Second Edition

Gauge Fields

Introduction to Quantum Theory

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EDITOR'S FOREWORD

The problem of communicating in a coherent fashion recent developments in the most exciting and active fields of physics continues to be with us. The enormous growth in the number of physicists has tended to make the familiar channels of communication considerably less effective. It has become increasingly difficult for experts in a given field to keep up with the current literature; the novice can only be confused. What is needed is both a consistent account of a field and the presentation of a definite "point of view" concerning it. Formal monographs cannot meet such a need in a rapidly developing field, while the review article seems to have fallen into disfavor. Indeed, it would seem that the people most actively engaged in developing a given field are the people least likely to write at length about it.

FRONTIERS IN PHYSICS was conceived in 1961 in an effort to improve the situation in several ways. Leading physicists frequently give a series of lectures, a graduate seminar, or a graduate course in their special fields of interest. Such lectures serve to summarize the present status of a rapidly developing field and may well constitute the only coherent account available at the time. Often, notes on lectures exist (prepared by the lecturer himself, by graduate students, or by postdoctoral fellows) and are distributed on a limited basis. One of the principal purposes of the FRONTIERS IN PHYSICS Series is to make such notes available to a wider audience of physicists.

It should be emphasized that lecture notes are necessarily rough and informal, both in style and content; and those in the series will prove no exception. This is as it should be. The point of the series is to offer new, rapid, more informal, and, it is hoped, more effective ways for physicists to teach one another. The point is lost if only elegant notes qualify.

The informal monograph, representing an intermediate step between lecture notes and formal monographs, offers an author the opportunity to present his views of a field which has developed to the point where a summation might prove extraordinarily fruitful but a formal monograph might be feasible or desirable.

During the past decade, the informal text monograph, *Gauge Fields*, has provided the reader with a lucid introduction to the role played by gauge fields in quantum field theory. As its eminent authors note, over the same period gauge invariant models have evolved from providing an attractive physical hypothesis into a working theory which describes accurately the physics of elementary particles at moderate energies. A second edition which contains both supplementary and improved material is therefore both timely and highly useful, and it gives me pleasure to welcome once more Drs. Faddeev and Slavnov to FRONTIERS IN PHYSICS.

DAVID PINES
Urbana, Illinois
September, 1990

Preface to the Second Revised (Russian) Edition

During the past ten years, since the first edition of this book, gauge invariant models of elementary particle interactions were transformed from an attractive plausible hypothesis into a generally accepted theory confirmed by experiments. It was therefore natural that the development of the methods of gauge fields attracted the attention of the great majority of specialists in quantum field theory. The new interesting lines of activity that arose in this period included the formulation of gauge theories on a lattice, the investigation of non-trivial classical solutions of the Yang-Mills equations and quantization in their neighborhood, the application of methods of algebraic topology in gauge field theory. In preparing the second edition of our book we were confronted with a difficult dilemma: either we were to extend the book significantly by including a serious discussion of the novel fields of research, or we would, in the main, adopt the same plan as for the first edition. We decided in favour of the latter version, since, in our opinion, the most promising issues mentioned above have not as yet attained a completed form. Besides, an exposition of these issues would require a significant extension of the mathematical apparatus utilized. Therefore, in the second edition we limited ourselves to presenting such supplements that are related in a natural way to the main content of the first edition, and we also introduced a number of improvements which, as we hope, should facilitate reading of the book and render it more self-consistent.

This Preface is being written just at a time, when hopes are arising that a more fundamental basis is to be developed for elementary particle theory, the theory of superstrings.

However, independently of whether these hopes come true, gauge field theory, clearly, describes the physics of elementary particles adequately at moderate energies. Besides, the methods applied in the field theory of relativistic strings represent a direct generalization of the methods of gauge field theory, to which this book is devoted. For this reason we consider a new edition of it to be useful, both for direct applications of the already developed gauge theory and for search of new ways.

Moscow - Leningrad, 1986

L. D. Faddeev
A. A. Slavnov

Preface to the Original (Russian) Edition

Progress in quantum field theory, during the last ten years, is to a great extent due to the development of the theory of Yang-Mills fields, sometimes called gauge fields. These fields open up new possibilities for the description of interactions of elementary particles in the framework of quantum field theory. Gauge fields are involved in most modern models of strong and also of weak and electromagnetic interactions. There also arise the extremely attractive prospects of unification of all the interactions into a single universal interaction.

At the same time the Yang-Mills fields have surely not been sufficiently considered in modern monographical literature. Although the Yang-Mills theory seems to be a rather special model from the of view of general quantum field theory, it is extremely specific and the models used in this theory are quite far from being traditional. The existing monograph of Konoplyova and Popov, "Gauge Fields", deals mainly with the geometrical aspects of the gauge field theory and illuminates the quantum theory of the Yang-Mills fields insufficiently. We hope that the present book to some extent will close this gap.

The main technical method, used in the quantum theory of gauge fields, is the path-integral method. Therefore, much attention is paid in this book to the description of this alternative approach to the quantum field theory. We have made an attempt to expound this method in a sufficiently self-consistent manner, proceeding from the fundamentals of quantum theory. Nevertheless, for a deeper understanding of the book it is desirable for the reader to be familiar with the traditional methods of quantum theory, for example, in the volume of the first four chapters of the book by N. N. Bogolubov and D. V. Shirkov, "Introduction to the Theory of Quantized Fields". In particular, we shall not go into details of comparing the Feynman diagrams to the terms of the perturbation-theory expansion, and of the rigorous substantiation of the renormalization procedure, based on the R-operation. These problems are not specific for the Yang-Mills theory and are presented in detail in the quoted monograph.

There are many publications on the Yang-Mills fields, and we shall not go into a detailed review of this literature to any extent. Our aim is to introduce the methods of the quantum Yang-Mills theory to the reader. We shall not discuss alternative approaches to this theory, but shall present in detail that approach, which seems to us the most simple and natural one. The applications dealt with in the book are illustrative in character and are not the last work to be said about applications of the Yang-Mills field to elementary-particle models. We do this consciously, since the phenomenological aspects of gauge theories are developing and changing rapidly. At the same time the technique of quantization and renormalization of the Yang-Mills fields has already become well established. Our book is mainly dedicated to these specific problems.

We are grateful to our colleagues of the V. A. Steclov Mathematical Institute in Moscow and Leningrad for numerous helpful discussions of the problems dealt with in this book.

We would especially like to thank D. V. Shirkov and O.I. Zav'ylov who read the manuscript and made many useful comments and E. Sh. Yegoryan for help in checking the formulas.

Moscow-Leningrad-Kirovsk L. D. Faddeev, A. A. Slavnov

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Introduction: Fundamentals of Classical Gauge Field Theory

1.1 Basic Concepts and Notation

The theory of gauge fields at present represents the widely accepted theoretical basis of elementary particle physics. Indeed, the most elaborate model of field theory, quantum electrodynamics, is a particular case of the gauge theory. Further, models of weak interactions have acquired an elegant and self-consistent formulation in the framework of gauge theories. The phenomenological four-fermion interaction has been replaced by the interaction with an intermediate vector particle, the quantum of the Yang-Mills field. Existing experimental data along with the requirement of gauge invariance led to the prediction of weak neutral currents and of new quantum numbers for hadrons.

Phenomenological quark models of strong interactions also have their most natural foundation in the framwork of a gauge theory known as quantum chromodynamics. This theory provides a unique possibility of describing, in the framework of quantum field theory, the phenomenon of asymptotic freedom. This theory also affords hopes of explaining quark confinement, although this question is not quite clear.

Finally, the extension of the gauge principle may lead to the gravitational interaction also being placed in the general scheme of Yang-Mills fields.

So the possibility arises of explaining, on the basis of one principle, all the hierarchy of interactions existing in nature. The term unified field theory, discredited sometime ago, now acquires a new reality in the framework of gauge field theories. In the formation of this picture a number of scientists took part. Let us mention some of the key dates.

In 1953 C. N. Yang and R. L. Mills, for the first time, generalized the principle of gauge invariance of the interaction of electric charges to the case of interacting isospins. In their paper, they introduced a vector field, which later became known as the Yang-Mills field, and within the framework of the classical field theory its dynamics was developed.

In 1967 L. D. Faddeev, V. N. Popov, and B. De Witt constructed a self-consistent scheme for the quantization of massless Yang-Mills fields. In the same year, S. Weinberg and A. Salam independently proposed a unified gauge model ofweak and electromagnetic interactions, in which the electromagnetic field and the field of the intermediate vector boson were combined into a multiplet of Yang-Mills fields. This model was based on the mechanism of mass generation for vector bosons as a result of a spontaneous symmetry breaking, proposed earlier by P. Higgs and T. Kibble.

In 1971 G. t'Hooft showed that the general methods of quantization of massless Yang-Mills fields may be applied, practically without any change, to the case of spontaneously broken symmetry. Thus the possibility was discovered of constructing a self-consistent quantum theory of massive vector fields, which are necessary for the theory of weak interactions and, in particular, for the Salam-Weinberg model.

By 1972 the construction of the quantum theory of gauge fields in the framework of perturbation theory was largely completed. In papers by A.A. Slavnov, by J. Taylor, by B. Lee and J. Zinn-Justin, and by G. t'Hooft and M. Veltman, various methods of invariant regularization were developed, the generalized Ward identities were obtained, and a renormalization procedure was constructed in the framework of perturbation theory. This led to the construction of a finite and unitary scattering matrix for the Yang-Mills field.

Since then, the theory of gauge fields has developed rapidly, both theoretically and phenomenologically. Such development led to the construction of a self-consistent theory of weak and electromagnetic interactions based on the Weinberg-Salam model, as well as to a successful description of hadron processes in the region of asymptotic freedom, where one can apply perturbation theory. From a purely theoretical point of view, profound relations were established of gauge theories with differential geometry and topology.

At present the main efforts are directed at the creation of computational methods not related to the expansion in the coupling constant. Along this way promising lines of activity are coming into being that raise great hopes. These hopes, however, have not been fully implemented yet. These include quantization in the neighborhood of nontrivial classical solutions (instantons), computations on large computers in the framework of the lattice approximation, application of methods of the theory of phase transitions, expansion in inverse powers of the number of colors, and a number of other methods.

Approaches are also being developed which combine utilization of the quantum theory of gauge fields and the dispersion technique (sum rules). In brief, hard work aimed at development of the theory of gauge fields is well under way.

From the above short historical survey we shall pass on to the description of the Yang-Mills field itself. For this, we must first recall some notation from the theory of compact Lie groups. More specifically, we shall be interested mainly in the Lie algebras of these groups. Let Ω be a compact semisimple Lie group, that is a compact group which has no invariant commutative (Abelian) subgroups. The number of independent parameters that characterize an arbitrary element of the group (that is, the dimension is equal to n. Among the representations of this group and its Lie algebra, there exists the representation of $n \times n$ matrices (adjoint representation). It is generated by the natural action of the group on itself by the similarity transformations

$$h \longrightarrow \omega h \omega^{-1}; \quad h, \omega \in \Omega. \tag{1.1}$$

Any matrix $\mathcal{F}$ in the adjoint representation of the Lie algebra can be represented by a linear combination of n generators,

$$\mathcal{F} = T^a \alpha^a. \tag{1.2}$$

For us it is essential that the generators T^a can be normalized by the condition

$$\mathrm{tr}\,(T^a T^b) = -2\delta^{ab}. \tag{1.3}$$

In this case the structure constants t^{abc} that take part in the condition

$$[T^a, T^b] = t^{abc} T^c, \tag{1.4}$$

are completely antisymmetric. The reader unfamiliar with the theory of Lie groups may keep in mind just these two relationships, which are actually a characterizing property of the compact semisimple Lie group.

A compact semisimple group is called simple if it has no invariant Lie subgroups. A general semisimple group is a product of simple groups. This means that the matrices of the Lie algebra in the adjoint representation have a blocked-diagram form, where each block corresponds to one of the simple factors. The generators of the group can be chosen so that each one has nonzero matrix elements only within one of the blocks. We shall always have in mind exactly such a choice of generators, in correspondence with the structure of the direct product.

The simplest example of such a group is the simple group $SU(2)$. The dimension of this group equals 3, and the Lie algebra in the adjoint representation is given by the antisymmetric 3×3 matrices; as generators the matrices

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad T^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad T^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{1.5}$$

can be chosen; the structure constants t^{abc} in this base coincide with the completely antisymmetric tensor ε^{abc}.

Besides semisimple compact groups, we shall also deal with the commutative (Abelian) group $U(1)$. The elements of this group are complex numbers, with absolute values equal to unity. The Lie algebra of this group is one-dimensional and consists of imaginary numbers or of real antisymmetric 2×2 matrices.

The Yang-Mills field can be associated with any compact semisimple Lie group. It is given by the vector field $\mathcal{A}_\mu(x)$, with values in the Lie algebra of this group. It is convenient to consider $\mathcal{A}_\mu(x)$ to be a matrix in the adjoint representation of this algebra. In this case it is defined by its coefficients $A^a_\mu(x)$:

$$\mathcal{A}_\mu(x) = A^a_\mu(x) T^a \tag{1.6}$$

with respect to the base of the generators T^a.

In the case of the group $U(1)$ the electromagnetic field $\mathcal{A}_\mu(x) = iA_\mu(x)$ is an analogous object.

We shall now pass on to the definition of the gauge group and its action on Yang-Mills fields. In the case of electrodynamics the gauge transformation is actually the well known gradient transformation

$$\mathcal{A}_\mu(x) \rightarrow \mathcal{A}_\mu(x) + i\partial_\mu \lambda(x). \tag{1.7}$$

Let us recall its origin in the framework of the classical field the y. The electromagnetic field interacts with charged fields, which are described by complex functions $\psi(x)$. In the equations of motion the field $\mathcal{A}_\mu(x)$ always appears in the following combination:

$$\nabla_\mu \psi = (\partial_\mu - \mathcal{A}_\mu)\psi = (\partial_\mu - iA_\mu)\psi. \tag{1.8}$$

The above gradient transformation provides the covariance of this combination with respect to the phase transformation of the fields ψ. If ψ transforms according to the rule

$$\begin{aligned} \psi(x) &\rightarrow e^{i\lambda(x)}\psi(x), \\ \bar{\psi}(x) &\rightarrow e^{-i\lambda(x)}\bar{\psi}(x), \end{aligned} \tag{1.9}$$

then $\nabla_\mu \psi$ transforms in the same way. Indeed,

$$(\partial_\mu - \mathcal{A}_\mu)\psi \rightarrow [\partial_\mu - i\partial_\mu \lambda(x) - \mathcal{A}_\mu(x)] e^{i\lambda(x)}\psi(x) = e^{i\lambda(x)}[\partial_\mu - \mathcal{A}_\mu(x)]\psi(x). \tag{1.10}$$

As a result, the equations of motion are also covariant with respect to the transformations (1.7) and (1.9); if the pair $\psi(x), \mathcal{A}_\mu(x)$ is a solution, then $e^{i\lambda(x)}\psi(x)$, $\mathcal{A}_\mu(x) + i\partial_\mu \lambda(x)$ is also a solution.

In other words, a local change in phase of the field $\psi(x)$, which can be considered to be the coordinate in the charge space, is equivalent to the appearance of an additional electromagnetic field. We see here a complete analogy with the weak equivalence principle in Einstein's theory of gravity, where a change of the coordinate system leads to the appearance of an additional gravitational field.

Extending this analogy further, one may formulate the relativity principle in the charge space. This principle was first introduced by H. Weyl in 1919: The field configurations $\psi(x), \mathcal{A}_\mu(x)$ and $\psi(x)e^{i\lambda(x)}, \mathcal{A}_\mu(x) + i\partial_\mu\lambda(x)$ described the same physical situation. If the construction of theory is based on this principle, then the above-described way of constructing the equations of motion in terms of covariant derivatives is the only possible one.

The generalization of this principle to the case of the more complicated charge space leads to the Yang-Mills theory. Examples of such charge (or internal, as they are often called) spaces are the isotopic space, the unitary-spin space in the theory of hadrons, and so on. In all these examples we deal with fields $\psi(x)$ that acquire values in the charge space, which itself is a representation space for some compact semisimple groups $\Omega(SU(2), SU(3)$, etc.). The equations of motion for the fields $\psi(x)$ contain the covariant derivative

$$\nabla_\mu = \partial_\mu - \Gamma(\mathcal{A}_\mu), \tag{1.11}$$

where $\Gamma(\mathcal{A}_\mu)$ is the representation of the matrix $\mathcal{A}_\mu$ corresponding to the given representation of the group Ω. For example, if $\Omega = SU(2)$ and the charge space corresponds to the two dimensional representation, then the above-mentioned generators T^a are represented by the Pauli matrices

$$\Gamma(T^a) = \frac{1}{2i}\tau^a, \tag{1.12}$$

where

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.13}$$

and in this case

$$\Gamma(\mathcal{A}_\mu) = \frac{1}{2i}\mathcal{A}_\mu^a \tau^a. \tag{1.14}$$

The transformation of the fields $\psi(x)$ analogous to the local phase transformation in electrodynamics has the following form:

$$\psi(x) \to \psi^\omega(x) = \Gamma[\omega(x)]\psi(x), \tag{1.15}$$

where $\omega(x)$ is a function of x which has its values in the group Ω. It is convenient to consider $\omega(x)$ to be a matrix in the adjoint representation of the group Ω. The derivative (1.11) will be covariant with respect to this transformation if the field $\mathcal{A}_\mu(x)$ transforms according to the rule

$$\mathcal{A}_\mu(x) \to \mathcal{A}_\mu^\omega(x) = \omega(x)\mathcal{A}_\mu(x)\omega^{-1}(x) + \partial_\mu\omega(x)\omega^{-1}(x). \tag{1.16}$$

It is not difficult to see that this transformation obeys the group law. The set of these transformations composes a group that may formally be denoted by

$$\tilde{\Omega} = \prod_x \Omega. \tag{1.17}$$

This group is called the group of gauge transformations.

Often it is convenient to deal with the infinitesimal form of the gauge transformation. Let the matrices $\omega(x)$ differ infinitesimally from the unit matrix

$$\omega(x) = 1 + \alpha(x) = 1 + \alpha^a(x)T^a, \tag{1.18}$$