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Elementary Topics in Differential Geometry

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Preface

In the past decade there has been a significant change in the freshman/sophomore mathematics curriculum as taught at many, if not most, of our colleges. This has been brought about by the introduction of linear algebra into the curriculum at the sophomore level. The advantages of using linear algebra both in the teaching of differential equations and in the teaching of multivariate calculus are by now widely recognized. Several textbooks adopting this point of view are now available and have been widely adopted. Students completing the sophomore year now have a fair preliminary understanding of spaces of many dimensions.

It should be apparent that courses on the junior level should draw upon and reinforce the concepts and skills learned during the previous year. Unfortunately, in differential geometry at least, this is usually not the case. Textbooks directed to students at this level generally restrict attention to 2-dimensional surfaces in 3-space rather than to surfaces of arbitrary dimension. Although most of the recent books do use linear algebra, it is only the algebra of $\mathbb{R}^3$. The student's preliminary understanding of higher dimensions is not cultivated.

This book develops the geometry of n-dimensional surfaces in $(n + 1)$-space. It is designed for a 1-semester differential geometry course at the junior-senior level. It draws significantly on the contemporary student's knowledge of linear algebra, multivariate calculus, and differential equations, thereby solidifying the student's understanding of these subjects. Indeed, one of the reasons that a course in differential geometry is so valuable at this level is that it does turn out students with a thorough understanding of several variable calculus.

Another reason that differential geometry regularly attracts students is that it contains ideas which are not only beautiful in themselves but are basic for both advanced mathematics and theoretical physics. It has been the author's experience that students taking his course have been more or less evenly divided between mathematics and physics majors. The approach adopted in this book, describing surfaces as solution sets of equations, seems to be especially attractive to physicists.

The book considers from the outset the geometry of orientable hypersurfaces in $\mathbb{R}^{n+1}$, exhibited as inverse images of regular values of smooth functions. By considering only such hypersurfaces for the first half of the book, it is possible to move rapidly into interesting global geometry without getting hung up on the development of sophisticated machinery. Thus, for example, charts (coordinate patches) are not introduced until after the initial discussions of geodesics, parallelism, curvature, and convexity. When charts are introduced, it is as a tool for computation. However, they then lead the development naturally into the study of focal points and surfaces of arbitrary codimension.

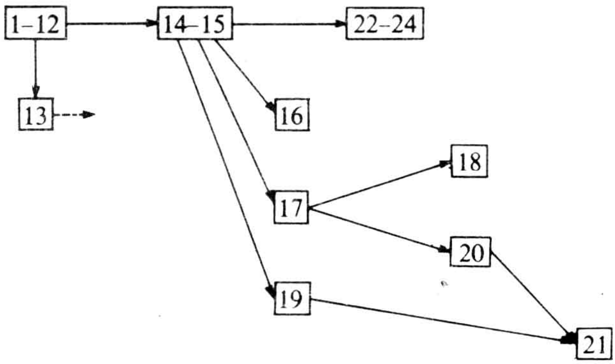
One of the advantages of treating the geometry of n-dimensions from the outset is that one can then illustrate each concept simultaneously in each of the low dimensions. Thus, for example, the student's understanding of the Gauss map and its (spherical) image is aided by the possibility of studying 1-dimensional examples, where the spherical image is a subset of the unit circle.

The main tool used in developing the theory is that of the calculus of vector fields. This seems to be the most natural tool for studying differential geometry as well as the one most familiar to undergraduate students of mathematics and physics. Differential forms are not introduced until fairly late in the book, and then only as needed for use in integration.

Students who have completed a good 2-year calculus sequence including linear algebra and differential equations should be adequately prepared to study this book. There are occasional places (e.g., in Chapter 13 on convexity) where some exposure to the ideas of mathematical analysis would be helpful, but not essential.

There is probably more material here than can be covered comfortably in one semester except by students with unusually strong backgrounds. Chapters 1–12, 14, 15, 22, and 23 contain the core of basic material which should be covered in every course. Most instructors will probably also want to cover at least parts of Chapters 17, 19, and 24.

The interdependence of the chapters is as follows:



A few concepts in the early part of Chapter 13 are used in later chapters but these may be studied, by those skipping Chapter 13, as needed.

Like the author of any textbook, I owe a considerable debt to researchers and textbook writers who have preceded me and to teachers, colleagues, and students who have influenced me. While I cannot explicitly acknowledge all these, I must at least credit M. do Carmo and E. Lima whose paper, Isometric immersions with semi-definite second quadratic forms, *Arch. Math.* *20* (1969) 173–175, inspired the treatment of convex surfaces in Chapter 13, and S. S. Chern whose paper, A simple intrinsic proof of the Gauss–Bonnet formula for closed Riemannian manifolds, *Ann. of Math.* (2) *45* (1944) 747–752, inspired the treatment of the Gauss–Bonnet theorem in Chapter 21. In addition, special thanks are due to Wolfgang Meyer whose comments on the manuscript have been extremely helpful.

Stony Brook, New York JOHN A. THORPE
November, 1978

Contents

Graphs and Level Sets 1

Associated with each real valued function of several real variables is a collection of sets, called level sets, which are useful in studying qualitative properties of the function. Given a function $f\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^{n+1}$, its *level sets* are the sets $f^{-1}(c)$ defined, for each real number c, by

$$f^{-1}(c) = \{(x_1, \ldots, x_{n+1}) \in U : f(x_1, \ldots, x_{n+1}) = c\}.$$

The number c is called the *height* of the level set, and $f^{-1}(c)$ is called the level set *at height* c. Since $f^{-1}(c)$ is the solution set of the equation $f(x_1, \ldots, x_{n+1}) = c$, the level set $f^{-1}(c)$ is often described as "the set $f(x_1, \ldots, x_{n+1}) = c$."

The "level set" and "height" terminologies arise from the relation between the level sets of a function and its graph. The *graph* of a function $f\colon U \to \mathbb{R}$ is the subset of $\mathbb{R}^{n+2}$ defined by

$$\text{graph}(f) = \{(x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} : (x_1, \ldots, x_{n+1}) \in U$$
$$\text{and } x_{n+2} = f(x_1, \ldots, x_{n+1})\}.$$

For $c \geq 0$, the level set of f at height c is just the set of all points in the domain of f over which the graph is at distance c (see Figure 1.1). For $c < 0$, the level set of f at height c is just the set of all points in the domain of f under which the graph lies at distance $-c$.

For example, the level sets $f^{-1}(c)$ of the function $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$ are empty for $c < 0$, consist of a single point (the origin) if $c = 0$, and for $c > 0$ consist of two points if $n = 0$, circles centered at the origin with radius $\sqrt{c}$ if $n = 1$, spheres centered at the origin with radius $\sqrt{c}$ if $n = 2$, etc (see Figures 1.1 and 1.2).

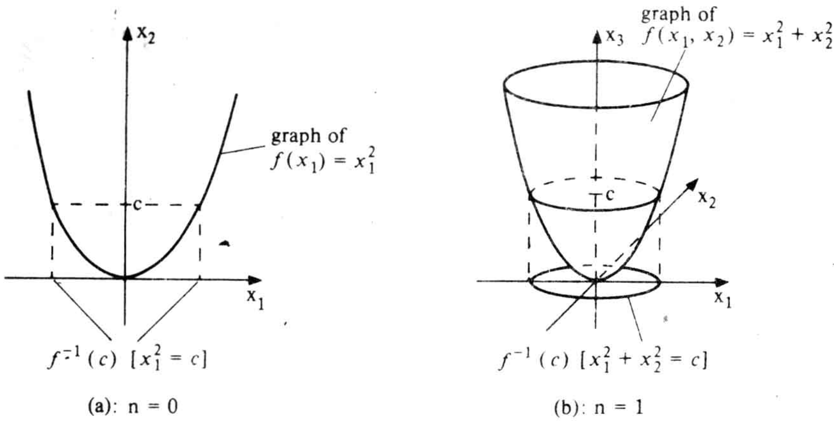


Figure 1.1 The level sets $f^{-1}(c)$ $(c > 0)$ for the function $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$.

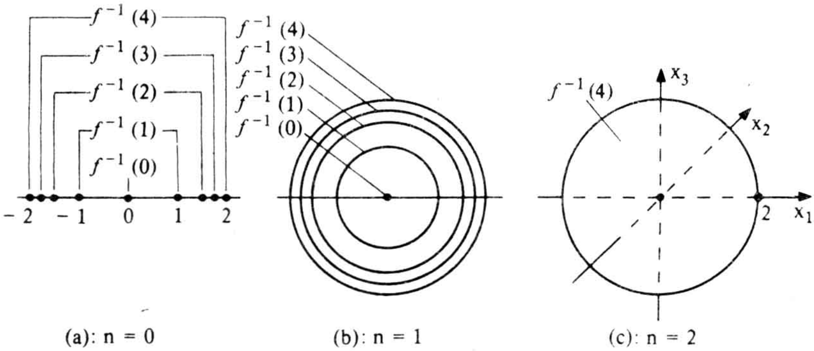


Figure 1.2 Level sets for the function $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$.

For $n = 1$, level sets are (at least for non-constant differentiable functions) generally curves in $\mathbb{R}^2$. These curves play the same roles as contour lines on a topographic map. If we think of the graph of f as a land, with local maxima representing mountain peaks and local minima representing valley bottoms, then we can construct a topographic map of this land by projecting orthogonally onto $\mathbb{R}^2$. Then all points on any given level curve $f^{-1}(c)$ correspond to points on the land which are at exactly height c above "sea level" $(x_3 = 0)$.

Just as contour maps provide an accurate picture of the topography of a land, so does a knowledge of the level sets and their heights accurately portray the graph of a function. For functions $f: \mathbb{R}^2 \to \mathbb{R}$, study of the level curves can facilitate the sketching of the graph of f. For functions $f: \mathbb{R}^3 \to \mathbb{R}$, the graph lies in $\mathbb{R}^4$, prohibiting sketches and leaving the level sets as the best tools for studying the behavior of the function.

One way of visualizing the graph of a function $f\colon U \to \mathbb{R}$, $U \subset \mathbb{R}^2$, given its level sets, is as follows. Think of a plane, parallel to the (x_1, x_2)-plane, moving vertically. When it reaches height c this plane, $x_3 = c$, cuts the graph of f in the translate to this plane of the level set $f^{-1}(c)$. As the plane moves, these sets generate the graph of f (see Figure 1.3).

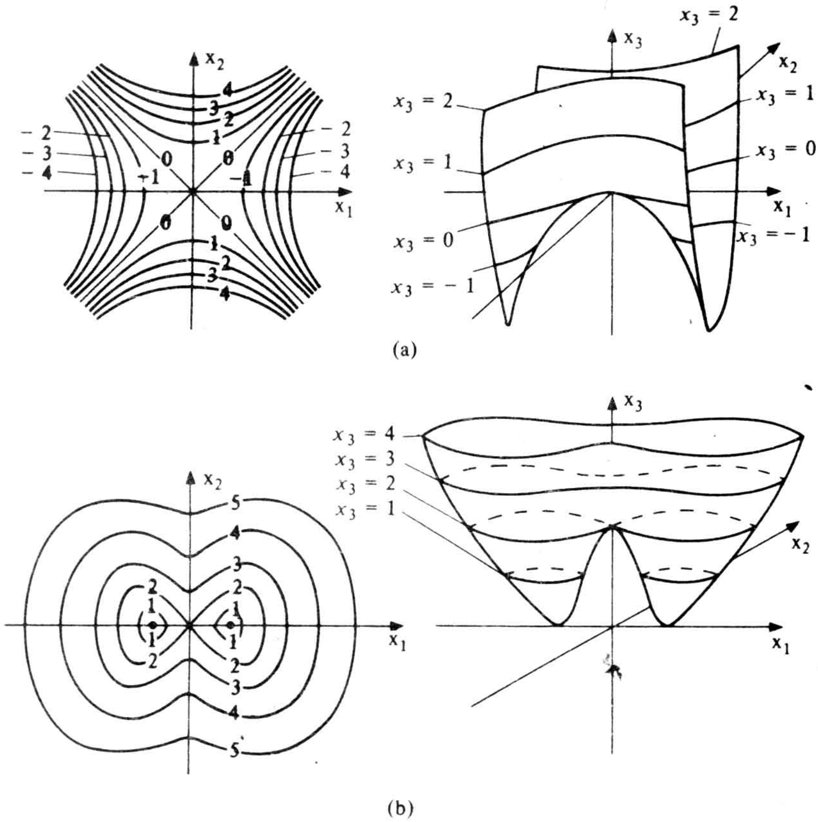


Figure 1.3 Level sets and graphs of functions $f\colon \mathbb{R}^2 \to \mathbb{R}$. The label on each level set indicates its height. (a) $f(x_1, x_2) = -x_1^2 + x_2^2$. (b) A function with two local minima.

The same principle can be used to help visualize level sets of functions $f\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^3$. Each plane $x_i =$ constant will cut the level set $f^{-1}(c)$ (c fixed) in some subset, usually a curve. Letting the plane move, by changing the selected value of the x_i-coordinate, these subsets will generate the level set $f^{-1}(c)$ (see Figure 1.4).

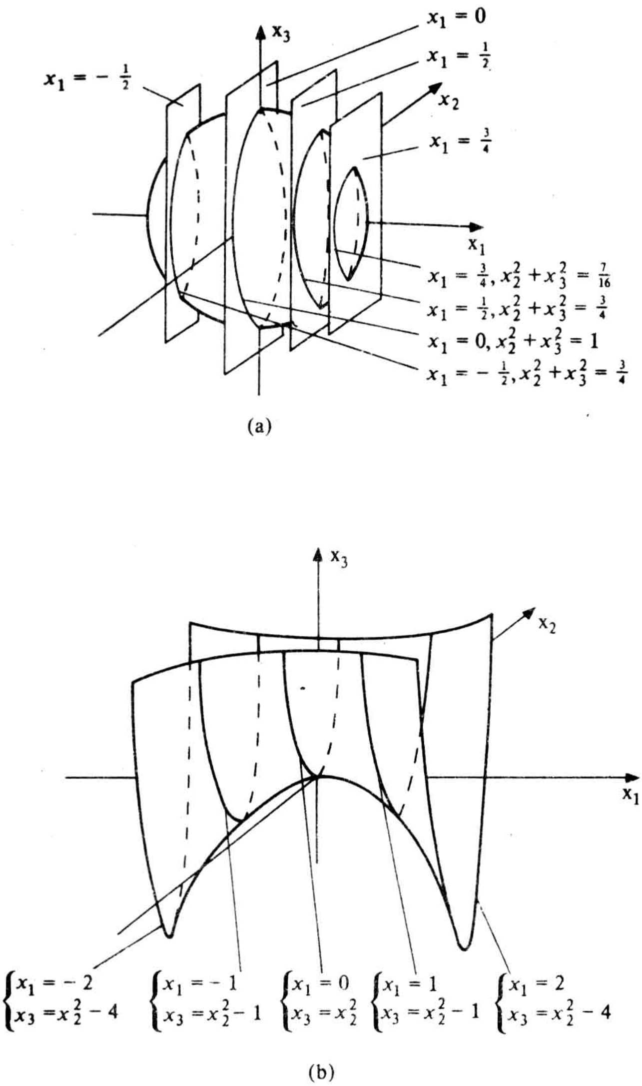


Figure 1.4 Level sets in $\mathbb{R}^3$, as generated by intersections with the planes $x_1 = \text{constant}$. (a) $x_1^2 + x_2^2 + x_3^2 = 1$. (b) $x_1^2 - x_2^2 + x_3 = 0$.

EXERCISES

In Exercises 1.1–1.4 sketch typical level curves and the graph of each function.

1.1. $f(x_1, x_2) = x_1$.

1.2. $f(x_1, x_2) = x_1 - x_2$.

1.3. $f(x_1, x_2) = x_1^2 - x_2^2$.

1.4. $f(x_1, x_2) = 3r^8 - 8r^6 + 6r^4$ where $r^2 = x_1^2 + x_2^2$. [*Hint*: Find and identify the critical points of f as a function of r.]

In exercises 1.5–1.9 sketch the level sets $f^{-1}(c)$, for $n = 0$, 1, and 2, of each function at the heights indicated.

1.5. $f(x_1, x_2, \ldots, x_{n+1}) = x_{n+1}$; $c = -1, 0, 1, 2$.

1.6. $f(x_1, x_2, \ldots, x_{n+1}) = 0x_1^2 + x_2^2 + \cdots + x_{n+1}^2$; $c = 0, 1, 4$.

1.7. $f(x_1, x_2, \ldots, x_{n+1}) = x_1 - x_2^2 - \cdots - x_{n+1}^2$; $c = -1, 0, 1, 2$.

1.8. $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 - x_2^2 - \cdots - x_{n+1}^2$; $c = -1, 0, 1$.

1.9. $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2/4 + \cdots + x_{n+1}^2/(n+1)^2$; $c = 1$.

1.10. Show that the graph of any function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$.

2 Vector Fields

The tool which will allow us to study the geometry of level sets is the calculus of vector fields. In this chapter we develop some of the basic ideas.

A *vector at a point* $p \in \mathbb{R}^{n+1}$ is a pair $\mathbf{v} = (p, v)$ where $v \in \mathbb{R}^{n+1}$. Geometrically, think of $\mathbf{v}$ as the vector v translated so that its tail is at p rather than at the origin (Figure 2.1). The vectors at p form a vector space $\mathbb{R}_p^{n+1}$ of

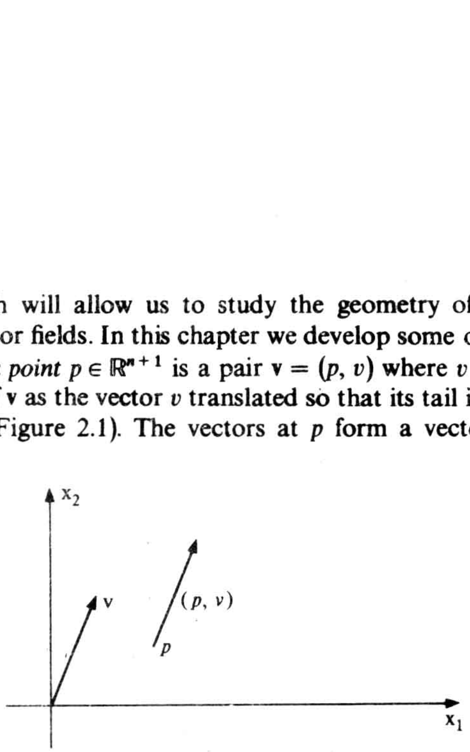


Figure 2.1 A vector at p.

dimension $n + 1$, with addition defined by $(p, v) + (p, w) = (p, v + w)$ (Figure 2.2) and scalar multiplication by $c(p, v) = (p, cv)$. The set $\{(p, v_1), \ldots, (p, v_{n+1})\}$ is a basis for $\mathbb{R}_p^{n+1}$ where $\{v_1, \ldots, v_{n+1}\}$ is any basis for $\mathbb{R}^{n+1}$. The set of all vectors at all points of $\mathbb{R}^{n+1}$ can be identified (as a set) with the Cartesian product $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{2n+2}$. However note that our rule of addition does not permit the addition of vectors at different points of $\mathbb{R}^{n+1}$.

Given two vectors (p, v) and (p, w) at p, their *dot product* is defined, using the standard dot product on $\mathbb{R}^{n+1}$, by $(p, v) \cdot (p, w) = v \cdot w$. When (p, v) and

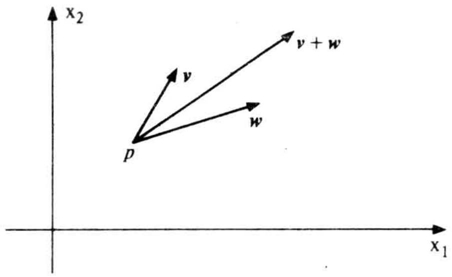


Figure 2.2 Addition of vectors at p.

$(p, v) \in \mathbb{R}_p^3$, $p \in \mathbb{R}^3$, the *cross product* is also defined, using the standard cross product on $\mathbb{R}^3$, by $(p, v) \times (p, w) = (p, v \times w)$.

Using the dot product, the *length* $\|\mathbf{v}\|$ of a vector $\mathbf{v} = (p, v)$ at p and the *angle* θ between two vectors $\mathbf{v} = (p, v)$ and $\mathbf{w} = (p, w)$ are defined by

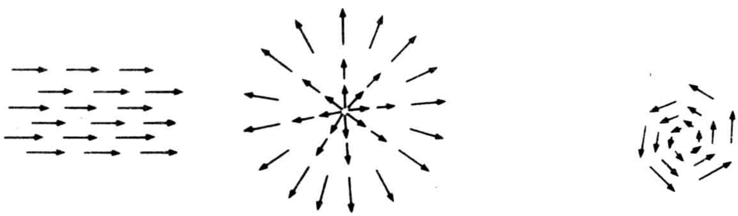
$$\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$$

$$\cos\theta = \mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\,\|\mathbf{w}\| \qquad 0 \leq \theta < \pi$$

A *vector field* $\mathbf{X}$ on $U \subset \mathbb{R}^{n+1}$ is a function which assigns to each point of U a vector at that point. Thus

$$\mathbf{X}(p) = (p, X(p))$$

for some function $X\colon U \to \mathbb{R}^{n+1}$. Vector fields on $\mathbb{R}^{n+1}$ are often most easily described by specifying this associated function X. Three typical vector fields on $\mathbb{R}^2$ are shown in Figure 2.3.



(a): $X(p) = (1,0)$ (b): $X(p) = p$ (c): $X(x_1, x_2) = (-x_2, x_1)$

Figure 2.3 Vector fields on $\mathbb{R}^2$: $\mathbf{X}(p) = (p, X(p))$.

We shall deal in this text mostly with functions and vector fields that are *smooth*. A function $f\colon U \to \mathbb{R}$ (U an open† set in $\mathbb{R}^{n+1}$) is smooth if all its partial derivatives of all orders exist and are continuous. A function $f\colon U \to \mathbb{R}^k$ is smooth if each component function $f_i\colon U \to \mathbb{R}$ ($f(p) = (f_1(p), \ldots, f_k(p))$ for $p \in U$) is smooth. A vector field $\mathbf{X}$ on U is smooth if the associated function $X\colon U \to \mathbb{R}^{n+1}$ is smooth.

† Recall that $U \subset \mathbb{R}^{n+1}$ is *open* if for each $p \in U$ there is an $\varepsilon > 0$ such that $q \in U$ whenever $\|q - p\| < \varepsilon$.

Associated with each smooth function $f\colon U \to \mathbb{R}$ (U open in $\mathbb{R}^{n+1}$) is a smooth vector field on U called the *gradient* ∇f of f, defined by

$$(\nabla f)(p) = \left(p, \frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_{n+1}}(p)\right).$$

We shall see that this vector field plays an important role in the study of the level sets of f.

Vector fields often arise in physics as velocity fields of fluid flows. Associated with such a flow is a family of parametrized curves called flow lines. These "flow lines" are in fact associated with any smooth vector field and are important in geometry as well as in physics. In geometry these flow lines are called "integral curves".

A *parametrized curve* in $\mathbb{R}^{n+1}$ is a smooth function $\alpha\colon I \to \mathbb{R}^{n+1}$, where I is some open interval in $\mathbb{R}$. By smoothness of such a function is meant that α is of the form $\alpha(t) = (x_1(t), \ldots, x_{n+1}(t))$ where each x_i is a smooth real valued function on I.

The *velocity vector* at time t ($t \in I$) of the parametrized curve $\alpha\colon I \to \mathbb{R}^{n+1}$ is the vector at $\alpha(t)$ defined by

$$\dot{\alpha}(t) = \left(\alpha(t), \frac{d\alpha}{dt}(t)\right) = \left(\alpha(t), \frac{dx_1}{dt}(t), \ldots, \frac{dx_{n+1}}{dt}(t)\right).$$

This vector is tangent to the curve α at $\alpha(t)$ (see Figure 2.4.). If $\alpha(t)$ represents

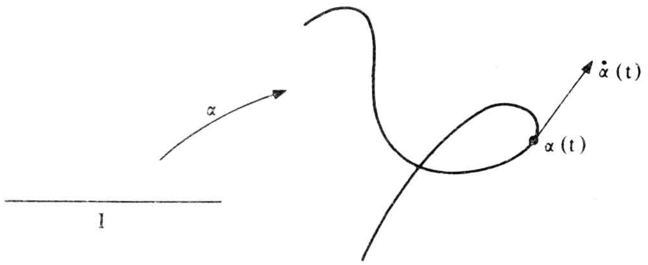


Figure 2.4 Velocity vector of a parametrized curve in $\mathbb{R}^2$.

for each t the position at time t of a particle moving in $\mathbb{R}^{n+1}$ then $\alpha(t)$ represents the velocity of this particle at time t.

A parametrized curve $\alpha\colon I \to \mathbb{R}^{n+1}$ is said to be an *integral curve* of the vector field $\mathbf{X}$ on the open set U in $\mathbb{R}^{n+1}$ if $\alpha(t) \in U$ and $\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$ for all $t \in I$. Thus α has the property that its velocity vector at each point of the curve coincides with the value of the vector field at that point (see Figure 2.5).

Theorem. *Let* $\mathbf{X}$ *be a smooth vector field on an open set* $U \subset \mathbb{R}^{n+1}$ *and let* $p \in U$. *Then there exists an open interval* I *containing* 0 *and an integral curve* $\alpha\colon I \to U$ *of* $\mathbf{X}$ *such that*

(i) $\alpha(0) = p$.