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PREFACE

It is a fundamental fact of nature that the space we live in is three-dimensional. Consequently, many branches of applied mathematics and theoretical physics are concerned with physical quantities defined in 3-space, as I shall call it; these subjects include Newtonian mechanics, fluid mechanics, theories of elasticity and plasticity, non-relativistic quantum mechanics, and many parts of solid state physics. The Greek geometers made the first systematic investigation of the properties of 'ordinary' 3-space, and their work is known to us mainly through the books of Euclid; our basic geometrical ideas about the physical world have their origins in Euclidean geometry. A major advantage of Euclid's work was its presentation as a deductive system derived from a small number of definitions and axioms (or 'basic assumptions'); although Euclid's axioms have turned out to be inadequate in a number of ways, he nevertheless provided us with a model of what a proper mathematical system should be [Reference P.1].

Through the introduction of coordinate systems, Descartes linked geometry with algebra [Reference P.2]; geometrical structures in 3-space such as lines, planes, circles, ellipses and spheres, were associated with algebraic equations involving three Cartesian coordinates (x, y, z). Then in the nineteenth century, Hamilton [Reference P.3] and Gibbs [Reference P.4] introduced two similar types of algebraic objects, 'quaternions' and 'vectors', which treated the three coordinates simultaneously; the rules of operation of these new sets of objects were different from those of real or complex numbers, giving rise to new types of 'algebra'; a more general algebra of N-dimensional space $(N = 3, 4, 5, \ldots)$ was introduced by Grassmann [Reference P.5]. Over several decades, the vector concept developed in two different ways: in a wide variety of physical applications, vector notation and techniques became, by the middle of this century, almost universal; on the other hand, pure mathematicians reduced vector algebra to an axiomatic system, and introduced wide generalisations of the concept of a three-dimensional 'vector space', not only to N-dimensional spaces, but also to Hilbert space and other infinite-dimensional metric spaces, and to topological spaces. These two developments proceeded largely independently, and many books dealing with the applications of vectors have approached the fundamentals of the subject intuitively rather than axiomatically, assuming some prior knowledge of Euclidean and Cartesian geometry. In recent decades, however, hard-and-fast distinctions between 'pure' and 'applied' mathematics have been disappearing; in particular, the concept of an abstract 'space', especially Hilbert space, has become familiar in many applications of mathematics, including quantum mechanics, numerical analysis and statistics, and in the study of differential and integral equations. Also, the concept of 'basic assumptions' or 'structure', in dealing with number systems, has taken its place in school mathematics [Reference P.6]; while these basic assumptions are not presented as a complete logical scheme in the way that Euclid intended, they nevertheless familiarise students with the concept of an axiomatic scheme. For these reasons, it seems appropriate to take account of both pure and applied mathematical points of view when treating the subject of 'vectors', which is now a fundamental part of both these modes of thought.

This book deals with vector algebra and analysis, and with their application to three-dimensional geometry and to the analysis of fields in 3-space. In order to bring out both the 'pure' and 'applied' aspects of the subject, my main objectives have been:

(i) to base the work on sound algebraic and analytic foundations;
(ii) to develop those intuitive relations between algebraic equations and geometrical concepts which are of fundamental importance in physical applications;
(iii) to establish standard vector techniques and theorems, giving numerous examples of their use.

In the first three chapters, the algebra of vectors is developed, based upon the axioms of vector space algebra; as the axioms are introduced, their geometrical interpretation is given, so that they can be understood intuitively. The axiomatic scheme is extended to provide a definition of Euclidean space, consisting of 'points' and 'displacements'; this provides an axiomatic basis for Euclidean geometry, linking it directly with the algebra of linear vector spaces. This linkage has the reciprocal advantage (not apparent in this book) that it enables geometrical intuitions to be developed in dealing with more general types of linear space, in particular with finite-dimensional spaces and Hilbert space. In the process of interpreting the algebraic axioms geometrically, algebraic definitions of elementary geometrical concepts such as 'length' and 'angle' have to be given and justified; we also define Cartesian or 'rectangular' coordinates and establish their fundamental properties, such as Pythagoras' theorem. By this means, Cartesian geometry and trigonometry, as well as Euclidean geometry, are seen to arise out of a single set of axioms. The first three chapters also develop the techniques of vector algebra, and apply them to problems in geometry, in particular the geometry of lines and planes.

The fourth chapter deals with transformations of the components of a vector in two or three dimensions, in particular with transformations representing rotations and reflections. A clear distinction is made between 'active transformations', due to a change of the vector itself, and 'passive transformations', due to change of the frame of reference. The idea of groups of transformations is introduced, and the study of rotations in two dimensions is linked with the intuitively familiar concept of 'addition of angles'. Transformations in 3-space are represented by 3×3 matrices. Although it has been assumed that the reader has some familiarity with matrices, the necessary theory of 3×3 matrices and their determinants has been developed in the first two sections of Chapter 4, using the properties of vectors established earlier. This emphasises the fact that vectors and matrix algebra are simply two different aspects of the algebra of vector spaces. It is of interest to note that this vectorial approach to matrix algebra can be made quite general, and is not restricted to 3×3 matrices.

The study of functions $f(x)$, where x is a variable lying in a continuous range, depends to a great extent upon the differential and integral calculus. When we study functions defined in 3-space, it is necessary to develop an extension of calculus appropriate to regions of this space. There are several difficult problems to solve before this extended calculus can be defined. First, we have to study how points in 3-space are specified by systems of coordinates; second, we have to give definitions of curves, surfaces and volume regions in 3-space; third, if we consider a specific surface or volume region, we need to define the 'boundary' of that region. These problems are dealt with in Chapter 5. Since points in 3-space are described by three coordinates, this work necessarily involves using analytic properties of functions of up to three variables, and of their derivatives and integrals. This raises a problem of presentation: establishing the necessary analytic properties of functions of one, two and three variables requires a substantial amount of work, whose incorporation in Chapter 5 would break the continuity of ideas developed there. Elementary analysis of functions of one variable is normally dealt with early in university mathematics courses, and is the subject of a large number of textbooks; so when I use an analytic property of one-variable functions, I simply quote a reference in one of the most readable elementary books on the subject, J. C. Burkill's *A First Course in Mathematical Analysis*. The analysis of functions of several variables is appreciably more complicated, and it is arguable that in an elementary textbook, we should not trouble about proofs of properties of partial derivatives and multiple integrals. In a book for students of mathematics, however, it is unsatisfactory to omit explanations simply because they are complicated. I have met this difficulty by establishing the essential properties of functions of *two* variables in Appendix A, to which reference is made when these properties are used in the main text; the necessary properties of functions of *three* variables are simple generalisations of those of two variables, and when they are used, I again refer to the analogous property of two-variable functions. A reader can therefore either accept the analytic properties assumed in the main text, or refer to Appendix A for a justification of these assumptions. By omitting this analytic detail from Chapter 5, it is possible to give a fairly detailed account of surfaces, volume regions, and especially of curves.

Scalars and vectors whose value depends upon their position in space are called scalar and vector 'fields', provided that they satisfy suitable analytic conditions. Since these fields in general depend upon three coordinates, variations in a field throughout 3-space depend upon the derivatives of the field with respect to three coordinates; certain combinations of derivatives, 'divergence', 'gradient' and 'curl', known as vector operators, are closely associated with physical concepts such as flux and vorticity. In the final chapter, these operators are defined and studied, and their physical significance is emphasised. As in Chapter 5, it is necessary to be careful over analytic details; the vector operators are defined in a mathematically sound way, but in the discussion of their physical significance, I have thought it best to omit some analytic details. One theorem (Stokes' theorem) is difficult to prove in full generality: its significance is brought out by proving it under special conditions in the main text; the general proof is given in Appendix B. The discussion of physical examples leads naturally to the introduction of the 'Laplacian' operator; this completes the definition and discussion of the principal differential operators used in a variety of branches of mathematical physics, and provides a natural point at which to end.

The first three chapters of this book arose out of a course of lectures given to first-year mathematics students at the University of Kent. Although the book is written primarily for students of mathematics in the early part of their University course, those interested in the more mathematical aspects of physics and engineering may prefer this treatment of vectors based on linear space algebra, since linear spaces have a rapidly widening relevance in these disciplines. A number of my former students have chosen to follow this approach in sixth-form mathematics teaching, and those studying advanced school mathematics may find that the first four chapters of the book provide a coherent picture of a number of sixth-form topics which are often treated separately.

While writing this book, I have had many helpful discussions with other members of staff of the School of Mathematics in the University of Kent. I am particularly indebted to Dr R. Hughes Jones for many exchanges of ideas, not only while the book was being written, but also when I was formulating the approach to Chapters 1–3. I am very grateful to Mrs Sandra Bateman and Miss Diane Mayes for their careful preparation of the manuscript, and for their patience in coping with a long series of additions and amendments. The Cambridge University Press have been most helpful and thorough in checking and tidying up the manuscript; I wish to thank them for their help, and also Miss Ruth Farwell for checking the examples and problems. I have been pleased to have the student's-eye comments of

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Roy Chisholm

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CONTENTS

1

Linear spaces and displacements

1.1 Introduction

Our understanding of the physical world depends to a great extent on making more or less exact measurements of a variety of physical quantities. All single measurements on a physical system consist of observing a single real number, and very often this single real number is, by itself, the value of an important physical quantity; examples are the measurement of a mass, a length, an interval of time, an electrical potential, the frequency or wavelength of an electromagnetic wave, a quantity of electrical charge, and the electric current in a wire. Physical quantities of this kind are called **scalar quantities**, or, more frequently, **scalars**. We shall make a distinction between these two expressions: 'scalar' will be used as a *mathematical* expression; scalars, for our purposes, are real algebraic variables λ, μ, . . . , which can, in general, take values in the whole range $(-\infty, \infty)$; they possess other properties which will be defined in Chapter 4, but for the present we shall regard them simply as real numbers. The expression 'scalar quantity' will refer to any specific physically measurable quantity, such as a mass or a charge, which is found experimentally to have the mathematical properties of a scalar. One important property of scalar quantities is that they are intrinsic properties of a physical system, and do not change if the whole physical system is translated to a different position in three-dimensional space, or is rotated in space. For example, if a metallic conductor is at a certain potential in an electric field produced by certain electric charges, this potential is unchanged if the conductor *and* the charges are translated or rotated as a whole, their relative positions remaining unchanged. Similarly, the mass of a body is independent of the position and orientation of the body in three-dimensional space.

Not all measured quantities are best understood as a single number. A change in position in three-dimensional space from a point P to a point Q, known as a **displacement**, depends upon a number (the distance from P to Q), but also depends upon the direction from P to Q. There are various ways of defining a displacement; the most familiar is to define a set of Cartesian axes, with the origin at P, as in Fig. 1.1. Then the displacement PQ is defined by giving the projections (x, y, z) of the line PQ on the three axes. Other examples of physical quantities with which we intuitively associate both a real number (the magnitude) and a direction in space are force, velocity, the electric, magnetic or gravitational field at a point in space, and the direction normal (that is, perpendicular) to a given plane in space. Physical quantities of this type are known as **vector quantities**; the corresponding abstract mathematical entities, whose properties we now start to define, are called **vectors**.

We shall define vectors by assuming that they obey certain basic algebraic equations, the **axioms** of vector algebra. From these axioms we shall be able to deduce the usual geometric properties of displacements in three-dimensional space; for example, we can show that the lengths PQ, x, y and z in Fig. 1.1 obey the Pythagorean relation

$$PQ^2 = x^2 + y^2 + z^2. \tag{1.1}$$

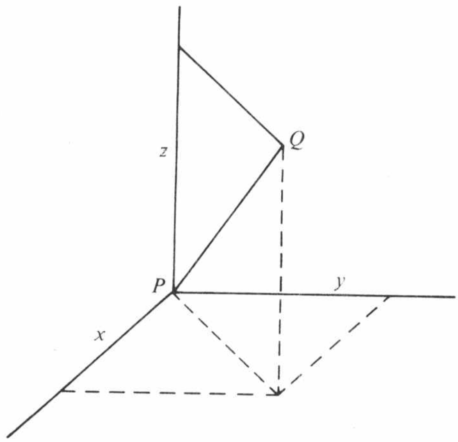

Fig. 1.1

The algebraic axioms are not necessarily associated with any geometrical interpretation; however, the interpretation of vectors as displacements is such a natural and familiar one that we inevitably think in geometrical terms when we discuss vectors; moreover, the geometric picture is a great aid to our intuition about vector quantities. So, on the one hand, we shall derive vector algebra from axioms written in algebraic form, and shall eventually deduce three-dimensional Euclidean geometry from these axioms; on the other hand, we shall, from the beginning, interpret the axioms and other equations intuitively in terms of three-dimensional geometry, with the vectors represented by displacements.

We denote vectors by symbols such as **a**, **b**, **r**, **n** and **u**. A set of vectors satisfying certain conditions is denoted by {**a**}, for example. Geometrically, a vector **a** is represented by a 'directed line' in space, as in Fig. 1.2. With any vector **a** we associate a unique non-negative

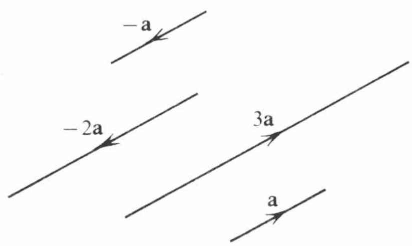

Fig. 1.2

real number a, called the **modulus** or **magnitude** of the vector. We frequently say that a is the 'length' of the vector; in saying this, we are using the geometrical interpretation of a vector as a spatial displacement. Although we do not give a definition of modulus or length a until Chapter 2, we shall use the concept in talking about the geometrical interpretation of vectors.

1.2 Scalar multiplication of vectors

A vector **a** can be multiplied by any real number λ to give another vector $\lambda\mathbf{a}$. If $\lambda > 0$ and if **a** represents a displacement, $\lambda\mathbf{a}$ is a displacement in the same direction as **a**, but with magnitude λa; so $1\mathbf{a}$

is simply **a** itself. The displacement 3**a** is indicated in Fig. 1.2. We have drawn the displacements representing 3**a** and **a** in different positions. As we shall discuss fully in §1.4, a displacement has a definite 'initial point' in space, and the displacement is 'from' this point; because displacements have a definite position in space, they are often referred to as 'fixed vectors'. The abstract vectors **a** and 3**a**, however, have no initial points in space – it is, in fact, rather misleading to represent them by directed lines in a diagram. In order to remind ourselves that vectors are *not* associated with points in space, we represent them (as in Fig. 1.2) by directed lines at arbitrarily chosen points; abstract vectors are for this reason sometimes called 'free vectors'.

Multiplication of a vector **a** by -1 gives a vector denoted by $-\mathbf{a}$; this vector is represented by a displacement of the same length as **a**, and in exactly the opposite direction, as indicated in Fig. 1.2. When $\lambda < 0$, the vector $\lambda\mathbf{a}$ is again represented by a displacement in the opposite direction to **a**; its length is $|\lambda|a$, where $|\lambda|$ is the absolute value of λ. For example, a displacement representing the vector $-2\mathbf{a}$ is as shown in Fig. 1.2; note that the arrows on $-\mathbf{a}$ and $-2\mathbf{a}$ are in the opposite sense to those on **a** and 3**a**.

The formal axioms governing multiplication by finite real scalars λ and μ are:

(1A) If **a** is a vector, and λ any real number, then $\lambda\mathbf{a}$ is a vector,

(1B) $$1\mathbf{a} = \mathbf{a}, \tag{1.2}$$

(1C) $$\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}.$$

The Axiom (1B) tells us that multiplication by unity does not change a vector **a**. Since $\lambda\mu = \mu\lambda$ on the right of Axiom (1C), we can extend the axiom to give

$$\lambda(\mu\mathbf{a}) = \mu(\lambda\mathbf{a}) = (\mu\lambda)\mathbf{a}. \tag{1.3}$$

So Axiom (1C) tells us that the order of multiplication by two scalars (λ and μ) does not matter, since the result is equivalent to multiplication by $\lambda\mu$. In formal language, (1.3) tells us that scalar multiplication of vectors is **associative** and **commutative**. If $\lambda \neq 0$ and $\mu = \lambda^{-1}$, (1.2) and (1.3) give

$$\lambda^{-1}(\lambda\mathbf{a}) = 1\mathbf{a} = \mathbf{a}.$$

This means that **a** is a scalar multiple of all vectors $\lambda\mathbf{a}$ ($\lambda \neq 0$). Geometrically, displacements corresponding to **a** and $\lambda\mathbf{a}$ are said to be 'parallel'; this is explained more fully in §1.4. Although we have referred to the modulus of a vector in discussion, we note that this has not been defined by the Axioms (1A)–(1C).

When $\lambda = 0$, the Axiom (1A) implies the existence of a vector $0\mathbf{a}$. A displacement corresponding to $0\mathbf{a}$ is of zero length, and so is no displacement at all; so for all vectors **a** we write

$$0\mathbf{a} = \mathbf{0} \tag{1.4}$$

defining the **zero vector 0**. The essential point of equation (1.4) is that **0** is the same vector, whatever **a** is. We formalize this into the axiom:

(1D) There is a unique vector **0**, called the zero vector, which satisfies

$$0\mathbf{a} = \mathbf{0},$$

for all vectors **a**.

The uniqueness of the zero vector is an important property of three-dimensional space, and is also a property of many more complicated 'spaces' occurring in mathematics and mathematical physics. Equation (1.4) is 'intuitively obvious', but this is only because of our everyday experience of displacements; in formulating an abstract mathematical theory of vectors, the obvious needs to be stated explicitly.

1.3 Addition and subtraction of vectors

The second set of axioms for vectors {**a**} define the laws of **addition of vectors**. They embody many familiar properties of displacements in space, and after stating the axioms, we shall discuss their geometric meaning. The operation of addition is denoted by the symbol '+'. It may appear confusing to use the same symbol for addition of numbers (scalars) and for addition of vectors; there are two reasons why confusion does not arise:

(i) the sum of two scalars $\lambda + \mu$ contains scalars (λ and μ), while the sum of two vectors $\mathbf{a}+\mathbf{b}$ contains vectors (**a** and **b**);

(ii) the axioms of addition and scalar multiplication of vectors are very similar to axioms of addition and multiplication of scalars.

The axioms of vector addition are:

(2A) $\mathbf{a}+\mathbf{b}$ is a vector, for any two vectors **a**, **b**,

(2B) $$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}, \tag{1.5}$$

(2C) $$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}, \quad (1.6)$$

(2D) $$(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}, \quad (1.7)$$

(2E) $$\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}. \quad (1.8)$$

The addition of **a** and **b** to give the **vector sum a+b** is represented in Fig. 1.3. The two vectors are represented by displacements **PQ** and **PS** from the point *P*. The point *R* is chosen so that *PQRS* is a parallelogram; then **PR** represents the vector sum **a+b**. Geometrically, this rule of combination, known as the **parallelogram law**, is obviously symmetrical between **a** and **b**; this symmetry is built in to vector algebra in Axiom (2B); algebraically, this axiom is known as the **commutative law of addition** of vectors. It has been already pointed out that representing vectors by displacements can be misleading; this shows up in Fig. 1.3, where it is more natural to think

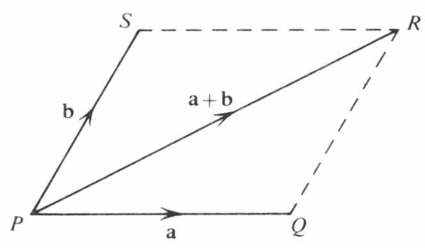

Fig. 1.3

of combining displacements **PQ** and **QR** to give the displacement **PR**. We shall see in §1.4, however, that this is not an accurate way of representing vector addition. A closer physical analogy to vector addition is the experimental law of combination of two forces acting at a point *P*: if they are represented by the vectors **a** and **b**, then they are equivalent to a force represented by **a+b**, also acting at *P*; Fig. 1.3 is then interpreted as the 'parallelogram of forces'.

Axiom (2C) is the **associative law of addition** of vectors; **a+(b+c)** is the vector formed by first adding **b** and **c** to give **(b+c)** and then adding this to **a**; this process is represented in Fig. 1.4, with **a+(b+c)** represented by **PT**. Likewise, **(a+b)+c** is represented by **PT** in Fig. 1.5. Axiom (2C) has the interpretation that the same displacement **PT** is defined by the two processes.