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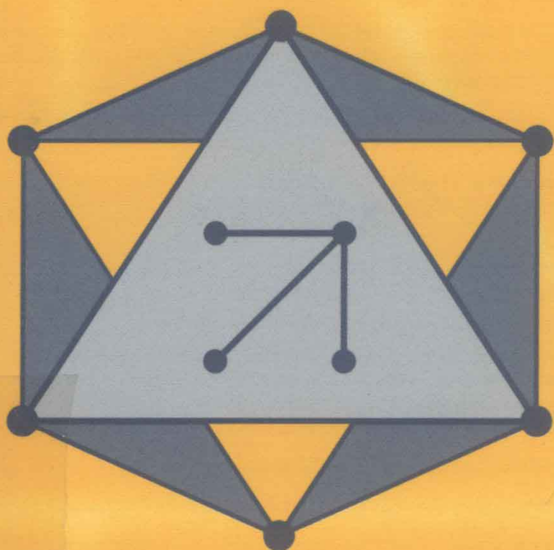
Jakob Jonsson

Simplicial Complexes of Graphs



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Preface

This book is a revised version of my 2005 thesis [71] for the degree of Doctor of Philosophy at the Royal Institute of Technology (KTH) in Stockholm. The whole idea of writing a monograph about graph complexes is due to Professor Anders Björner, my scientific advisor. I am deeply grateful for all his comments, remarks, and suggestions during the writing of the thesis and for his very careful reading of the manuscript.

I spent the first years of my academic career at the Department of Mathematics at Stockholm University with Professor Svante Linusson as my advisor. He is the one to get credit for introducing me to the field of graph complexes and also for explaining the fundamentals of discrete Morse theory, the most important tool in this book. Most of the work presented in Chapters 17 and 20 was carried out under the inspiring supervision of Linusson.

The opponent (critical examiner) of my thesis defense was Professor John Shareshian; the examination committee consisted of Professor Boris Shapiro, Professor Richard Stanley, and Professor Michelle Wachs. I am grateful for their valuable feedback that was of great help to me when working on this revision.

The work of transforming the thesis into a book took place at the Technische Universität Berlin and the Massachusetts Institute of Technology. I thank Björner and Professor Günter Ziegler for encouraging me to submit the manuscript to Springer.

Some chapters in this book appear in revised form as journal papers: Chapters 4, 17, and 20 are revised versions of a paper published in the Journal of Combinatorial Theory, Series A [67]. Chapter 5 is a revised version of a paper published in the Electronic Journal of Combinatorics [70]. Chapter 26 is a revised version of a paper published in the SIAM Journal of Discrete Mathematics [72]. I am grateful to several anonymous referees and editors representing these journals, and also to anonymous referees representing the FPSAC conference, who all provided helpful comments and suggestions.

In addition, I thank two anonymous reviewers for this series for providing several useful comments on the manuscript and the editors at Springer for showing patience and being of great help during the preparation of the manuscript.

Finally, and most importantly, I thank family and friends for endless support.

For the reader's convenience, let me list the major revisions compared to the thesis version of 2005:

- Chapter 1 has been extended with a more thorough discussion about applications of graph complexes to problems in other areas of mathematics.
- Recent results about the matching complex M_n and the chessboard complex $\mathsf{M}_{m,n}$ have been incorporated into Sections 11.2.3 and 11.3.2.
- Section 15.4 has been updated with a more precise statement about the Euler characteristic of the complex $\mathsf{DGr}_{n,p}$ of digraphs that are graded modulo p and a shorter proof of a formula for the Euler characteristic of $\mathsf{DGr}_n = \mathsf{DGr}_{n,n+1}$.
- Section 16.3 has been updated with a proof that the complex NXM_n of noncrossing matchings is semi-nonevasive.
- Section 18.5 is new and contains a brief discussion about the complex of disconnected hypergraphs.
- Section 19.4 is new and contains a generalization of the complex NC_n^2 of not 2-connected graphs along with yet another method for computing the homotopy type of NC_n^2. The theory in this section is applied in Section 22.2, which is also new and contains a discussion about the complex DNSC_n^2 of not strongly 2-connected digraphs.
- At the end of Section 23.3, we discuss a recent observation due to Shareshian and Wachs [121] about a connection between the complex NEC_{kp+1}^p of not p-edge-connected graphs on $kp+1$ vertices and the poset $\Pi_{kp+1}^{1 \bmod p}$ of set partitions on $kp+1$ elements in which the size of each part is congruent to 1 modulo p.

Cambridge, MA, *Jakob Jonsson*
March 2007

Summary. Let G be a finite graph with vertex set V and edge set E. A *graph complex* on G is an abstract simplicial complex consisting of subsets of E. In particular, we may interpret such a complex as a family of subgraphs of G. The subject of this book is the topology of graph complexes, the emphasis being placed on homology, homotopy type, connectivity degree, Cohen-Macaulayness, and Euler characteristic.

We are particularly interested in the case that G is the complete graph on V. *Monotone graph properties* are complexes on such a graph satisfying the additional condition that they are invariant under permutations of V. Some well-studied monotone graph properties that we discuss in this book are complexes of matchings, forests, bipartite graphs, disconnected graphs, and not 2-connected graphs. We present new results about several other monotone graph properties, including complexes of not 3-connected graphs and graphs not coverable by p vertices.

Imagining the vertices as the corners of a regular polygon, we obtain another important class consisting of those graph complexes that are invariant under the natural action of the dihedral group on this polygon. The most famous example is the associahedron, whose faces are graphs without crossings inside the polygon. Restricting to matchings, forests, or bipartite graphs, we obtain other interesting complexes of noncrossing graphs. We also examine a certain "dihedral" variant of connectivity.

The third class to be examined is the class of digraph complexes. Some well-studied examples are complexes of acyclic digraphs and not strongly connected digraphs. We present new results about a few other digraph complexes, including complexes of graded digraphs and non-spanning digraphs.

Many of our proofs are based on Robin Forman's discrete version of Morse theory. As a byproduct, this book provides a loosely defined toolbox for attacking problems in topological combinatorics via discrete Morse theory. In terms of simplicity and power, arguably the most efficient tool is Forman's divide and conquer approach via decision trees, which we successfully apply to a large number of graph and digraph complexes.

Contents

Part I

Introduction and Basic Concepts

1

Introduction and Overview

This book focuses on families of graphs on a fixed vertex set. We are particularly interested in *graph complexes*, which are graph families closed under deletion of edges. Equivalently, a graph complex Δ has the property that if $G \in \Delta$ and e is an edge in G, then the graph obtained from G by removing e is also in Δ. Since the vertex set is fixed, we may identify each graph in Δ with its edge set and hence interpret Δ as a simplicial complex. In particular, we may realize Δ as a geometric object and hence analyze its topology. Indeed, this is the main purpose of the book.

Fig. 1.1. Δ contains all graphs isomorphic to one of the four illustrated graphs.

As an example, consider the simplicial complex Δ of graphs G on the vertex set $\{1, 2, 3, 4\}$ with the property that some vertex is contained in all edges in G. This means that G is isomorphic to one of the graphs in Figure 1.1. Denoting the edge between i and j as ij, we obtain that

$$\begin{aligned}\Delta = \{&\emptyset, \{12\}, \{13\}, \{14\}, \{23\}, \{24\}, \{34\}, \{12, 13\}, \{12, 14\}, \{13, 14\},\\ &\{12, 23\}, \{12, 24\}, \{23, 24\}, \{13, 23\}, \{13, 34\}, \{23, 34\}, \{14, 24\},\\ &\{14, 34\}, \{24, 34\}, \{12, 13, 14\}, \{12, 23, 24\}, \{13, 23, 34\}, \{14, 24, 34\}\}.\end{aligned}$$

See Figure 1.2 for a geometric realization of Δ. It is easy to see that Δ is homotopy equivalent to a one-point wedge of three circles.

Monotone Graph Properties

In the above example, note that a given graph G belongs to Δ if and only if all graphs isomorphic to G belong to Δ. Equivalently, Δ is invariant under the

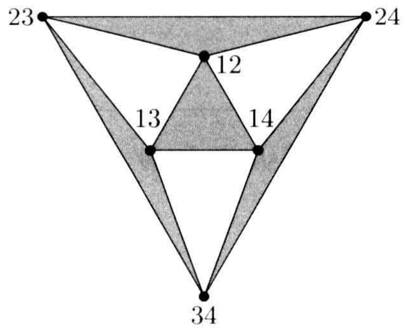

Fig. 1.2. Geometric realization of the complex Δ.

action of the symmetric group on the underlying vertex set. A family of graphs satisfying this condition is a *graph property*. We will be mainly concerned with graph properties that are also graph complexes, hence closed under deletion of edges. We refer to such graph properties as *monotone* graph properties.

In this book, we discuss and analyze the topology of several monotone graph properties, some examples being matchings, forests, bipartite graphs, non-Hamiltonian graphs, and not k-connected graphs; see Chapter 7 for a summary. Some results are our own, whereas others are due to other authors. We restrict our attention to topological and enumerative properties of the complexes and do not consider representation-theoretic aspects of the theory.

Remark. Some authors define monotone graph properties to be graph properties closed under *addition* of edges. While such graph properties are not simplicial complexes, they are quotient complexes of simplicial complexes and hence realizable as geometric objects; see Sections 3.2 and 3.5.

Other Graph Complexes

Monotone graph properties are not the only interesting graph complexes. For example, for any monotone graph property Δ and any graph G, one may consider the subcomplex $\Delta(G)$ consisting of all graphs in Δ that are also subgraphs of G; this is the induced subcomplex of Δ on G. In some situations, $\Delta(G)$ is interesting in its own right; we would claim that this is the case for complexes of matchings, forests, and disconnected graphs. In other situations, $\Delta(G)$ is of use in the analysis of the larger complex Δ; one example is the complex of bipartite graphs.

With graph properties being invariant under the action of the symmetric group, a natural generalization would be to replace the symmetric group with a smaller group. In this book, we concentrate on the dihedral group D_n. This group acts in a natural manner on the family of graphs on the vertex set $\{1, \ldots, n\}$: Represent the vertices as points evenly distributed in a clockwise manner around a unit circle and identify a given edge with the line segment between the two points representing the endpoints of the edge. We refer to this representation of a graph as the *polygon representation*; the vertices are the corners in a regular polygon. The action of the dihedral group consists of rotations and reflections, and combinations thereof, of this polygon. The associahedron is probably the most well-studied graph complex with a natural dihedral action. Some other interesting "dihedral" graph complexes are complexes of noncrossing matchings, noncrossing forests, and graphs with a disconnected polygon representation. See Chapter 8 for more information.

Finally, we mention complexes of directed graphs; we refer to such complexes as *digraph complexes*. Some important examples are complexes of directed forests and acyclic digraphs. We also discuss some directed variants of the property of being bipartite and the property of being disconnected. See Chapter 9 for an overview.

Remark. As is obvious from the discussion in this section, our graph complexes are completely unrelated to Kontsevich's graph complexes [83, 85].

Discrete Morse Theory

The most important tool in our analysis is Robin Forman's discrete version of Morse theory [48, 49]. As we describe in more detail in Chapter 4, one may view discrete Morse theory as a generalization of the concept of collapsibility. A complex Δ is collapsible to a smaller complex Σ if we can transform Δ into Σ via a sequence of elementary collapses. An elementary collapse is a homotopy-preserving operation in which we remove a maximal face τ along with a codimension one subface σ such that the resulting complex remains a simplicial complex (i.e., closed under deletion of elements).

To better understand the generalization, we first interpret a collapse as a giant one-step operation in which we perform all elementary collapses *at once*, rather than one by one. This way, a collapse from a complex Δ to a subcomplex Σ boils down to a partial matching on Δ such that Σ is exactly the family of unmatched faces. Dropping the condition that the unmatched faces must form a simplicial complex, we obtain discrete Morse theory.

More precisely, under certain conditions on a given matching – similar to the ones that we would need on a matching corresponding to an ordinary collapse – Forman demonstrated how to build a cell complex homotopy equivalent to Δ using the unmatched faces as building blocks. Indeed, this very construction is the main result of discrete Morse theory. As an immediate corollary of Forman's construction, we obtain upper bounds on the Betti numbers.

Remark. We should mention that some aspects of the above interpretation of discrete Morse theory are due to Chari [32]. In addition, while we discuss