ADVANCED PRACTICAL PHYSICS FOR STUDENTS

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WITH 394 DIAGRAMS AND ILLUSTRATIONS

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PREFACE

THE course of Practical Physics described in this book is based upon that followed in King's College, London, by students who have completed their Intermediate Course, and who are proceeding to a Pass or Honours Degree. This has been extended, and it is hoped that the book will be useful to a wider circle of students of Physics than those immediately concerned with University Examinations.

A number of well-known Physicists have contributed to the development of the King's College course, amongst whom we may mention Professors H. A. Wilson, C. G. Barkla, H. S. Allen, and W. Wilson, who formerly worked here in the Wheatstone Laboratory, and Professor O. W. Richardson, the present occupant of the chair.

The general aim has been to provide with each experiment a short theoretical treatment which will enable the student to perform the experiment without immediate reference to theoretical treatises. To aid this scheme an introductory chapter in the Calculus has been included. This chapter is an innovation in a book of this type, but it is hoped that the student will find here a bridge over that period during which his Physics demands more advanced mathematics than his systematic study of that subject has yet given him.

We take this opportunity of expressing our gratitude to Professor O. W. Richardson, who has allowed us to make use of laboratory manuscripts and results of experiments. We are also greatly indebted to our colleagues and to Mr. G. Williamson, who have given us many suggestions, and to the Honours students of the past session who have supplied us with numerical and graphical results. We have been greatly helped by the ready assistance on the part of The Cambridge and Paul Scientific Instrument Co.; Messrs. Elliot Bros., Gambrel, Ltd., Adam Hilger, Ltd., W. G. Pye & Co., and the Weston Electric Co., who supplied us with the blocks for many of the illustrations.

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CONTENTS

ADVANCED PRACTICAL PHYSICS FOR STUDENTS

INTRODUCTION TO DIFFERENTIAL AND INTEGRAL CALCULUS

The Differential Calculus

§1. Any quantity x which may assume a series of values is called a variable quantity or simply a variable, and if its value does not depend on that of any other quantity it is called an independent variable.

On the other hand a quantity y, which bears a particular relation to x, assumes values which depend on the values of x, and for this reason is called a dependent variable. We may have for example:

$$y = 2x - 3.$$

Here y takes values which depend in a quite definite manner on those of x.

We may also have a dependence defined by the relations:

$$y = \sin x, \quad y = \log x, \quad \text{and} \quad y = e^x.$$

Such expressions as $2x - 3$, $\sin x$, $\log x$, etc., are called functions of x, and when we say that y is a function of x we mean that y depends on the values that x assumes.

In case we do not specify definitely how y and x are related we write

$$y = f(x).$$

$f(x)$ denotes any function of x.

It is often convenient in Physics to show by means of a diagram the relation between two variables y and x. For example, a record may be required of the atmospheric pressure at various times. Such a record is drawn automatically by a self-recording barometer so that it can be seen how the pressure and time are related. Here we have as independent variable the time and the dependent variable is the barometric pressure.

In fig. 1, the curve represents the relation between x and y and the shape depends on the way y and x are connected.

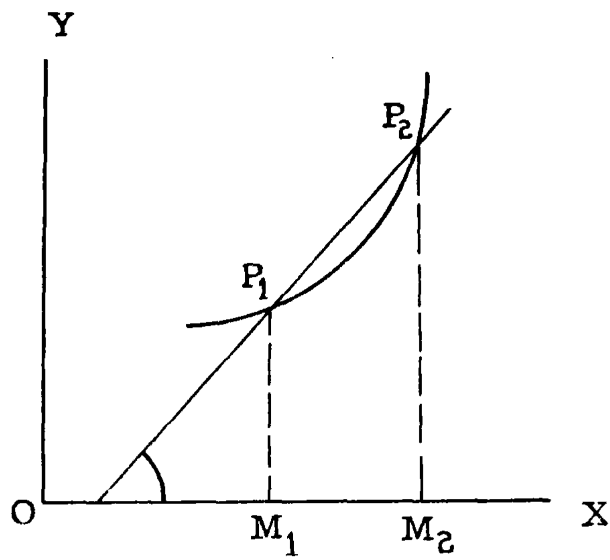

FIG. 1

If $y = 2x - 3$, the curve becomes a straight line, and if $y = \sin x$, we have the familiar sine curve, fig. 2.

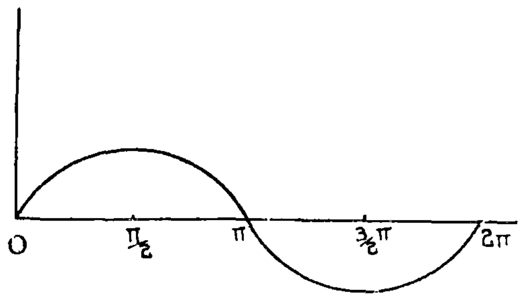

FIG. 2

A function is said to be a continuous function of a variable when the graph representing it is a curve in which there is no sudden change in value of the ordinate at any point. In such a curve, if we approach a point where $x = a$, from left to right, we find a certain value for y, and if we approach the point from right to left we find the same value. In fig. 3 we have an example of a function which is discontinuous at $x = 0$. If we approach the origin from left to right the value of y is very great and negative in sign, while in approaching from the right y is very large and positive. In nature we are chiefly concerned with continuously varying quantities. If a train is at rest at a station at a particular instant, and is observed to be moving with a velocity of ten miles per hour ten minutes later, it must have

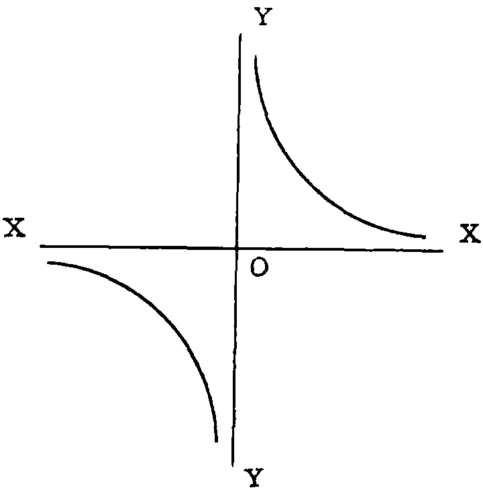

FIG. 3

possessed every possible velocity between zero and ten miles per hour during the interval.

The speed is continuous, and if it depends on the lapse of time from the start it is said to be a continuous function of the time.

We do not contemplate the possibility that the train could possess a speed of five miles per hour at one instant and at the next without any interval whatever a velocity of six miles per hour. If this were possible we should describe the speed as discontinuous, because it had no value between five and six. If this were so we should consider that our powers of observation were at fault, and we should describe the motion as changing very rapidly between five and six miles per hour; so rapidly that we had failed to detect the lapse of time in which the change took place.

Discontinuous functions are of frequent occurrence in Mathematics. Consider as an example the case of $y = \frac{1}{x}$.

When x is a very small positive number, let us say $\frac{1}{10^6}$, y is large and has the value 10^6.

On the other hand if $x = -\frac{1}{10^6}$, y is large in magnitude but negative, it equals -10^6.

As x passes through the value zero y suddenly leaps from an enormously large negative value to a very great positive value, and has no value between.

This is represented in the diagram, fig. 3. The curve has two branches: they are the two parts of the rectangular hyperbola

$$xy = 1.$$

We shall not be concerned with such functions so we dismiss them briefly. It is to be borne in mind that our applications of the Calculus are to continuous functions only. The results we obtain must not be applied to discontinuous functions without closer examination.

It is important to understand the meaning of the limit of a function.

Suppose y depends on x, and that as x approaches the value a, y approaches the value b.

b is called the limit of y as x approaches a, and we write:

$$\underset{x \longrightarrow a}{\text{Lim.}}\, y \longrightarrow b$$

If reference be made to fig. 1, as x approaches the value OM_1 y approaches the value M_1P_1 and M_1P_1 is actually the value of y when $x = a$.

Cases occur in which the conception of a limit is not so simple. If we examine the curve

$$y = \frac{1}{x}$$

in the neighbourhood of the origin as $x \longrightarrow 0$, we obtain a different value of y according as we begin on the right or left of the origin.

On account of the discontinuity the limit of y as x approaches zero is not definite.

Another case occurs in which x may continue to increase to any extent while y continually approaches some particular value.

We may turn once more to the curve

$$y = \frac{1}{x}$$

As x gets larger and larger, y gets smaller and smaller approaching the value zero.

We may get as near zero as we please by making x larger. For example, we may make y as small as one-millionth by choosing $x = 10^6$.

This is a very important point in defining a limit. It must be possible to get as close to the limiting value as we please by choosing x properly, although it may not actually be possible to cause y to attain the limit. We have in our example a case in point. y is never zero however large x may be, but it is possible to make y nearer and nearer zero by increasing x.

The former definition of the limit of a function is not very satisfactory. A limit is accurately defined as follows:

The limit of a function of x is some number, b, such that as x approaches a particular value, a, the difference between b and the function may be made as small as we please by taking x sufficiently near a.

§2. In describing natural phenomena by means of equations, simplifications are often brought about by neglecting certain terms in comparison with other more important terms.

Suppose an equation is obtained which we may write:

$$A_1 + B_1 + C_2 + D_2 = E_1 + F_3.$$

The suffix numbers denote the order of importance of the terms; that is to say, 1 denotes that the term is to be regarded as of first importance, or it is of the first order of magnitude. The 2 and 3 denote that the terms are only of second and third degrees of importance, they are of the second and third orders.

If we wish to include terms of the first and second orders we omit F_3, while if only terms of the first order are to be considered the equation becomes:

$$A_1 + B_1 = E_1.$$

Great care has to be exercised in thus drawing up a scale of magnitude, and this leads to a short consideration of infinitesimals.

Suppose a quantity X is divided into 1000 equal parts, these again subdivided in the same way, and so on. We then have a series of values:

$$X, \frac{X}{10^3}, \frac{X}{10^6}, \text{ etc.},$$

which provides a scale of magnitude.

If circumstances do not permit of accurate observation of quantities less than those of the same order as X we regard $\frac{X}{10^3}, \frac{X}{10^6}$, etc., as negligible.

Generally, if f is a small fraction, i.e. small compared with unity:

$$fX, f^2X, f^3X, \text{ etc.}$$

are all small compared with X, and are said to be small quantities of the first, second, third, etc., orders. If these small quantities have zero limits they are called infinitesimals.

In equations between infinitesimals only the terms of the lowest order are to be retained, i.e. the terms of greatest magnitude.

This is made clearer by an example which has important Physical applications.

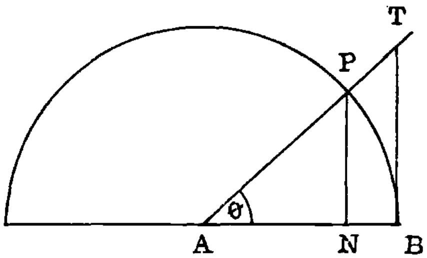

FIG. 4

In fig. 4, AB represents the radius of an arc, BP, of a circle which subtends an angle θ at A.

PN is normal to AB.

BT is also normal to AB cutting AP produced in T.

It may be regarded as an axiom that:

$$PN < \text{arc } PB < BT.$$

We shall examine the order of the differences between these quantities if θ be regarded as of the first order of small quantities.

By expansion of sin θ and cos θ in powers of θ we have:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\dots\dots \quad (1)$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\dots\dots \quad (2)$$

If only small quantities of the first order are retained:

$$\left.\begin{aligned} \sin\theta &= \theta. \\ \cos\theta &= 1. \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (3)$$

$$PN = a \sin\theta.$$
$$PB = a\,\theta.$$

$$\therefore PB - PN = a(\theta - \sin\theta) = a\left\{\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \dots\right\}$$

This difference is of the third order.

Thus up to considerations of magnitude of the third order:

$$PB = PN.$$

Again

$$BT = a \tan \theta.$$
$$= a\left(\theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots\right),$$

as may be shown by division of the expressions for sin θ and cos θ.

Thus BT − arc PB = a quantity of the third order of magnitude.

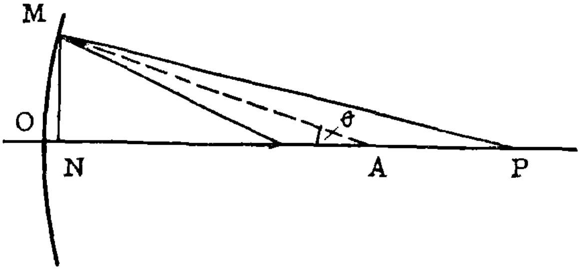

FIG. 5

BN = a − AN = $a(1 - \cos\theta)$ = a quantity of the second order of magnitude.

Thus, if we regard θ as of the first order and retain only this order in our equations we may write :

$$BN = 0, \quad PN = \text{arc } PB = BT.$$

and with the exception of BN = 0 this is true for the case when second order quantities are retained.

Extensive use is made of these relations in Geometrical Optics in the first study of reflection and refraction in mirrors and lenses.

In the case of a mirror, for example (see fig. 5), when the angle θ is small, i.e. when the rays from an object, P, strike the mirror at M not far from the pole, O, we establish certain formulæ by assuming that O and N may be regarded as being coincident.

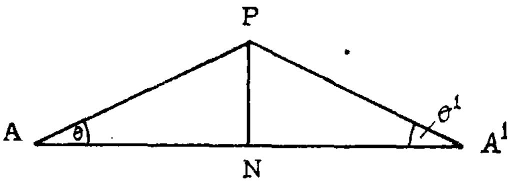

FIG. 6

This is because we do not retain quantities of an order higher than θ. Thus NO = 0 by the foregoing considerations.

Another important case is the calculation of the order of the difference between the sum of two sides of a triangle and the base when the base angles are of the first order of small quantities.

Referring to fig. 6, we have as for fig. 4,

$$PA - AN = \text{a quantity of the second order in } \theta,$$

and similarly $PA^1 - A^1N$ is of the second order in θ^1

$\therefore$ $PA + PA^1 - (AN + A^1N)$ = a quantity of the second order since θ and θ^1 are by the data small quantities of the first order.

$$\therefore PA + PA^1 = AN + A^1N = AA^1$$

to the first order.

This result is made use of in the establishment of Fermat's Law of the extreme path which plays a fundamental part in Optics. (See for example Houstoun's "Treatise on Light," p. 17.)

We consider as a final example the difference between a chord and an arc both subtending the same small angle θ at the centre of a circle.

Thus, again referring to fig. 4, we require the difference between chord BP and arc BP.

$$\text{arc BP} - \text{chord BP} = a\theta - 2a\sin\frac{\theta}{2}.$$

$$= a\left\{\theta - 2\left(\frac{\theta}{2} - \frac{1}{3!}\frac{\theta^3}{8} + \ldots\right)\right\}$$

$$= \text{a quantity of the third order.}$$

We can thus regard the chord and arc as equal up to and including quantities of the second order.

It should be noted that the successive orders are vanishingly small with regard to the terms earlier in the scale, e.g. in comparing

$$a\theta,\quad b\theta^2,\quad c\theta^3,$$

if θ is of the first order, $\dfrac{b\theta^2}{a\theta} = \dfrac{b\theta}{a}$ so that as $\theta \longrightarrow 0$ $\quad b\theta^2 \longrightarrow 0$ infinitely more rapidly than $a\theta$, and the same holds for any two consecutive terms in the scale.

The ratio of two quantities of the same order will be a finite quantity—not a vanishing or negligible quantity, but the ratio of two quantities of differing order (higher order $\div$ lower order) is vanishingly small.

We are concerned with small variations of this kind in the differential Calculus.

§3. The Differential Coefficient

Let y be a function of x, and suppose x varies by a small quantity which we denote by δx. In consequence of this variation y will vary a small quantity, say δy.

The ultimate ratio $\frac{\delta y}{\delta x}$ when δx becomes very small is called the differential coefficient of y with respect to x. It is denoted by $\frac{d}{dx}y$ and written $\frac{dy}{dx}$ and sometimes denoted simply by Dy.

In accordance with our notation we may write :

$$\frac{dy}{dx} = \underset{\delta x \longrightarrow 0}{\text{Lim.}} \frac{\delta y}{\delta x}.$$

In general the quantities δy and δx are of the same order of magnitude, and $\frac{dy}{dx}$ is a finite quantity.

In order to illustrate this, consider the relation :

$$y = 2x - 3.$$

Let x become $x + \delta x$, then the new value of y is

$$2(x + \delta x) - 3$$

i.e.
$$\begin{aligned} y + \delta y &= 2(x + \delta x) - 3 \\ &= y + 2\delta x. \\ \therefore \delta y &= 2\delta x. \\ \therefore \frac{\delta y}{\delta x} &= 2. \end{aligned}$$

Now no matter how small δx becomes, the ratio is always 2, for δy is of the same order as δx, and their ratio is finite and equal to 2.

We have a simpler case still in the differential coefficient of a constant.

A constant is a number that does not depend on the variable.

Thus, if $y =$ A it does not matter how x varies, y still remains $=$ A. Thus there is no change δy corresponding to a change δx.

Hence $\frac{dy}{dx} = 0$ if y is a constant.

It should be noted that $\frac{dy}{dx}$ does not mean $dy \div dx$. $\frac{dy}{dx}$ is a short notation for the operation of finding the ultimate ratio $\frac{\delta y}{\delta x}$. Nevertheless Physicists continually appear to use the coefficient as if it meant $dy \div dx$, and it is not a rare occurrence to find an equation :

$$\frac{dy}{dx} = x^2$$

written alternatively $dy = x^2\,dx$.

This is, in fact, a very convenient mode of expressing the result, and it means that dy and dx now no longer retain the same significance. The second of these means:

$$\delta y = x^2 \delta x.$$

We have in the equation $\dfrac{dy}{dx} = x^2$ an expression of the rate of variation of y with respect to x at a particular point on the curve, which represents graphically the relation between y and x.

The alternative equation means that in the neighbourhood of this point we can calculate a small change δy corresponding to a small change δx. This point rarely causes difficulty in practice, and it is obviously inconvenient to change to and fro from d to δ, but to be strictly accurate we must bear the distinction in mind.

The definition of the differential coefficient gives the clue to its determination. We will not determine its value for more than one or two cases but be content with reference to a table of values of the important coefficients.

The Differential Coefficient for x^n where n is any Number.

Write:

$$y = x^n.$$

$$y + \delta y = (x + \delta x)^n = x^n \left(1 + \frac{\delta x}{x}\right)^n$$

$$= x^n \left\{1 + n\frac{\delta x}{x} + \frac{n(n-1)}{1\cdot 2} \cdot \left(\frac{\delta x}{x}\right)^2\right\} + \ldots$$

$$= x^n + nx^{n-1} \cdot \delta x + \frac{n(n-1)}{1\cdot 2} \cdot x^{n-2}(\delta x)^2 + \ldots$$

$$\therefore \quad \delta y = nx^{n-1}\delta x + \frac{n(n-1)}{1\cdot 2} \cdot x^{n-2}(\delta x)^2 + \ldots$$

$$\therefore \quad \frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{1\cdot 2} \cdot x^{n-2}\,\delta x + \text{higher powers of } \delta x.$$

δx is a quantity which we have called infinitesimal. In the next step of finding the limit of $\dfrac{\delta y}{\delta x}$ we shall suppose δx a quantity of the first order of magnitude. It is therefore infinitesimally small with regard to the finite quantity nx^{n-1}.

We thus neglect all quantities of higher order than nx^{n-1} and have:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = nx^{n-1}.$$

Differential Coefficient of sin x.

$$y + \delta y = \sin (x + \delta x) = \sin x \cos \delta x + \cos x \sin \delta x.$$

We need retain only quantities of the first order on the right.

Thus: $\cos \delta x = 1, \quad \sin \delta x = \delta x,$ by equation (3).

$$\therefore \quad y + \delta y = \sin x + \cos x \cdot \delta x.$$

$$\therefore \quad \delta y = \cos x \cdot \delta x.$$

$$\text{or} \quad \frac{dy}{dx} = \cos x.$$

Similarly $\frac{d}{dx} \cos x = - \sin x.$

Differential Coefficient of log x.

$$y + \delta y = \log (x + \delta x) = \log x\left(1 + \frac{\delta x}{x}\right)$$

$$= \log x + \log\left(1 + \frac{\delta x}{x}\right).$$

$$= \log x + \left\{\frac{\delta x}{x} - \frac{1}{2}\left(\frac{\delta x}{x}\right)^2 + \ldots\right\}$$

Retaining quantities of first order only:

$$\delta y = \delta x \cdot \frac{1}{x}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{x}$$

or

$$\frac{d}{dx} \log x = \frac{1}{x}.$$

The same method of treatment can be applied to other cases. In the case of a function $f(x)$ we write:

$$\frac{dy}{dx} = \lim_{\delta x \longrightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

The Differential Coefficient of the Sum of two Functions.

If $y = \sin x + \cos x$

we have $\frac{dy}{dx} = \cos x - \sin x.$

From the definition it follows that the differential coefficient is the sum of the differential coefficients of $\sin x$ and $\cos x$.

In the general case if:

$$y = y_1 + y_2$$

where y_1 and y_2 are any two functions of x:

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}.$$

And similarly if $y = y_1 - y_2$

$$\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{dy_2}{dx}.$$

Differential Coefficient of a Product.

Let $$y = y_1 y_2$$
where y_1 and y_2 are any two functions:
e.g. we might have:

$$y = \sin x \times x^n,$$

$\sin x$ and x^n are two functions of x.

Suppose that x becomes $x + \delta x$ and in consequence y increases to $y + \delta y$, y_1 to $y_1 + \delta y_1$, y_2 to $y_2 + \delta y_2$.

Then

$$y + \delta y = (y_1 + \delta y_1)(y_2 + \delta y_2)$$
$$= y_1 y_2 + y_1 \delta y_2 + \delta y_1 y_2 + \delta y_1 \delta y_2.$$

Since $y = y_1 y_2$

$$\therefore \delta y = y_1 \delta y_2 + \delta y_1 y_2 + \delta y_1 \delta y_2.$$

$\delta y_1 \delta y_2$ is a term of the second order, and the other terms are of the first order.

Thus we need not retain it.

Dividing throughout by δx.

$$\frac{\delta y}{\delta x} = y_1 \frac{\delta y_2}{\delta x} + \frac{\delta y_1}{\delta x} \cdot y_2.$$

Hence in the limit:

$$\frac{dy}{dx} = y_1 \frac{dy_2}{dx} + y_2 \frac{dy_1}{dx}.$$

In a product we differentiate one factor at a time, leaving the others unchanged, and add all the resulting expressions together.

This is true for any number of factors, as may be shown in the same way.

Thus, if

$$y = y_1 y_2 y_3 y_4.$$

$$\frac{dy}{dx} = \frac{dy_1}{dx} \cdot y_2 y_3 y_4 + y_1 \frac{dy_2}{dx} y_3 y_4 + y_1 y_2 \frac{dy_3}{dx} \cdot y_4 + y_1 y_2 y_3 \frac{dy_4}{dx}.$$

e.g. $y = \sin x \times x^n$

$$\frac{dy}{dx} = \sin x \cdot n x^{n-1} + \cos x \cdot x^n.$$

The Differential Coefficient of a Quotient.

We use the same notation as before and apply the same principles.

$$y = \frac{y_1}{y_2}$$

$$y + \delta y = \frac{y_1 + \delta y_1}{y_2 + \delta y_2}$$

$$\therefore \quad \delta y = \frac{y_1 + \delta y_1}{y_2 + \delta y_2} - \frac{y_1}{y_2} = \frac{y_2\delta y_1 - y_1\delta y_2}{y_2^2\left(1 + \frac{\delta y_2}{y_2}\right)}$$

$$= \frac{(y_2\delta y_1 - y_1\delta y_2)}{y_2^2}\left(1 - \frac{\delta y_2}{y_2}\right)$$

$$= \frac{y_2\delta y_1 - y_1\delta y_2}{y_2^2} \text{ (retaining only terms of first order).}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{y_2\frac{\delta y_1}{\delta x} - y_1\frac{\delta y_2}{\delta x}}{y_2^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}}{y_2^2}$$

e.g.
$$y = \tan x = \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \sec^2 x.$$

Differential Coefficient of a Function of a Function.

The expression :

$$y = a\sin^2 x + b\sin x + c$$

in which a, b, c are constant quantities, is a function of $\sin x$, $\sin x$ is itself a function of x.

Thus, y is a function of a function of x.

We proceed to determine the differential coefficient $\frac{dy}{dx}$ in this complex case.

Before attacking the general problem we will consider a special case.

Let $y = \log \sin x$.
and write $z = \sin x$.