mu(\phi)$, where ϕ is a continuous function on X greater than or equal to f.

Proof. For every continuous function ϕ, the inequality $\mu(f) \leqq \mu(\phi)$ results from the positivity of μ.
Consider now, for every natural number n, the function f_n which is equal to Sup(f,-n), and is so upper semi-continuous and bounded.
Sequence (f_n) then decreases to f. According to the monotone convergence theorem, the integral of f is the limit of the integral of f_n when n

tends to infinity.
We have only then to prove the results for bounded upper semi-continuous functions. By adding a constant we can even restrict the proof to the case of non-negative functions.
To achieve such a proof, let g be a non-negative upper semi-continuous function on X.
For every positive real number δ, g_δ is the function given by

$$g_\delta = \delta . \sum_{n=0}^{\infty} 1_{\{g \geqq n\delta\}}$$

In fact, there is a finite number of terms in the sum that are different from 0 since g is bounded. The last subscript is $n_0 = E(\mathrm{Sup}(g)/\delta)$.
This function is greater than or equal to g, and this inequality holds:

$$\mu(g_\delta - g) \leqq \delta . \mu(1)$$

Given a positive real number ε, select a δ less than $\varepsilon/2\mu(1)$.

Since μ is inner regular, every compact set $K_n = \{g \geqq n\delta\}$ has an open neighborhood O_n such that $\mu(O_n - K_n) < \eta$.
Then, there exists a continuous function ϕ_n, whose values lie between 0 and 1, and which is equal to 1 at every point of K_n and to 0 at every point outside of O_n.
The positive real number η is then an upper bound for

$$\mu(\phi_n) - \mu(1_{K_n})$$

Choose η less than $\varepsilon/2\ \mathrm{Sup}(g)$ and consider the function

$$\phi = \delta . (1 + \sum_{n=1}^{n_0} \phi_n)$$

This function is greater than or equal to g at every point in X, and the inequality $\mu(\phi - g) < \varepsilon$ holds, which concludes the proof.

1.3.6. A version of Dini's lemma. Let $(f_i)_{i \in I}$ be a set of upper semi-continuous functions on a Hausdorff compact set X such that every pair of elements has a common lower bound in the set, i.e. that this set is directed for the order $\geqq$.
Let f be the greatest lower bound of all these functions.
The following minimax result holds in this situation:

$$\operatorname*{Sup}_{x\in X} f(x) = \operatorname*{Inf}_{i\in I} \operatorname*{Sup}_{x\in X} f_i(x)$$

To demonstrate this consider first the obvious inequality

$$\operatorname*{Sup}_{x\in X} f(x) \leqq \operatorname*{Inf}_{i\in I} \operatorname*{Sup}_{x\in X} f_i(x)$$

In order to prove the converse inequality denote by a the following number:

$$a = \operatorname*{Inf}_{i\in I} \operatorname*{Sup}_{x\in X} f_i(x)$$

If a is equal to $-\infty$, there is nothing to prove.
Otherwise, the compact sets $f_i^{-1}([a,+\infty[)$ are non-empty and their family has the non-empty intersection property.
At every point x of the intersection of all these compact sets, the limit function f takes a values f(x) which is less than or equal to a; and this concludes the proof.

1.3.7. Corollary. Given a convex and compact set of Radon probability measures on a compact Hausdorff set X, and an upper semi-continuous function f on X, the function $\mu \longrightarrow \mu(f)$ on K is also upper semi-continuous.
Moreover,

$$\operatorname*{Sup}_{\mu\in K} \mu(f) = \operatorname*{Inf}_{\phi\geqq f} \operatorname*{Sup}_{\mu\in K} \mu(\phi)$$

where the infimum is taken on the set of all continuous functions on X bounded below by f.

Proof. Corollary 1.3.5 implies that the function $\mu \longrightarrow \mu(f)$ is an infimum of a set of continuous functions, and thereby upper semi-continuous.
The minimax result is obtained by application of lemma 1.3.6.