THE THEORY OF TOPOLOGICAL SEMIGROUPS

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PREFACE

Our goal is to survey the field of topological semigroups, and to make this body of information accessible to the average graduate student. As a step in this direction, we offer the present volume, which lays the groundwork and covers several special avenues of recent research. The spirit of our exposition follows that of A. D. Wallace, through whose influence both in his research and by way of his students, the subject has developed since 1953. At that time his invited address to the American Mathematical Society stimulated much activity along the lines of the question "What topological spaces admit a continuous associative multiplication with unit?" As noted by Wallace, the answers to these questions are likely to involve more algebra and topology than was the case for compact groups, where there is a representation theory due to the presence of Haar measure. Our coverage here includes background material, internal structure, products, quotients, and semigroups with some special algebraic or topological property. We defer to a subsequent volume some of the aspects which rely on cohomology or category theory.

This material can serve as a text for an introduction to topological semigroups. It is not intended as a research tract, but rather to expose various aspects of the subject.

In the initial chapter we discuss the fundamental concepts of topological semigroups, their substructures, and maps between semigroups. Most of the notions here were introduced in the 1955 Wallace notes, and these will be further explored in the remaining chapters. Featured in the final section of this chapter is the Lawson-Madison theorem which generalizes an earlier result of Wallace on compact semigroups to k_ω-semigroups. We present their proof for locally compact σ-compact semigroups.

In the second chapter we demonstrate various techniques of developing new examples of semigroups from existing ones, and discuss the properties of these new semigroups inherited from the old semigroups. Constructions include free topological semigroups, Bohr compactifications, and products of various types.

Monothetic semigroups, which were characterized by the work of Koch, Numakura, and Hewitt, are discussed in the first section of the third chapter. The Wallace-Rees-Suschkewitch structure theorem for compact completely simple semigroups appears in this chapter. Most of the remaining portions of the chapter are devoted to algebraic considerations of Green's relations and quasi-orders.

Contributions of Clifford, Faucett, Mostert and Shields, Cohen and Krule, and Phillips are featured in the fourth chapter on interval semigroups (threads). Their structure and the nature of congruences on interval semigroups are characterized in this brief chapter.

The fifth chapter features the Carruth-Lawson proof of the classical Mostert-Shields theorem on the existence of one parameter semigroups in certain compact monoids. An important example due to Hunter appears at the end of the chapter.

In the final chapter we present results on compact divisible semigroups. This chapter features the results of Keimel, the contributions of Hudson and Hofmann, the structure theorems of Brown and Friedberg, and the Hildebrant characterization of compact subunithetic semigroups (the atoms of compact divisible semigroups).

More than a reference source for the text, the bibliography of this book is one of the most complete guides to the literature in topological semigroups to date, listing nearly 400 articles and books.

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CONTENTS

Chapter 1

FUNDAMENTAL CONCEPTS

In this chapter we present some concepts which we consider to be fundamental to the study of topological semigroups. Many of the results pertain to compact semigroups, as will be the situation for the entire book. However, some of the theorems are purely algebraic in nature or apply to a more general class of topological semigroups. The algebraic results are included to make the chapter as self-contained as we consider feasible. Some knowledge of elementary topology will be assumed. The notation in this chapter will be used throughout the book.

SEMIGROUPS

A *semigroup* is a non-empty set S together with an associative multiplication $(x,y) \to xy$ from $S \times S$ into S. The associative condition on S states that $x(yz) = (xy)z$ for each $x, y, z \in S$. If A and B are subsets of S, we use the notation $AB = \{ab : a \in A \text{ and } b \in B\}$. If S has a Hausdorff topology such that $(x,y) \mapsto xy$ is continuous, with the product topology on $S \times S$, then S is called a *topological semigroup*. The condition that the multiplication on S is continuous is equivalent to the condition that for each $x, y \in S$ and each open set W in S with $xy \in W$, there exist open sets U and V in S such that $x \in U$, $y \in V$, and $UV \subset W$.

If the word "semigroup" appears with a topological adjective, then "topological semigroup" is implied. For example, the statement "S is a compact semigroup" means that S is a compact topological semigroup. Observe that any semigroup can be made into a topological semigroup by giving it the discrete topology, and thus a finite semigroup is a compact semigroup.

Relations are one of the fundamental notions in the study of topological semigroups. We recall some of the basic concepts pertaining to relations. A *relation on a set* X is a subset of $X \times X$. The *diagonal* of X is the relation $\Delta(X) = \{(x,x) : x \in X\}$. When no confusion seems likely, we write simply Δ for the diagonal relation. If A and B are relations on X, then the *composition* of A with B is $A \circ B = \{(x,y) : (x,z) \in A \text{ and } (z,y) \in B \text{ for some } z \in X\}$. The *converse* of a relation R on X is $R^{-1} = \{(x,y) \in X \times X : (y,x) \in R\}$. A relation R on X is *reflexive* if $\Delta \subset R$, *symmetric* if $R = R^{-1}$, *transitive* if $R \circ R \subset R$, and an *equivalence* if R is reflexive, symmetric, and transitive. A relation R on X is *anti-symmetric* if $R \cap R^{-1} \subset \Delta$, a *quasi-order* if R is reflexive and transitive, and a *partial order* if R is an anti-symmetric quasi-order. If R is a relation on a set X, then $R^{(n)}$ is defined by $R^{(1)} = R$ and $R^{(n+1)} = R^{(n)} \circ R$ for each positive integer n. The *transitive closure* of R is defined $Tr(R) = \cup \{R^{(n)} : n \text{ a positive integer}\}$. It is readily seen that $Tr(R)$ is the smallest transitive relation of X containing R.

Although nets are not an essential part of the study of topological semigroups, they sometimes serve as a convenient tool in establishing results about topological semigroups. We recall a few basic notions about nets and directed sets.

A *directed set* is a pair $(D,\leqslant)$, where D is a non-empty set and $\leqslant$ is a reflexive and transitive relation on D such that for $\alpha, \beta \in D$ there exists $\gamma \in D$ such that $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$. Notice that we indicate that $(\alpha,\gamma) \in \leqslant$ in the traditional manner by $\alpha \leqslant \gamma$. A subset C of D is *cofinal* in D if for each $\alpha \in D$, there exists $\beta \in C$ such that $\alpha \leqslant \beta$, and a subset R of D is *residual* in D if there exists $\delta \in D$ such that if $\delta \leqslant \gamma$ in D, then $\gamma \in R$. If no confusion seems likely, we suppress the mention of $\leqslant$ and write simply D for the directed set.

A *net* in a set X is a function from a directed set D into X. We denote the image of $\alpha \in D$ by x_α and the net itself by $\{x_\alpha\}_{\alpha \in D}$ or simply $\{x_\alpha\}$ when no confusion seems likely. If T is a subset of X, then the net $\{x_\alpha\}_{\alpha\in D}$ is *frequently* in T provided $\{\alpha \in D : x_\alpha \in T\}$ is cofinal in D, and $\{x_\alpha\}_{\alpha\in D}$ is *eventually* in T if $\{\alpha \in D : x_\alpha \in T\}$ is residual in D. If X is a space, $x \in X$, and $\{x_\alpha\}$ is a net in X, then $\{x_\alpha\}$ *clusters* to x if $\{x_\alpha\}$ is frequently in each neighborhood of x, and $\{x_\alpha\}$ *converges* to x if $\{x_\alpha\}$ is eventually in each neighborhood of x. In the latter case we write $\{x_\alpha\} \to x$. A *subnet* of a net $f : (D,\leqslant) \to X$ is a pair (g,ψ), where $g : (E,\prec) \to X$ is a net and $\psi : E \to D$ is a function such that $g = f \circ \psi$ and for each $\alpha \in D$, there exists $\beta \in E$ such that if $\beta \prec \gamma$ in E, then $\alpha \leqslant \psi(\gamma)$ in D.

There are some useful topological results pertaining to nets. A net in a Hausdorff space converges to at most one point; A space X is compact if and only if each net in X has a convergent subnet; A subset C of a space X is closed if and only if for each net $\{x_\alpha\}$ in C converging to a point $x \in X$, we have $x \in C$; and a function f from a space X into a space Y is continuous if and only if $\{x_\alpha\} \to x$ in X implies $\{f(x_\alpha)\} \to f(x)$ in Y. Convergence can be replaced by clustering in Y or by clustering in both X and Y in the last result.

If S is a Hausdorff space endowed with an associative multiplication, then continuity of multiplication on S is equivalent to the statement that for nets in S, $\{x_\alpha\} \to x$ and $\{y_\alpha\} \to y$ implies that $\{x_\alpha y_\alpha\} \to xy$. Convergence of the net $\{x_\alpha y_\alpha\}$ can be replaced by clustering if one of the nets $\{x_\alpha\} \to x$ or $\{y_\alpha\} \to y$ is assumed only to cluster to the given point.

The following is one of the more useful topological results in the area of topological semigroups [Wallace, 1951]:

1.1 Theorem. Let X, Y, and Z be spaces, A a compact subset of X, B a compact subset of Y, $f : X \times Y \to Z$ a continuous function, and W an open subset of Z containing $f(A \times B)$. Then there exists an open set U in X and an open set V in Y such that $A \subset U$, $B \subset V$, and $f(U \times V) \subset W$.

Proof. Since f is continuous, $f^{-1}(W)$ is an open set in $X \times Y$ containing $A \times B$. For each (x,y) in $A \times B$, there exist open sets M and N in X and Y, respectively, such that $x \in M$, $y \in N$, and $M \times N \subset f^{-1}(W)$. Since B is compact, for a fixed $x \in A$, there are open sets

$M_1,\ldots,M_n$ in X containing x and corresponding open sets $N_1,\ldots,N_n$ in Y such that $B \subset Q = N_1 \cup \cdots \cup N_n$. Let $P = M_1 \cap \cdots \cap M_n$. Then P is open in X, Q is open in Y, $x \in P$, $B \subset Q$, and $P \times Q \subset f^{-1}(W)$. Since A is compact, there exists open sets $P_1,\ldots,P_m$ in X and corresponding $Q_1,\ldots,Q_m$ open in Y such that $B \subset V = Q_1 \cap \cdots \cap Q_m$ and $A \subset U = P_1 \cup \cdots \cup P_m$. It follows that U and V are the required open sets. ■

In 1.1, if one has the additional hypothesis that X is locally compact, then U can be chosen so that $\overline{U}$ is compact, and likewise if Y is locally compact, then V can be chosen so that $\overline{V}$ is compact.

1.2 Theorem. Let A and B be subsets of a topological semigroup S.

(a) If A and B are compact, then AB is compact.

(b) If A and B are connected, then AB is connected.

Proof. This is immediate from the fact that $AB = m(A \times B)$, where $m : S \times S \to S$ is multiplication. ■

Observe that "compact" or "connected" can be replaced in 1.2 by any topological property which is productive and preserved by continuous functions, e.g., "arcwise connected".

1.3 Theorem. Let A and B be subsets of a topological semigroup S. Then:

(a) If B is closed, then $\{x \in S : xA \subset B\}$ is closed;

(b) If B is compact, then $\{x \in S : A \subset xB\}$ is closed;

(c) If B is compact, then $\{x \in S : xA \subset Bx\}$ is closed;

(d) If A is compact and B is open, then $\{x \in S : xA \subset B\}$ is open; and

(e) If A is compact and B is closed, then $\{x \in S : xA \cap B \neq \square\}$ is closed.

Proof. To prove (a), let $C = \{x \in S : xA \subset B\}$ and fix $y \in S\backslash C$. Then there exists $a \in A$ such that $ya \in S\backslash B$. Since $S\backslash B$ is open, there exist open sets M and N in S such that $y \in M$, $a \in N$, and $MN \subset S\backslash B$. In particular, we have that $Ma \subset S\backslash B$. It follows that $y \in M \subset S\backslash C$, $S\backslash C$ is open, and hence C is closed.

To prove (b), let $F = \{x \in S : A \subset xB\}$ and fix $y \in S\backslash F$. Then $a \notin yB$ for some $a \in A$. Since yB is compact, there exist disjoint open sets M and N such that $a \in M$ and $yB \subset N$. In view of 1.1, we see that there exist open sets U and V such that $y \in U$, $B \subset V$, and $UV \subset N$. In particular, $UB \subset N$, $UB \cap M = \square$, $a \notin UB$, $U \subset S\backslash F$, $S\backslash F$ is open, and hence F is closed.

To prove (c), we apply 1.1 and the continuity of multiplication in S to conclude that $\{x \in S : xA \subset Bx\}$ is closed if B is compact.

To prove (d), we again apply 1.1.

To prove (e), we apply (d). ∎

To illustrate the usefulness of nets in topological semigroups, we present a net argument as an alternate proof of 1.3 (a).

To prove (a) using nets, let $\{x_\alpha\}$ be a net in $C = \{x \in S : xA \subset B\}$ such that $\{x_\alpha\} \to x$ with $x \in S$, and let $a \in A$. Then $\{x_\alpha a\} \to xa$ in B, and since B is closed, $xa \in B$. We obtain that $xA \subset B$, $x \in C$, and hence C is closed.

If S is a semigroup, $a \in S$, and n is a positive integer, then a^n is defined recursively by $a^1 = a$ and $a^{k+1} = a^k a$.

An element e of a semigroup S is called an *idempotent* if $e^2 = e$. The set of idempotents of S is denoted by E(S), or when no confusion seems likely, simply by E.

The set of idempotents of a semigroup may be empty, as is the case for the additive semigroup of positive integers. We will show later that if S is a compact semigroup, then $E \neq \square$. Moreover, in any topological semigroup S, we have that E is closed. The latter result we obtain as a consequence of the following topological result:

<u>*1.4 Theorem*</u>. Let X be a Hausdorff space and $f : X \to X$ a continuous function. Then the set of fixed points of f is closed in X.

Proof. Define $g : X \to X \times X$ by $g(x) = (f(x),x)$. Then g is continuous and so $g^{-1}(\Delta(X)) = \{x \in X : f(x) = x\}$ is closed, since the diagonal of a Hausdorff space X is closed in $X \times X$. ∎

1.5 Theorem. If S is a topological semigroup, then E(S) is a closed subset of S.

Proof. This is immediate from 1.4 and the observation that E is the set of fixed points of the continuous function $x \to x^2$. ■

1.6 Theorem. Let S be a topological semigroup. For e, $f \in E(S)$, define $e \leqslant f$ if $ef = fe = e$. Then $\leqslant$ is a partial order on E and is a closed subspace of $S \times S$.

Proof. The argument that $\leqslant$ is a partial order on E is straight forward. To see that $\leqslant$ is closed in $S \times S$, let $\{(e_\alpha, f_\alpha)\} \to (e,f)$ in $S \times S$ with $\{(e_\alpha, f_\alpha)\}$ in $\leqslant$. Then $\{e_\alpha\} \to e$ and $\{f_\alpha\} \to f$, and since E is closed in S, we have that $(e,f) \in E \times E$. Moreover, $e_\alpha f_\alpha = f_\alpha e_\alpha = e_\alpha$ for all α, so that the continuity of multiplication yields that $ef = fe = f$. We conclude that $(e,f) \in \leqslant$ and $\leqslant$ is closed. ■

If S is a semigroup and $a \in S$, then the function $x \to xa$ is called *right translation* by a and is denoted ρ_a, and $x \to ax$ is called *left translation* by a and is denoted λ_a. It is clear that if S is a topological semigroup and $a \in S$, then both ρ_a and λ_a are continuous. Moreover, in the case that S is a topological semigroup and $e \in E$, we have that ρ_e is a retraction of S onto Se, λ_e is a retraction of S onto eS, and $\rho_e \circ \lambda_e = \lambda_e \circ \rho_e$ is a retraction of S onto eSe.

A semigroup S is said to be *abelian* (or *commutative*) if $ab = ba$ for all $a, b \in S$.

An element e of a semigroup S is called a *left identity* for S if $ex = x$ for all $x \in S$, a *right identity* for S if $xe = x$ for all $x \in S$, and an *identity* for S if e is both a left and right identity. Observe that if e is either a left or right identity for S, then $e \in E$. Moreover, if S has a left identity e and a right identity f, then $e = ef = f$ is an identity for S. Thus each semigroup S can have at most one identity.

A semigroup which has an identity is called a *monoid*.

If S is a [topological] semigroup, we can adjoin an identity 1 to S [discretely] to form a new [topological] semigroup $T = S \cup \{1\}$. Note that if S is a compact semigroup, then T is a compact semigroup.

If S is a [topological] semigroup, then $S^1 = S$ if S has an identity, and $S \cup \{1\}$ [with 1 adjointed discretely] otherwise.

An element f of a semigroup S is called a *left zero* for S if $fx = f$ for all $x \in S$, a *right zero* for S if $xf = f$ for all $x \in S$, and a *zero* for S if f is both a left and right zero for S. As in the case of a left or right identity, a left or right zero for a semigroup S is an idempotent, and S can have at most one zero. If S is a [topological] semigroup, we can adjoin [discretely] a zero element in a manner analogous to that of adjoining an identity. We also define S^0 to be the obvious analog of S^1.

If S is a semigroup with a zero 0, and x is an element of S such that $x^n = 0$ for some positive integer n, then x is called a *nilpotent element* of S.

A *band* is a semigroup S such that $S = E(S)$, and a *semilattice* is a commutative band. Semilattices have been the topic for extensive study in both of the fields of algebraic and topological semigroups. A chapter of Volume II will be devoted to this topic.

We now turn to the presentation of a collection of examples of topological semigroups. Notation used in these examples will be applied throughout this book.

Let $\mathbb{N}$ be the additive semigroup of positive integers with the discrete topology. Then $\mathbb{N}$ is a non-compact abelian topological semigroup. Observe that $E(\mathbb{N}) = \square$. Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ denote the one-point compactification of $\mathbb{N}$ with $x + \infty = \infty + x = \infty$ for each $x \in \mathbb{N}^*$. Then $\mathbb{N}^*$ is a compact abelian semigroup with $E(\mathbb{N}^*) = \{\infty\}$, and ∞ is a zero.

Let $\mathbb{H}$ denote the additive semigroup of non-negative real numbers with the usual topology. Then $\mathbb{H}$ is a locally compact σ-compact connected abelian semigroup with $E(\mathbb{H}) = \{0\}$, and 0 is an identity. Let $\mathbb{H}^* = \mathbb{H} \cup \{\infty\}$ denote the one-point compactification of $\mathbb{H}$ with $\infty + x = x + \infty = \infty$ for each $x \in \mathbb{H}^*$. Then $\mathbb{H}^*$ is a compact connected abelian semigroup with $E(\mathbb{H}^*) = \{0,\infty\}$, 0 is an identity, and ∞ is a zero.

One can convert any non-empty Hausdorff space S into a topological semigroup by declaring that $xy = x$ for all $x, y \in S$. A semigroup with this multiplication is called a *left zero semigroup*. If we define $xy = y$ for all $x, y \in S$, then S is called a *right zero semigroup*. For a left zero semigroup, the multiplication on S is simply first projection $\pi_1 : S \times S \to S$, and second projection if S is a right zero semigroup.

If S is a Hausdorff space, $z \in S$, and we define $xy = z$ for all $x, y \in S$. Then S is called a *zero semigroup* , and is an abelian topological semigroup with zero z and $E = \{z\}$.

There are three fundamental types of semigroups on an interval which, as we will see later, are the building blocks of what we will call I-semigroups. We present these three basic examples now.

Let $I_u = [0,1]$ be the real unit interval with the usual topology and usual multiplication. Then I_u is a compact abelian semigroup called the *usual interval*. Note that 0 is a zero, 1 is an identity, $E = \{0,1\}$, and I_u has no nilpotent elements except 0. We shall show later that I_u and $\mathbb{H}^*$ are the same semigroup in a certain sense.

Let $I_n = [1/2,1]$ with the usual topology and multiplication $(x,y) \mapsto \min\{x,y\}$. Then I_m is a compact semilattice, 0 is a zero, and 1 is an identity. The semigroup I_m is called the *min interval*.

Let $I_n = \{1/2,1\}$ with the usual topology and multiplication $(x,y) \to \max\{1/2,xy\}$, where xy is the usual product of x and y. Then I_n is a compact abelian semigroup, $E = \{1/2,1\}$, 1/2 is a zero, 1 is an identity, and each element of $I_n \backslash \{1\}$ is nilpotent. The semigroup I_n is called the *nilpotent interval*.

SUBSEMIGROUPS

A considerable portion of the study of topological semigroups deals with determining the algebraic and topological structure of subsemigroups of a given class of topological semigroups. For example, each compact connected monoid contains an irreducible subsemigroup, and one of the major results in compact semigroup theory is that irreducible semigroups are abelian [Hofmann and Mostert, 1966]. In this section we develop the concept of a subsemigroup and give some examples.

If A is a subset of a semigroup S and $n \in \mathbb{N}$, then A^n is defined recursively by $A^1 = A$ and $A^{k+1} = A^kA$.

A *subsemigroup* of a semigroup S is a non-empty subset T of S such that $T^2 \subset T$.

Observe that a subsemigroup T of a [topological] semigroup S is itself a [topological] semigroup under the restriction of the multiplication on S to $T \times T$.

If A is a subset of a semigroup S, then the set T of all finite products of elements of A is the smallest subsemigroup of S containing A and T is called the *subsemigroup of* S *generated by* A. Observe that $T = \cup \{A^n : n \in \mathbb{N}\}$.

It is evident that the intersection (if non-empty) of a collection of subsemigroups of a semigroup S is again a subsemigroup of S. To see that the union of subsemigroups need not be a subsemigroup, consider the semigroup $\mathbb{N}$ and the semigroups $\{2n : n \in \mathbb{N}\}$ and $\{3n : n \in \mathbb{N}\}$.

Certain subsemigroups of a given semigroup appear with such regularity in the remainder of this book that they are isolated here and some of their basic properties are presented.

If S is a semigroup and $x \in S$, then $\theta(x) = \{x^n : n \in \mathbb{N}\}$ is an abelian subsemigroup of S and is the subsemigroup of S generated by x. If $A \subset S$, then $Z(A) = \{b \in S : ba = ab \text{ for all } a \in A\}$ is called the *centralizer* of A in S, and $N(A) = \{b \in S : bA = Ab\}$ is called the *normalizer* of A in S. The set $Z(S)$ is called the *center* of S. If $N(A) = S$, then A is said to be *normal* in S. If $N(S) = S$, then S is called a *normal semigroup*.

If S is a topological semigroup and A and B are subsets of S, observe that $\overline{A}\,\overline{B} \subset \overline{AB}$. If $\overline{A}$ and $\overline{B}$ are both compact then $\overline{A}\,\overline{B} = \overline{AB}$ since in this case, $\overline{A}\,\overline{B}$ is closed and contains AB.

From the preceding observations one can conclude that if T is a subsemigroup of a topological semigroup S, then $\overline{T}$ is also a subsemigroup of S. Moreover, if T is abelian, then so is $\overline{T}$. To see this, let $f : S \times S \to S \times S$ be defined by $f(x,y) = (xy,yx)$. Then f is continuous and hence $f^{-1}(\Delta)$ is closed. Now if T is abelian, then

$T \times T \subset f^{-1}(\Delta)$, and since $\overline{T} \times \overline{T} = \overline{T \times T} \subset f^{-1}(\Delta)$, we have that $\overline{T}$ is abelian.

If A is a subset of a topological semigroup S, then the smallest closed subsemigroup of S containing A is the closure of the subsemigroup of S generated by A. In particular, if $x \in S$, then $\Gamma(x) = \overline{\theta(x)}$ is a closed subsemigroup of S called the *monothetic subsemigroup of* S *with generator* x. If $S = \Gamma(x)$, then S is called a *monothetic semigroup*.

1.7 Theorem. Let S be a topological semigroup, $x \in S$, A a non-empty subset of S, and $e \in E(S)$. Then:

(a) $\theta(x)$ is an abelian subsemigroup of S and is the smallest subsemigroup of S containing x;

(b) $\Gamma(x)$ is an abelian subsemigroup of S and is the smallest closed subsemigroup of S containing x;

(c) eS is a closed subsemigroup of S with left identity e;

(d) Se is a closed subsemigroup of S with right identity e;

(e) $eSe = eS \cap Se = \{x \in S : ex = x = xe\}$ is a closed subsemigroup of S with identity e;

(f) Z(A) is a closed subsemigroup of S if $Z(A) \neq \square$;

(g) $Z(A) \subset N(A)$;

(h) N(A) is a subsemigroup of S if $N(A) \neq \square$, and is closed if A is compact;

(i) AS and SA are subsemigroup of S, and if S and A are compact, then AS and SA are closed; and

(j) if $E(S) \subset N(S)$, then $E(S) \subset Z(S)$. ■

We close this section by presenting some examples of semigroups and mentioning certain of their distinguished subsemigroups.

Let $\mathbb{C}$ denote the space of complex numbers with complex multiplication. Then $\mathbb{C}$ is an abelian topological semigroup with a zero and an identity and no other idempotents. The non-zero elements of $\mathbb{C}$ form a non-closed subsemigroup as do the non-zero real elements of