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Norbert Knarr

Translation Planes

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Foundations and Construction Principles

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Introduction

An affine plane $\mathcal{A}$ is called a translation plane if the translation group of $\mathcal{A}$ operates transitively on the point set of $\mathcal{A}$. The fundamental results on translation planes were obtained by André in 1954. The translation group of $\mathcal{A}$ is isomorphic to the additive group of a vector space V over a skewfield, and the points of $\mathcal{A}$ can be identified with the elements of V in such a way that the lines of $\mathcal{A}$ are cosets of subspaces of V. The lines through the origin form a spread $\mathcal{B}$ of V, i.e. any two elements of $\mathcal{B}$ are complementary and the elements of $\mathcal{B}$ cover V. Hence, translation planes can be investigated using tools from linear algebra and projective geometry.

Primarily, we will be concerned with spreads of a 4-dimensional vector space. Equivalently, we can study systems of lines of a 3-dimensional projective space which are mutually disjoint and cover the space. These are also called spreads. First, we investigate spreads of 3-dimensional projective spaces over arbitrary skewfields. In later chapters we restrict our attention to topological spreads of real and complex projective spaces, for which our methods work especially well.

The first chapter contains an introduction to the theory of translation planes. We assume that the reader knows the basic facts about projective and affine planes. For arbitrary projective planes, the relevant definitions and theorems are given without proof. However, all results dealing directly with translation planes are proved explicitly.

Furthermore, we give a brief account of the theory of topological translation planes.

In the second chapter we discuss several possibilities for the description of spreads of 3-dimensional projective spaces.

Let $\mathcal{B}$ be a spread of a 3-dimensional projective space $\mathcal{P}$ and choose a line $S \in \mathcal{B}$. Let E_1, E_2 be distinct planes of $\mathcal{P}$ both of which contain S and let p be a point of $\mathcal{P}$ which is not contained in the union of E_1 and E_2. The affine plane obtained from E_1 by deleting the line S is denoted by E_1'. With each line $G \in \mathcal{B} \setminus \{S\}$ we associate the point $G \cap E_1$ and the image of the point $G \cap E_2$ under the projection from E_2 to E_1 with center p. Since $\mathcal{B}$ is a spread, this defines a bijective mapping $f : E_1' \to E_1'$. We show that the mappings obtained in this way are generalizations of the transversal mappings invented by Ostrom for the description of spreads of finite projective spaces. If $\mathcal{B}$ is a dual spread instead of a spread, i.e. if every plane of $\mathcal{P}$ contains precisely one element of $\mathcal{B}$, then the same construction yields a mapping which is defined only on a subset of E_1'. It turns out that these mappings generalize the transversal mappings introduced by Betten for the study of topological spreads of the 3-dimensional real projective space. In order to distinguish them from the transversal mappings we call them $*$-transversal. We show that there exists a bijective correspondence between the set of all transversal or $*$-transversal mappings of an affine plane over a skewfield F and the set of all spreads of a 3-dimensional projective space $\mathcal{P}$ over F which contain a fixed line of $\mathcal{P}$.

If the skewfield F admits an extension skewfield L which has rank 2 as a right vector space over F, then the affine plane over F can be identified with L. Moreover, the graphs of transversal mappings of the affine plane over F can be viewed as subsets of the affine plane over L. According to Bruen, who studied this process for finite fields, we call the resulting sets indicator sets. We prove that a subset $\mathcal{J}$ of the affine plane over L is an indicator set if and only if each line of the affine plane whose slope is contained in $F \cup \{\infty\}$ intersects $\mathcal{J}$ in precisely one point.

If F is commutative and L is a separable quadratic extension field of F, then we can associate an inversive plane $\Sigma(F, L)$ with the pair of fields (F, L). The point set of $\Sigma(F, L)$ is the projective line $L \cup \{\infty\}$ and the circles of $\Sigma(F, L)$ are the images of $F \cup \{\infty\}$ under the group $\mathrm{PGL}_2(L)$. So the idea suggests itself to define also indicator sets with respect to other circles of $\Sigma(F, L)$. A natural candidate is the unit circle $L_1 = \{z \in L \mid z\overline{z} = 1\}$, where $\overline{}$ denotes the involutorial F-automorphism of L. Indicator sets with respect to the unit circle are called L_1-indicator sets. Using a suitably defined Cayley transformation, we set up a bijective correspondence between the indicator sets and the L_1-indicator sets of the affine plane over L. If the 4-dimensional F-vector space underlying $\mathcal{P}$ is identified with $L \times L$, then almost all elements of $\mathcal{B}$ become graphs of linear mappings of the 2-dimensional F-vector space L. We show that algebraically L_1-indicator sets lead to the decomposition of these linear mappings into an L-linear and an L-antilinear part. This is similar to the Wirtinger calculus in complex analysis, where the real differential of a real differentiable mapping $f : \mathbb{C} \to \mathbb{C}$ is decomposed into its complex linear and its complex antilinear part.

In section 2.6 we examine the case $F = \mathbb{R}$ and $L = \mathbb{C}$ more closely. We show that $\mathbb{C}_1$-indicator sets in the complex affine plane can also be viewed as images of spreads of the 3-dimensional real projective space under the kinematic mapping of Blaschke and Grünwald. This connection is further investigated in chapter 3.

Since not every commutative field F admits a separable quadratic extension field, in section 2.7 we replace the field L by the ring $A = A(F)$ of double numbers over F. We introduce A_1-indicator sets and we show that every spread of the 4-dimensional F-vector space A^2 is associated with an A_1-indicator set.

In the third chapter we relate the theory developed in chapter 2 to the theory of kinematic spaces. Among other things we show that the fundamental properties of transversal mappings and of L_1- and A_1-indicator sets can also be obtained using this theory.

In chapter 4 we study the behaviour of L_1- and A_1-indicator sets under the application of linear mappings. Furthermore, we compute the quasifields associated with an L_1- or A_1-indicator set and we investigate the relation between algebraic properties of an L_1- or A_1-indicator set and geometric properties of the corresponding translation plane. In particular, we characterize the L_1- and A_1-indicator sets which lead to pappian planes and planes of Lenz type V.

In section 4.4 spreads covered by reguli are investigated. We introduce parabolic and hyperbolic flocks of reguli. Then we show that a spread $\mathcal{B}$ contains a parabolic or hyperbolic flock of reguli if and only if the collineation group of the translation plane associated with $\mathcal{B}$ contains a subgroup acting in a certain special way on $\mathcal{B}$. This generalizes results which were obtained by Gevaert and Johnson for finite projective spaces.

Let S be a line of a 3-dimensional projective space $\mathcal{P}$ over a field F. It follows from the theory of L_1- and A_1-indicator sets that the reguli of $\mathcal{P}$ which contain S can be naturally identified with certain lines of a 4-dimensional affine space over F. Using this identification, we show that if a spread $\mathcal{B}$ is covered by reguli all of which contain S, then $\mathcal{B}$ contains either a hyperbolic or a parabolic flock of reguli. As a corollary, it follows that the translation plane associated with $\mathcal{B}$ is pappian if and only if for each $Z, W \in \mathcal{B}$ there exists a regulus $\mathcal{R}$ of $\mathcal{P}$ with $Z, W \in \mathcal{R}$ and $\mathcal{R} \subset \mathcal{B}$.

In chapter 5 we study topological spreads of a 4-dimensional real vector space using $\mathbb{C}_1$-indicator sets. We show that every $\mathbb{C}_1$-indicator set is the graph of a contraction $\varphi : \mathbb{C} \to \mathbb{C}$ which in addition satisfies a condition at infinity. Furthermore, we derive necessary and sufficient conditions for the existence of parabolic or hyperbolic flocks of reguli in topological spreads. In particular, we prove that a topological spread $\mathcal{B}$ of a 4-dimensional real vector space contains a parabolic flock of reguli if and only if the corresponding translation plane admits a 1-dimensional group of shears.

The locally compact 4-dimensional translation planes with an at least 7-dimensional collineation group were determined by Betten. Using the method of $\mathbb{C}_1$-indicator sets we derive simplified descriptions for some of these planes.

In chapter 6 we classify the locally compact planes of Lenz type V whose kernel is isomorphic to $\mathbb{C}$. Planes of this type are associated with topological spreads of a 4-dimensional complex vector space. We describe these planes using the double numbers over $\mathbb{C}$. This enables us to show that each of these planes can be obtained from two complex 2×2-matrices B and C which satisfy the condition $|m^t B m| < |m^t C m^*|$ for all $m \in \mathbb{C}^2 \setminus \{0\}$, where * denotes componentwise complex conjugation. It turns out that there exist two families of planes, which depend on either 1 or 3 real parameters. The 1-parameter family contains the planes over the division algebras of Rees; their collineation group is 17-dimensional. The 3-parameter family contains a 1-parameter subfamily of planes with an 18-dimensional collineation group and a 2-parameter subfamily with a 16-dimensional collineation group. The other planes of this family have a 15-dimensional collineation group. Up to now, only the locally compact 8-dimensional translation planes with an at least 17-dimensional collineation group had been classified by Hähl.

In chapter 7 we investigate topological spreads of 8- and 16-dimensional real vector spaces. Generalizing the results of chapter 5 we show that with every contraction of one of the normed real division algebras $\mathbb{H}$ or $\mathbb{O}$ which in addition satisfies a condition at infinity one can associate a topological spread of the real vector space $\mathbb{H}^2$ or $\mathbb{O}^2$, respectively. In contrast to the 4-dimensional case, not all topological spreads of these vector spaces can be obtained in this way. The locally compact 16-dimensional translation planes with an at least 38-dimensional collineation group were determined by Hähl. It turns out that these planes can be obtained from contractions of $\mathbb{O}$ which depend only on the real part or only on the absolute value of their argument.

1. Foundations

1.1 Translation Planes and Spreads

In this first section we give a comprehensive introduction to the theory of translation planes. For more detailed accounts the reader is referred to the original paper by André [1], or to the books by Lüneburg [74], Pickert [84] and Hughes-Piper [61].

We assume that the reader is familiar with the fundamentals of the theory of projective and affine planes. We shall use the following notation. $\mathcal{P} = (P, \mathcal{L})$ is a projective plane with point set P and line set $\mathcal{L}$. The line joining two distinct points $p, q \in P$ is denoted by $p \vee q$. Dually, the intersection point of two distinct lines $L, M \in \mathcal{L}$ is denoted by $L \wedge M$. The collineation group of $\mathcal{P}$ is denoted by $\Sigma := \Sigma(\mathcal{P})$. For $p \in P$ the group of all collineations of $\mathcal{P}$ with center p is defined by

$$\Sigma_{[p]} = \{\sigma \in \Sigma \,|\, \sigma(M) = M \text{ for all } M \in \mathcal{L} \text{ with } p \in M\}.$$

Dually, for $L \in \mathcal{L}$ the group of all collineations with axis L is defined by

$$\Sigma_{[L]} = \{\sigma \in \Sigma \,|\, \sigma(q) = q \text{ for all } q \in P \text{ with } q \in L\}.$$

Furthermore, we put

$$\Sigma_{[p,L]} = \Sigma_{[p]} \cap \Sigma_{[L]}$$

for $p \in P$ and $L \in \mathcal{L}$. A collineation σ has a center if and only if it has an axis. If the center is on the axis σ is called an *elation*; otherwise σ is a *homology*. The center and the axis of a non-identity central-axial collineation are unique. A central-axial collineation $\sigma \in \Sigma_{[p,L]}$ is determined by the image of one point of $\mathcal{P}$ which is not on L and different from p.

If $\sigma \in \Sigma$ is a collineation of $\mathcal{P}$, then ${\Sigma_{[p,L]}}^{\sigma} = \sigma\Sigma_{[p,L]}\sigma^{-1} = \Sigma_{[\sigma(p),\sigma(L)]}$ for all $p \in P$ and $L \in \mathcal{L}$. For all $M, L \in \mathcal{L}$ the set $\Sigma_{[M,L]} = \cup_{p \in M} \Sigma_{[p,L]}$ is a group.

The group $\Sigma_{[p,L]}$ is called a *(linearly) transitive group of central-axial collineations* if $\Sigma_{[p,L]}$ acts transitively on the point sets $M \setminus \{p, L \wedge M\}$, where $M \in \mathcal{L}$ is a line incident with p. If $\Sigma_{[p,L]}$ is linearly transitive we also say that the plane $\mathcal{P}$ is (p, L)*-transitive*. The *Lenz-Barlotti figure* of $\mathcal{P}$ is defined by

$$\mathrm{LBF}(\mathcal{P}) = \{(p, L) \in P \times \mathcal{L} \,|\, \mathcal{P} \text{ is } (p, L) - \text{transitive}\}.$$

Three points of $\mathcal{P}$ form a proper triangle if they are distinct and non-collinear. The point $p \in P$ is a center of the two proper triangles p_1, p_2, p_3 and q_1, q_2, q_3 if the lines $p_i \vee q_i$ contain the point p for $i = 1, \ldots, 3$. Dually, $L \in \mathcal{L}$ is an axis of the proper triangles p_1, p_2, p_3 and q_1, q_2, q_3 if the points $r_i = (p_j \vee p_k) \wedge (q_j \vee q_k)$ are contained in L for $\{i, j, k\} = \{1, 2, 3\}$. Let $p \in P, L \in \mathcal{L}$. We say that $\mathcal{P}$ is (p, L)-*desarguesian* if the following holds. Let p_1, p_2, p_3 and q_1, q_2, q_3 be proper triangles with center p and assume that $r_1, r_2 \in L$. Then also $r_3 \in L$, i.e. L is an axis of the triangles.

The plane $\mathcal{P}$ is (p, L)-transitive if and only if it is (p, L)-desarguesian.

If $\mathrm{LBF}(\mathcal{P}) = P \times \mathcal{L}$, i.e. $\mathcal{P}$ is (p, L)-desarguesian for all $(p, L) \in P \times \mathcal{L}$, then $\mathcal{P}$ is called desarguesian. The desarguesian planes are precisely the planes over skewfields.

If $\mathrm{LBF}(\mathcal{P}) = \{(p, L) \in P \times \mathcal{L} \mid p \in L\}$, then $\mathcal{P}$ is called a *Moufang plane*.

The plane $\mathcal{P}$ is called *pappian* if the following holds. Let p_1, p_2, p_3 and q_1, q_2, q_3 be triples of collinear points and put $r_i = (p_j \vee q_k) \wedge (q_j \vee p_k)$ for $\{i, j, k\} = \{1, 2, 3\}$. Then the points r_1, r_2, r_3 are collinear. The *Theorem of Hessenberg* says that every pappian plane is also desarguesian. Moreover, the pappian planes are precisely the planes over commutative fields.

Since we are usually dealing with affine planes, the following conventions turn out to be convenient. If W is a line of an affine plane $\mathcal{A} = (P, \mathcal{L})$, then w denotes the improper point of W, i.e. $W \cup \{w\}$ is a line of the projective extension of $\mathcal{A}$. The improper line of $\mathcal{A}$ is denoted L_∞. The elements of $\Sigma_{[L_\infty]}$ are called *dilatations* of $\mathcal{A}$ and the dilatations whose center is on L_∞ are called *translations*. The elations of $\mathcal{A}$ whose center is on L_∞ but whose axis is different from L_∞ are called *shears* of $\mathcal{A}$.

DEFINITION 1.1. Let $\mathcal{A} = (P, \mathcal{L})$ be an affine plane. $\mathcal{A}$ is called a *translation plane* if the translation group of $\mathcal{A}$ operates transitively on P.

Since a translation is determined by the image of one point, in fact the action is regular.

A projective plane $\mathcal{P} = (P, \mathcal{L})$ is called a translation plane if there exists a line $L \in \mathcal{L}$ such that the affine plane obtained from $\mathcal{P}$ by deleting L is a translation plane. This is equivalent to $\{(p, L) \mid p \in L\} \subseteq \mathrm{LBF}(\mathcal{P})$.

LEMMA 1.2. *Let $\mathcal{P} = (P, \mathcal{L})$ be a projective plane, and let $p, q \in P$ and $L, M \in \mathcal{L}$ such that $p \in M$ and $q \in L$. Assume moreover that $p \neq q$ or $L \neq M$. Then the groups $\Sigma_{[p,L]}$ and $\Sigma_{[q,M]}$ centralize each other.*

Proof. Since $p \neq q$ or $L \neq M$ we have $\Sigma_{[p,L]} \cap \Sigma_{[q,M]} = \{1\}$. Let $\sigma \in \Sigma_{[p,L]}$ and $\tau \in \Sigma_{[q,M]}$. Then $\tau\sigma\tau^{-1} \in \Sigma_{[\tau(p),\tau(L)]} = \Sigma_{[p,L]}$ since $\tau(p) = p$ and $\tau(L) = L$. Hence $\tau\sigma\tau^{-1}\sigma^{-1} \in \Sigma_{[p,L]}$. Exchanging the roles of σ and τ we get $\sigma\tau\sigma^{-1}\tau^{-1} \in \Sigma_{[q,M]}$ and hence $(\sigma\tau\sigma^{-1}\tau^{-1})^{-1} = \tau\sigma\tau^{-1}\sigma^{-1} \in \Sigma_{[p,L]} \cap \Sigma_{[q,M]} = \{1\}$. □

COROLLARY 1.3. *The translation group of a translation plane is abelian.*

Proof. Let $\sigma, \tau \in \Sigma_{[L_\infty, L_\infty]} \setminus \{1\}$ be translations of $\mathcal{A}$. If σ and τ have different centers they commute by Lemma 1.2. So we may assume that σ and τ have the same center $p \in L_\infty$. Let $q \in L_\infty \setminus \{p\}$ and $\delta \in \Sigma_{[q, L_\infty]} \setminus \{1\}$, then the center of $\tau\delta$ is different from p and q. Thus $\sigma(\tau\delta) = (\tau\delta)\sigma = \tau\sigma\delta$ by Lemma 1.2 and hence $\sigma\tau = \tau\sigma$. □

DEFINITION 1.4. Let F be a skewfield and let V be a vector space over F. A collection $\mathcal{B}$ of subspaces of V with $|\mathcal{B}| \geq 3$ is called a *partial spread* of V if the following condition is satisfied:

(P1) For any two different elements $U_1, U_2 \in \mathcal{B}$ we have $V = U_1 \oplus U_2$.

A partial spread is called a *spread* of V if it also satisfies

(P2) Every vector $x \in V \setminus \{0\}$ is contained in an element of $\mathcal{B}$.

If $\mathcal{B}$ is a spread, the element of $\mathcal{B}$ whose existence is required by (P2) is uniquely determined by (P1).

The elements of a spread $\mathcal{B}$ are also called the components of $\mathcal{B}$.

THEOREM 1.5. *Let $\mathcal{B}$ be a spread of a vector space V over a skewfield F. Put $P = V$ and $\mathcal{L} = \{U + x \,|\, U \in \mathcal{B}, x \in V\}$. Then $\mathcal{A} = \mathcal{A}(\mathcal{B}) = (P, \mathcal{L})$ is a translation plane. The translation group of $\mathcal{A}$ is isomorphic to $(V, +)$.*

Proof. We show first that $\mathcal{A}$ is an affine plane. Since $|F| \geq 2$ and $|\mathcal{B}| \geq 3$ every line of $\mathcal{A}$ contains at least 2 points and every point of $\mathcal{A}$ is on at least 3 lines. Let $x, y \in V$ be distinct points of $\mathcal{A}$. Then there exists precisely one element $U \in \mathcal{B}$ such that $x - y \in U$. Thus $U + x = U + y \in \mathcal{L}$ is the unique line of $\mathcal{A}$ connecting x and y. Let $x \in V$ and $U + y \in \mathcal{L}$. The lines through x are precisely the sets $W + x, W \in \mathcal{B}$. Such a line is parallel to $U + y$ if and only if $U = W$ since otherwise $U + W = V$. So there is a unique line through x which is parallel to $U + y$. Hence $\mathcal{A}$ is an affine plane.

Obviously, the set $\{\tau_y : V \to V : x \mapsto x + y \,|\, y \in V\}$ is a transitive translation group of $\mathcal{A}$, and this group is isomorphic to $(V, +)$. □

It was proved by André [1] that the converse of Theorem 1.5 is also true, i.e. every translation plane can be obtained from a spread of a suitable vector space.

Let $\mathcal{A} = (P, \mathcal{L})$ be a translation plane. Since the translation group $\Sigma_{[L_\infty, L_\infty]}$ is abelian, it will be written additively.

LEMMA 1.6. *Let $p, q \in L_\infty$ be distinct points. Then $\Sigma_{[L_\infty, L_\infty]} = \Sigma_{[p, L_\infty]} + \Sigma_{[q, L_\infty]}$ and the sum is direct.*

Proof. As $\Sigma_{[p, L_\infty]} \cap \Sigma_{[q, L_\infty]} = \{0\}$ the sum is certainly direct. Since the action of $\Sigma_{[L_\infty, L_\infty]}$ on P is regular, it is sufficient to show that $\Sigma_{[p, L_\infty]} + \Sigma_{[q, L_\infty]}$ acts transitively on P. Let $x, y \in P$. Put $z = (x \vee p) \wedge (y \vee q)$. Then there are

$\sigma \in \Sigma_{[p,L_\infty]}$ and $\tau \in \Sigma_{[q,L_\infty]}$ such that $\sigma(x) = z$ and $\tau(z) = y$. Consequently, $(\tau + \sigma)(x) = y$, and hence $\Sigma_{[p,L_\infty]} + \Sigma_{[q,L_\infty]}$ is transitive on P. □

We are now in position to prove André's representation theorem for translation planes.

THEOREM 1.7. *Let $\mathcal{A} = (P, \mathcal{L})$ be a translation plane. Put $V = \Sigma_{[L_\infty,L_\infty]}$ and $\mathcal{B} = \{\Sigma_{[p,L_\infty]} \mid p \in L_\infty\}$. Let the kernel of $\mathcal{A}$ be defined by $K(\mathcal{A}) = \{\delta \in \mathrm{End}(V) \mid \delta(U) \subseteq U$ for all $U \in \mathcal{B}\}$, where $\mathrm{End}(V)$ denotes the endomorphism ring of V. Then $K(\mathcal{A})$ is a skewfield, V is a left vector space over $K(\mathcal{A})$ and $\mathcal{B}$ is a spread of V. Moreover, the translation planes $\mathcal{A}(\mathcal{B})$ and $\mathcal{A}$ are isomorphic.*

Proof. We show first that $K(\mathcal{A})$ is a ring. Let $\gamma, \delta \in K(\mathcal{A})$ and let $U \in \mathcal{B}$. Then $(\gamma - \delta)(U) \subseteq \gamma(U) + \delta(U) \subseteq U + U \subseteq U$ and $(\gamma \circ \delta)(U) = \gamma(\delta(U)) \subseteq \gamma(U) \subseteq U$. This shows that $K(\mathcal{A})$ is a subring of $\mathrm{End}(V)$ and hence is a ring. Also, V naturally is a left module over $K(\mathcal{A})$ and the elements of $\mathcal{B}$ are $K(\mathcal{A})$-submodules of V. Although we do not yet know that $K(\mathcal{A})$ is a skewfield, we can construct the incidence structure $\mathcal{A}(\mathcal{B})$ as in Theorem 1.5. From Lemma 1.6 we infer that $\mathcal{B}$ satisfies (P1) and since every translation has a center on L_∞, condition (P2) is satisfied as well. Let $p \in P$ and define $\sigma : V \to P$ by $\sigma(\tau) = \tau(p)$. Then σ is bijective since V is sharply transitive on P. Moreover, σ induces an isomorphism between $\mathcal{A}(\mathcal{B})$ and $\mathcal{A}$, as is easily seen. Hence $\mathcal{A}(\mathcal{B})$ is an affine plane.

So it remains to show that $K(\mathcal{A})$ is a skewfield.

Let $\delta \in K(\mathcal{A})\setminus\{0\}$, where 0 denotes the zero endomorphism. Assume that δ is not injective. Then there exists $x \in V \setminus \{0\}$ such that $\delta(x) = 0$. Let $U \in \mathcal{B}$ be the component with $x \in \mathcal{B}$. Let $y \in V \setminus U$ and let W and Z denote the unique elements of $\mathcal{B}$ for which $y \in W$ and $y + x \in Z$. Then W and Z are distinct and since $\delta(y) = \delta(y + x) \in \delta(W) \cap \delta(Z) \subseteq W \cap Z = \{0\}$, we get $\delta(y) = 0$. Applying this argument once more we conlude that δ is the zero endomorphism, contradicting our assumption. Hence δ is injective.

We show next that δ is also surjective. Let $x \in V \setminus \{0\}$ and let $U \in \mathcal{B}$ with $x \in U$. Choose $y \in V \setminus U$ and let $W \in \mathcal{B}$ with $y \in W$. Then $\delta(y) \neq x$ and hence there is a unique $Z \in \mathcal{B}$ with $\delta(y) - x \in Z$. Since $\mathcal{A}(\mathcal{B})$ is an affine plane and $U \neq Z$, there exists $z \in (Z + y) \cap U$. This implies $z - y \in Z$ and hence $\delta(z) - \delta(y) \in Z$. As $\delta(y) - x \in Z$ and Z is a subgroup of V we thus get $\delta(z) - x \in Z$. On the other hand, we have $z \in U$ and hence $\delta(z) \in U$. It follows that $\delta(z) - x \in U \cap Z = \{0\}$ and hence δ is surjective.

Let $U \in \mathcal{B}$, then $\delta^{-1}(U) = \delta^{-1}(\delta(U)) = U$. Thus $\delta^{-1} \in K(\mathcal{A})$ and $K(\mathcal{A})$ is a skewfield.

By definition of $K(\mathcal{A})$, the elements of $\mathcal{B}$ are $K(\mathcal{A})$-subspaces of V, hence $\mathcal{B}$ is a spread of V. □

Since we let our mappings operate from the left on their arguments, the translation group of a translation plane $\mathcal{A}$ naturally becomes a left vector space over the kernel of $\mathcal{A}$. It is also possible to use a right vector space for the representation of $\mathcal{A}$. To this end we replace the skewfield $(K(\mathcal{A}), +, \cdot)$ by its *opposite skewfield* $(\widetilde{K(\mathcal{A})}, \tilde{+}, \tilde{\cdot})$, where $a \tilde{+} b = a + b$ and $a \tilde{\cdot} b = b \cdot a$ for $a, b \in K(\mathcal{A})$. Every left vector space V over $K(\mathcal{A})$ becomes a right vector space over $\widetilde{K(\mathcal{A})}$ if we define $x \tilde{\cdot} c = c \cdot x$ for $x \in V, c \in \widetilde{K(\mathcal{A})} = K(\mathcal{A})$. The subspaces of the left $K(\mathcal{A})$-vector space V coincide with the subspaces of the right $\widetilde{K(\mathcal{A})}$-vector space V. Hence, $\mathcal{B}$ is also a spread of the right $\widetilde{K(\mathcal{A})}$-vector space V.

DEFINITION 1.8. Let $\mathcal{A}$ be a translation plane and let F be a skewfield. We say that $\mathcal{A}$ admits a *representation* over F if there exists a vector space V over F and a spread $\mathcal{B}$ of V such that $\mathcal{A}$ is isomorphic to $\mathcal{A}(\mathcal{B})$.

PROPOSITION 1.9. *Let $\mathcal{A}$ be a translation plane and let F be a skewfield. Then $\mathcal{A}$ admits a representation over F if and only if F is isomorphic or antiisomorphic to a subskewfield of $K(\mathcal{A})$. More precisely: $\mathcal{A}$ admits a representation in a left (right) vector space over F if and only if F is isomorphic (antiisomorphic) to a subskewfield of $K(\mathcal{A})$. In particular, $K(\mathcal{A})$ and $\widetilde{K(\mathcal{A})}$ are the largest skewfields over which $\mathcal{A}$ admits a representation.*

Proof. Let F be a subskewfield of $K(\mathcal{A})$. Since every left vector space over $K(\mathcal{A})$ also is a left vector space over F, the translation plane $\mathcal{A}$ admits a representation over F and hence also over any skewfield isomorphic to F. A similar argument applies to subskewfields of $\widetilde{K(\mathcal{A})}$.

Assume now that $\mathcal{A}$ admits a representation over a skewfield F. Let V be a vector space over F and let $\mathcal{B}$ be a spread of V such that $\mathcal{A}$ is isomorphic to $\mathcal{A}(\mathcal{B})$.

If V is a left vector space we define $K' = \{\delta_c : V \to V : x \mapsto cx \mid c \in F\}$. Then F' is a subskewfield of $K(\mathcal{A}(\mathcal{B}))$ which is isomorphic to F. Hence $K(\mathcal{A})$ contains a subskewfield which is isomorphic to F.

If V is a right vector space we define instead $K' = \{\delta_c : V \to V : x \mapsto xc \mid c \in F\}$. Then F' is a subskewfield of $K(\mathcal{A}(\mathcal{B}))$ which is antiisomorphic to F. Hence $K(\mathcal{A})$ contains a subskewfield which is antiisomorphic to F. $\square$

If $K(\mathcal{A})$ is a field, the distinction between left and right vector spaces vanishes. By Wedderburn's theorem, cf. e.g. [62: p.453], every finite skewfield is commutative, hence the kernel of a finite translation plane is a field. The same holds for locally compact connected translation planes with the exception of the quaternion plane, cf. Proposition 1.29.

Let $\mathcal{P}$ be the projective space associated with the vector space V and let $\mathcal{B}$ be a spread of V. Viewed projectively, $\mathcal{B}$ has the following properties:

(S1) Any two distinct subspaces $U_1, U_2 \in \mathcal{B}$ intersect trivially and span $\mathcal{P}$.

(S2) Every point of $\mathcal{P}$ is contained in an element of $\mathcal{B}$.

A system of subspaces of $\mathcal{P}$ satisfying (S1) and (S2) will be called a *spread* of the projective space $\mathcal{P}$.

1.2 Quasifields and Spread Sets

Let $\mathcal{B}$ be a spread of the F-vector space V and let $W, S \in \mathcal{B}$ be distinct. Then V is the direct sum of W and S and for every $U \in \mathcal{B} \setminus \{S\}$ we have $U \cap S = \{0\}$. Hence, every component $U \in \mathcal{B} \setminus \{S\}$ is the graph of a linear mapping $\lambda_U : W \to S$. In particular, λ_W is the zero mapping. It follows easily from (P1) and (P2) that the set $\mathcal{M} = \{\lambda_U : W \to S \mid U \in \mathcal{B} \setminus \{S\}\}$ has the following characteristic properties:

(L1) For all $U, Z \in \mathcal{B} \setminus \{S\}$ with $U \neq Z$ the mapping $\lambda_U - \lambda_Z$ is bijective.

(L2) For all $x \in W \setminus \{0\}$ the mapping $\varrho_x : \mathcal{B} \setminus \{S\} \to S : U \mapsto \lambda_U(x)$ is surjective.

Since the vector spaces W and S are isomorphic, they usually are identified. This motivates the following

Definition 1.10. Let X be a vector space over a skewfield F. A collection of linear mappings $\mathcal{M} \subseteq \mathrm{End}_F(X)$ is called a *spread set* of X if the following conditions are satisfied:

(M1) For any two distinct elements $\lambda_1, \lambda_2 \in \mathcal{M}$ the mapping $\lambda_1 - \lambda_2$ is bijective.

(M2) For all $x \in X \setminus \{0\}$ the mapping $\varrho_x : \mathcal{M} \to X : \lambda \mapsto \lambda(x)$ is surjective.

It follows from (M1) that the mappings ϱ_x considered in (M2) are injective for $x \in X \setminus \{0\}$. Hence, if (M2) is satisfied, they are even bijective.

Actually, it is sufficient to require that X is an abelian group instead of a vector space and that $\mathcal{M} \subseteq \mathrm{End}(X)$ satisfies (M1) and (M2). It then follows from Theorem 1.7 that X is a vector space over a suitable skewfield and that the elements of $\mathcal{M}$ are linear mappings.

Proposition 1.11. *Let $X \neq \{0\}$ be a vector space over a skewfield F and let $\mathcal{M} \subset \mathrm{End}_F(X)$ be a spread set of X. Put $V = X \times X$ and $S = \{0\} \times X$. For $\lambda \in \mathcal{M}$ let $U_\lambda = \{(x, \lambda(x)) \mid x \in X\}$ denote the graph of λ. Then $\mathcal{B}(\mathcal{M}) = \{S\} \cup \{U_\lambda \mid \lambda \in \mathcal{M}\}$ is a spread of V. Conversely, every spread $\mathcal{B}$ of V with $S \in \mathcal{B}$ is obtained from a spread set of X in the way just described.*

Proof. Let $\mathcal{M}$ be a spread set of X and define $\mathcal{B}$ as in the proposition.

Let $\lambda, \mu \in \mathcal{M}$ be distinct. We have to show that $V = U_\lambda \oplus U_\mu$. Let $(w, z) \in V$. Then we have

$$\begin{aligned}(w, z) &= (x, \lambda(x)) + (y, \mu(y)) \\ &= (x + y, \lambda(x) + \mu(y)) \\ &= (w, \lambda(x) + \mu(w - x)) \\ &= (w, \lambda(x) - \mu(x) + \mu(w)).\end{aligned}$$

Since $\lambda - \mu$ is bijective, this equation has a unique solution $x \in X$. So $y \in X$ is determined uniquely as well and hence $V = U_\lambda \oplus U_\mu$.

Let $(w, z) \in V \setminus \{(0,0)\}$. If $w = 0$ then $(w, z) \in S$. So assume $w \neq 0$. We need to find $\lambda \in \mathcal{M}$ such that $(w, z) \in U_\lambda$. The equation

$$(w, z) = (x, \lambda(x)) = (w, \lambda(w)) = (w, \varrho_w(\lambda))$$

has a solution $\lambda \in \mathcal{M}$ since ϱ_w is surjective. So (w, z) is contained in an element of $\mathcal{B}$ and hence $\mathcal{B}$ is a spread of V.

Assume now that $\mathcal{B}$ is a spread of V with $S \in \mathcal{B}$. Let $U \in \mathcal{B} \setminus \{S\}$. Since $V = X \times X$ and $U \cap S = \{(0,0)\}$, there exists a linear mapping $\lambda : X \to X$ such that $U = U_\lambda$. By reversing the arguments given above it is easily seen that $\mathcal{M} = \{\lambda \in \mathrm{End}_F(X) \mid U_\lambda \in \mathcal{B} \setminus \{S\}\}$ is a spread set of X, and obviously we have $\mathcal{B}(\mathcal{M}) = \mathcal{B}$. □

It follows from elementary linear algebra that if a vector space V contains a spread $\mathcal{B}$ then there exists a vector space X such that V can be identified with $X \times X$ and $S = \{0\} \times X \in \mathcal{B}$. Hence, every spread can be obtained from a suitable spread set.

Definition 1.12. Let Q be a set equipped with two binary operations $+, \circ : Q \times Q \to Q$. For $a \in Q$ we define the mappings $\lambda_a, \varrho_a : Q \to Q$ by $\lambda_a(x) = a \circ x$ and $\varrho_a(x) = x \circ a$, respectively. Then $(Q, +, \circ)$ is called a *right quasifield* if the following axioms are satisfied:

- (Q1) $(Q, +)$ is an abelian group.
- (Q2) $x \circ 0 = 0 \circ x = 0$ for all $x \in Q$.
- (Q3) There exists an element $1 \in Q \setminus \{0\}$ such that $1 \circ x = x \circ 1 = x$ for all $x \in Q$.
- (Q4) For all $m, x, y \in Q$ we have $(x + y) \circ m = x \circ m + y \circ m$.
- (Q5) For all $m, n \in Q$ with $m \neq n$ the mapping $\varrho_m - \varrho_n : Q \to Q : x \mapsto x \circ m - x \circ n$ is bijective.
- (Q6) For all $x \in Q \setminus \{0\}$ the mapping $\lambda_x : Q \to Q : m \mapsto x \circ m$ is surjective.

The *kernel* of Q is defined by $K(Q) = \{k \in Q \mid k \circ (x + y) = k \circ x + k \circ y$ and $k \circ (x \circ y) = (k \circ x) \circ y$ for all $x, y \in Q\}$.

The axioms for a left quasifield are obtained from (Q1) - (Q6) by exchanging the factors in all products that appear.

It follows from (Q5) that the mappings λ_x are injective for $x \in Q \setminus \{0\}$. Hence, they are even bijective if (Q6) is satisfied.

Lemma 1.13. *Let Q be a right quasifield with kernel $K(Q)$. Then $K(Q)$ is a skewfield and Q is a left vector space over $K(Q)$. Moreover, the mappings $\varrho_m : Q \to Q : x \mapsto x \circ m$ are linear over $K(Q)$ for all $m \in Q$.*

Proof. Let $k, l, x \in Q$ with $k + l = 0$. Then we have $k \circ x + l \circ x = (k + l) \circ x = 0 \circ x = 0$. It follows that $-(k \circ x) = (-k) \circ x$.

Let $k, l \in K(Q)$ and $x, y \in Q$. Then we have

$$\begin{aligned}(k - l) \circ (x + y) &= k \circ (x + y) - l \circ (x + y)\\ &= k \circ x + k \circ y - l \circ x - l \circ y\\ &= (k - l) \circ x + (k - l) \circ y.\end{aligned}$$

Also we have

$$\begin{aligned}(k - l) \circ (x \circ y) &= k \circ (x \circ y) - l \circ (x \circ y)\\ &= (k \circ x) \circ y - (l \circ x) \circ y\\ &= (k \circ x - l \circ x) \circ y\\ &= ((k - l) \circ x) \circ y.\end{aligned}$$

This shows $(k - l) \in K(Q)$. By a similar argument, also $k \circ l \in K(Q)$.

Let $x, y \in Q$. From (Q3) we infer that $1 \circ (x + y) = x + y = (1 \circ x) + (1 \circ y)$ and $1 \circ (x \circ y) = x \circ y = (1 \circ x) \circ y$. It follows that $1 \in K(Q)$.

By definition of $K(Q)$, the multiplication of Q restricted to $K(Q)$ is associative, Moreover, $K(Q)$ satisfies both distributive laws. Hence, $K(Q)$ is a ring.

Let $k \in K(Q) \setminus \{0\}$. By (Q6) there exists $l \in Q$ such that $k \circ l = 1$. Let $x, y \in Q$. Then we get

$$\begin{aligned}k \circ (l \circ (x \circ y)) &= (k \circ l) \circ (x \circ y) = (x \circ y)\\ &= ((k \circ l) \circ x) \circ y = (k \circ (l \circ x)) \circ y = k \circ ((l \circ x) \circ y).\end{aligned}$$

It follows that $l \circ (x \circ y) = (l \circ x) \circ y$. By a similar argument also $l \circ (x + y) = l \circ x + l \circ y$ and hence $l \in K(Q)$. Thus $K(Q)$ is a skewfield.

It follows from the definition of $K(Q)$ that Q is a left vector space over $K(Q)$.

From (Q4) we infer that the mappings $\varrho_m : Q \to Q : x \mapsto x \circ m$ are additive for all $m \in Q$. Let $x, m \in Q$ and $k \in K(Q)$. Then we have

$$\varrho_m(k \circ x) = (k \circ x) \circ m = k \circ (x \circ m) = k \circ \varrho_m(x).$$

Hence ϱ_m is linear for all $m \in Q$. □

PROPOSITION 1.14. *Let Q be a right quasifield and define $\mathcal{M} = \mathcal{M}(Q) = \{\varrho_m : Q \to Q \mid m \in Q\}$. Then $\mathcal{M}$ is a spread set of the $K(Q)$-vector space Q. Let $\mathcal{A} = \mathcal{A}(Q) = \mathcal{A}(\mathcal{B}(\mathcal{M}(Q)))$ denote the translation plane associated with the spread $\mathcal{B}(\mathcal{M}(Q))$. Then we have $K(\mathcal{A}) = \{\delta_k : Q^2 \to Q^2 : (x, y) \mapsto (k \circ x, k \circ y) \mid k \in K(Q)\}$. In particular, the skewfields $K(Q)$ and $K(\mathcal{A})$ are isomorphic.*

Proof. By Lemma 1.13, the elements of $\mathcal{M}$ are linear mappings of the left $K(Q)$-vector space Q.

Obviously, (Q5) is equivalent to (M1) and (Q6) is equivalent to (M2). Hence $\mathcal{M}$ is a spread set.