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Abstract

Let X be a finite group. Denote by O(X) its maximal normal subgroup of odd order. A 2-component of X is a perfect subnormal subgroup L of X such that L/O(L) is quasisimple. The principal result of this paper is to classify groups G with a unique 2-component L(G) for which $O(C_G(E)) \not\subseteq O(G)$ for some elementary 2-subgroup E. A corollary of this result is that the 2-components of the centralizer of an involution in any finite group G with O(G) = 1 are quasisimple (which is the B-Conjecture). The classification first obtains a characterization of quasisimple groups G such that $C_G(u)$ has a 2-component L for some involution u for which L/O(L) is a Chevelley group over a field of odd characteristic. This characterization is based on Michael Aschbacher's characterization of Chevalley groups. Having obtained this, the bulk of the argument proceeds with an analysis of the situation when the centralizers of involutions have centralizers with 2-components of different types.

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PART I

CHARACTERIZATION OF CHEVALLEY GROUPS AND OTHER LOCALLY $\mathcal{E}$-UNBALANCED GROUPS

0. Introduction. A perfect group L such that $L/Z(L)$ is simple is said to be *quasisimple*. Designate by $O(X)$ the maximal normal subgroup of odd order of a group X. A *2-component* of X is a perfect subnormal subgroup L such that $L/O(L)$ is quasisimple. If $O(L) = 1$, L is said to be a *component* of X. Let $\mathfrak{X}$ be a set of quasisimple groups. A 2-component L with the property that $L/O(L) = X_\alpha/O(X_\alpha)$ for some $X_\alpha \in \mathfrak{X}$ is said to have *type* $\mathfrak{X}$. Denote by $\mathcal{L}(X)$ the set of 2-components of a group X, and set $L(X) = \langle L | L \in \mathcal{L}(X)\rangle$. Let $\mathcal{I}(G)$ and $\mathcal{E}(G)$ respectively denote the set of involutions and the set of nontrivial elementary subgroups of a group G. Set

$$(0.1) \qquad \mathcal{L}(\mathcal{I}(G)) = \cup\{\mathcal{L}(C_G(t)) \mid t \in \mathcal{I}(G)\},$$

$$(0.2) \qquad \mathcal{L}(\mathcal{E}(G)) = \cup\{\mathcal{L}(C_G(E)) \mid E \in \mathcal{E}(G)\}.$$

The following statement has singled out an important step in the classification of simple finite groups.

B-CONJECTURE. *The elements of* $\mathcal{L}(\mathcal{I}(G/O(G))$ *are quasisimple for any finite group* G.

A proof of this statement will be presented in this monograph. This result plays a key role in the characterization of simple finite groups G by providing the first step in the analysis of the centralizers of involutions of $\mathrm{Aut}(G)$.

Let L be a 2-component of a group X and set $\bar{N}_L = \mathrm{Aut}_X(L/O(L)) = N_X(L)/C_X(L/O(L))$. We say that L is *locally balanced* [*locally $\mathcal{E}$-balanced*] if $O(C_{\bar{N}_L}(t)) = 1$ for all $t \in \mathcal{I}(N_X(L))$ [$O(C_{\bar{N}_L}(E)) = 1$ for all $E \in \mathcal{E}(N_X(L))$]. It is a consequence of Proposition 2 of [33] that $O(C_G(t))$ determines a signalizer functor if the elements of $\mathcal{L}(\mathcal{I}(G))$ are locally balanced for a finite group G. It then follows from Aschbacher's classification [1] of groups with a proper 2-generated core that $O(C_G(t)) \subseteq O(G)$ for all $t \in \mathcal{I}(G)$ or that the maximal rank of an element of $\mathcal{E}(G)$ is 3.

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In this case, results of Gorenstein and Harada [29] provide a classification of the elements of $\mathcal{L}(G)$ which verifies the B-Conjecture. The principal result of this work is Theorem I which provides a characterization of groups with locally $\mathcal{E}$-unbalanced components. This provides a proof of the B-Conjecture since this condition can then be seen to hold in all groups G in which $\mathcal{L}(\mathcal{I}(G))$ contains locally unbalanced elements.

To describe the types of these 2-components, we use a standard notation for the classical (cf. [28]), alternating and sporadic groups (cf. [28]). When a simple group of type X has a unique perfect central extension with center of order 2, we denote this extension by $\hat{X}$. In addition, denote by TL(3,4) any perfect central extension of SL(3,4) by a 2-group. Denote by Chev(p) the set of quasisimple groups Chevalley groups and their twisted analogues defined over a field of characteristic p. Exceptional and twisted exceptional Chevalley groups will be denoted by a standard notation (cf. [13]). By Chev*(p), we designate the set Chev(p) with the groups $PSL(2,p^n)$ omitted when $p > 3$. When $p = 3$, we exclude PSL(3,3), PSU(3,3), ${}^2G_2(3^{2n+1})$, $G_2(3)$, PSL(4,3), PSp(4,3), and PSU(4,3).

The types of locally unbalanced 2-components in $\mathcal{L}(\mathcal{I}(G))$ for a group G fall into three classes:

(0.3) $\mathcal{B}_a = \{PSL(2,q),\ q \text{ odd};\ TL(3,4),\ He\}$,

(0.4) $\mathcal{B}_b = \{A_n, \hat{A}_n,\ n \text{ odd},\ n \geq 7\}$,

(0.5) $\mathcal{B}_c = \{Chev^*(p),\ p \text{ odd}\}$.

Set $\mathcal{B} = \mathcal{B}_a \cup \mathcal{B}_b \cup \mathcal{B}_c$. Set

$$\mathcal{C}_o = \mathcal{B} \cup \{M_{12}, \hat{M}_{12}, J_2, \hat{J}_2, HS, \hat{HS}, Sz, \hat{Sz}, ON, Co_3, Ly, {}^2G_2(3^{2n+1})\}.$$

The principal result of this work is the following.

THEOREM I. *Let G be a finite group with a unique 2-component L(G) Suppose* $O(C_G(E)) \not\subseteq O(G)$ *for some* $E \in \mathcal{E}(G)$. *Then L(G) has type* $\mathcal{C}_o$.

COROLLARY. *The B-Conjecture holds.*

Let G be a minimal counterexample to Theorem I. It follows that L(G) is simple, $C_G(L(G)) = 1$, the locally $\mathcal{E}$-unbalanced elements of $\mathcal{L}(\mathcal{E}(G))$ have type $\mathcal{C}_o$, and the B-Conjecture holds in proper sections of G. The contradiction is obtained by applying various characterization theorems to G. This paper is divided into two parts. Part II is more complex than Part I and provides the establishment of the B-Conjecture for the group G. Part I provides the characterization of G under this condition, which immediately lead to a standard element in $\mathcal{L}(\mathcal{I}(G))$ by virtue of Aschbacher's

Component Theorem [2]; many of these results are of interest in themselves independently of the role they play in the establishment of the B-Conjecture. It is a consequence of the induction hypothesis on G that the standard subgroup has type $\mathscr{C}$ where the class $\mathscr{C}$ contains $\mathscr{C}_0$ (cf. (0.7)). But also in Part I a characterization of L(G) is obtained when $\mathscr{L}(\mathscr{I}(G))$ has an element of type Chev*(p), and this characterization is carried out without the use of the B-Conjecture except for four types listed in (0.7). This result allows us to restrict the locally $\mathscr{E}$-unbalanced elements in $\mathscr{L}(\mathscr{E}(G))$ in Part II. The introduction to Part II contains a more complete analysis of the strategy of this proof. But the essential idea is to obtain the B-Conjecture in such a form that the reduction to the analysis of groups with standard subgroups is as direct as possible.

We now state more precisely the main theorems which are proved in this work. First set[1]

(0.7) $\mathscr{C} = \mathscr{C}_0 \cup \{J_1, J_3, Co_1, \hat{C}o_1, Mc, F_5, Chev(3), SL(2,4^n), PSL(3,2^n), PSU(3,4), PSL(4,2^n), Sp(4,4^n), PSL(5,2^n), SU(5,2), G_2(2^n), \hat{G}_2(4), n \geq 1; A_{2n}, n \geq 4\}$.

The following theorem provides the final step in characterization of a minimal counterexample to Theorem This theorem is proved in Part I by the consideration of a succession of standard subgroup problems. In the second section of Part I, we present a characte This theorem is proved in Part I by the consideration of a succession of standard subgroup problems. In the second section of Part I, we present a characterization of L(G) when $\mathscr{L}(\mathscr{I}(G))$ contains an element L of type

This theorem is proved in Part I by the consideration of a succession of standard subgroup problems. In the second section of Part I, we present a characterization of L(G) when $\mathscr{L}(\mathscr{I}(G))$ contains an element L of type PSL(2,q) or A_7 in a form needed for Part II. This argument relies on [43] for the case where $|L|_2 = 8$. The cases where $\mathscr{L}(\mathscr{I}(G))$ contains a standard element of type He, ON, Co_3, and F_5 are treated in §3. These follow from Corollary 3.3 of Part I which gives a short uniform treatment based on Theorem 3.2, which has an immediate and direct application to a variety of other standard subgroup problems that are irrelevant to this paper. The remaining cases with standard components of sporadic type where our ori-

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[1] In a previous version of this paper, we did not include the elements $G_2(3)$, SL(3,3), SU(3,3), $G_2(2^n)$, $G_2(4)$, Co_1, and $\hat{C}o_1$ in the class $\mathscr{C}$ because we used a proof of Theorem 3.4 which did not require the inclusion of these groups. But now the argument is based on a paper of Michael Aschbacher [5], which uses this larger class of groups and deals with this question more effectively at the expense of considering more standard form problems. As a result, it is necessary to quote a larger number of characterization theorems, but these must be considered in the general problem of the classification of finite simple groups in any event.

ginal arguments showed no advantage over the present literature are now treated by quoting that literature.

The case where some member of $\mathcal{L}(\mathcal{I}(G))$ has type Chev*(p) is considered in the last section of Part I and is the most important result in Part I. Here we obtain a characterization of L(G) when $\mathcal{L}(\mathcal{I}(G))$ has an element of type Chev*(p). The argument is based on Michael Aschbacher's characterization [2] of $\langle L^G \rangle$ in groups G where L is an intrinsic element of $\mathcal{L}(\mathcal{I}(G))$ of type SL(2,q), q > 3; a 2-component of $C_G(t)$, $t \in \mathcal{I}(G)$, is called *intrinsic* if $t \in L$. The B-Conjecture is not required for this argument except for the case where the elements of $\mathcal{L}(\mathcal{I}(G))$ of type Chev*(p) have type $\mathfrak{D}$ where[2]

(0.8) $$\mathfrak{D} = \{PSL(4,3), PSU(4,3), \Omega^-(6,3), \Omega(7,3), \Omega^-(8,3), P\Omega^+(8,3)\}.$$

The class $\mathfrak{D}$ has to be considered because in the characterization of groups of type Chev*(3), there exists groups G with an intrinsic element in $\mathcal{L}(\mathcal{I}(G))$ of type $\Omega^-(6,3)$ but no intrinsic element of type SL(2,q).

For a group X, set

(0.9) $$\mathcal{I}(X) = \{j \in X \mid j^2 \in Z(X)\},$$

(0.10) $$\mathcal{L}(\mathcal{I}(X)) = \cup\{\mathcal{L}(C_X(j)) \mid j \in \mathcal{I}(X)\}.$$

The precise result which we establish is the following theorem.

THEOREM III. *Let G be a finite group with L(G) quasisimple. Assume that the B-Conjecture is satisfied in each proper section of G, that Theorem II holds in proper sections of G/Z(G), and that $\mathcal{L}(\mathcal{I}(G))$ contains an element of type Chev*(p) for some odd prime p. Then either L(G) has type Chev*(p) or one of the following holds.*

(i) L(G) is not simple and the elements of $\mathcal{L}(\mathcal{I}(G))$ of type Chev(p) have type SL(2,q), q = 5,7 or 9 and L(G) has type $\hat{A}_n$, n odd, TL(3,4), or $\hat{A}_n$, n even, according as q = 5,7 or 9, respectively.*

(ii) L(G) is simple and the elements of $\mathcal{L}(\mathcal{I}(G))$ of type Chev(p) have type $\mathfrak{D}$ or SL(2,7), but $\mathcal{L}(\mathcal{I}(G))$ contains no intrinsic element of type SL(2,7).*

THEOREM IV. *Assume the B-Conjecture and the conditions of Theorem III. Then L(G) has type Chev*(3) under condition (ii) of Theorem III.*

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[2]Here $\Omega^+(2n,q)$ and $\Omega^-(2n,q)$ denote the commutator subgroups of the orthogonal groups $O^+(2n,q)$ and $O^-(2n,q)$ respectively defined over a vector space of dimension 2n over the finite field $\mathbb{F}_q$ of q elements relative to a quadratic form of Witt index n and n-1. We denote by $\Omega(2n+1,q)$ the commutator subgroup of the orthogonal group O(2n+1),q) defined on a vector space of dimension 2n+1 over $\mathbb{F}_q$

Theorem III together with the results of Solomon [57] allow us to assume that the locally $\mathcal{E}$-unbalanced elements of $\mathcal{L}(\mathcal{E}(G))$ in the proof of Theorem I have type $\mathcal{C}_o^*$ where

(0.11) $\mathcal{C}_o^* = \{\mathcal{C}_o \setminus (\{Chev^*(p) | p \text{ odd}\} \cup \{A_{2n+1} | n \le 3\} \cup \{Ly\})\} \cup \mathcal{D}_o \cup \{SL(2,7)\}$

and where

(0.12) $\mathcal{D}_o = \{PSL(4,3), PSU(4,3), \Omega^-(6,3), \Omega^-(8,3), P\Omega^+(8,3)\}$.

The set $\mathcal{D}_o$ is the subset of $\mathcal{D}$ consisting of those types which give rise to locally $\mathcal{E}$-unbalanced elements of $\mathcal{L}(\mathcal{I}(G))$ (cf. Proposition 2.11 of Part II). In Part II, Theorem I is proved by using signalizer functors.

The characterization of Chevalley groups plays a key role in the organization of this material. At the Sapporo Conference, during a conversation with Michael Aschbacher, it was realized that the inductive approach of characterizing groups with locally $\mathcal{E}$-unbalanced elements could be made more effective by utilizing a characteriztion of Chevalley groups based on Aschbacher's fundamental paper together also with Solomon's characterization of groups with elements of $\mathcal{L}(\mathcal{I}(G))$ of type $\hat{A}_n$. These characterizations are based on a signalizer functor appearing in Aschbacher's paper and consequently do not involve the B-Conjecture. This constructon works well with intrinsic elements of $\mathcal{L}(\mathcal{I}(G))$.

The characterization of Chevalley groups appearing in Part I depends upon reducing the situation to a configuration in the centralizer of an involution which charaterizes Spin(7,q) when $q > 3$. The essence of this argument is given in [66]. Shortly after these ideas were developed, John Thompson introduced to this problem a variant of this approach which introduced the concept of an unbalanced triple (a,x,J) where J is an element of $\mathcal{L}(C_G(a))$ of type $Chev^*(p)$ which is $\langle x\rangle$-invariant for $a \in \mathcal{I}(G)$ and $x \in \mathcal{I}(C_G(a))$ and $L = [L, O(C_L(x)) \cap C_L(a)]O(L)$. This idea was developed further by Burgoyne [10]. Then it was followed by Solomon [57], who developed the idea of a minimal unbalancing triple with Robert Gilman [22]. This is explained in [14]. The present work uses the signalizer functor method to the maximum extent in order to provide the B-Conjecture. This simplifies the characterization problems appearing in Part I. A more complete survey of the approach used in this paper and a discussion of its significance appears in the introduction to Part II.

The author gratefully acknowledges the advice and suggestions given to him by Michael Aschbacher. His encouragement and interest played an essential role in completion of this paper. His suggestions in regard to the material in Part I provided the basis for a substantial simplification of the argument. This is particularly true for Theorem 3.2 which was developed from an argument communicated to the author by him.

1. Preliminary concepts and results. Because the notation and concepts required in Part II are more complex than those required for this part, we introduce here only what is needed here.

Let X be a finite group. The subgroup $L(X)$ is discussed in [33] and [34] (cf. also Section 3, Part II). In particular, Theorem 3.1 of [33] asserts that

(1.1) $$L(C_X(T)) \subseteq L(X)$$

for any 2-subgroup T of X. Then $L(C_X(T)) = L(C_{L(X)}(T))$, and this implies the following proposition.

PROPOSITION 1.1. *Let* $L_1 \in \mathcal{L}(C_X(t))$ *where* $t \in \mathcal{I}(X)$. *Then there exists* $L_2 \in \mathcal{L}(X)$ *such that* $L_1 = L(C_{L_2L_2^t}(t))$.

Let $\bar{L}_i = L_i/O(L_i)$ and denote images by bars for $i = 1,2$. When $L_2 \neq L_2^t$, the mapping $x \mapsto \bar{x}\bar{x}^t$ induces an epimorphism $L_2 \to L_1/O(L_1)$ whose kernel is contained in $Z^*(L_2)$. Suppose that $L_i \in \mathcal{L}(C_X(t_i))$ where $t_i \in \mathcal{I}(X)$, $i = 1,2$, and that $[t_1,t_2] = 1$ with $[L_1,t_2] \subseteq O(L_1)$; set $L_{12} = L(C_{L_1}(t_2))$ so that $L_1 = L_{12}O(L_1)$. Then write

(1.2) $$L_1 \hookrightarrow L_2 \quad \text{or} \quad L_1 \downarrow L_2L_2^t$$

according as $L_2 = L_2^{t_1}$ or $L_2 \neq L_2^{t_1}$ when by Proposition 1.1, $L_{12} = L(C_{L_2L_2^{t_1}}(t_1))$. Then L_1 is a homomorphic image of L_2 when $L_2^{t_1} \neq L_2$.

Let $L_i \in \mathcal{L}(C_X(t_i))$ where $t_i \in \mathcal{I}(X)$, $i = 1,2$, $[t_1,t_2] = 1$, and $[L_1/O(L_1),t_2] = [L_2/O(L_2),t_1] = 1$. Set $L_{12} = L(C_{L_1}(t_2))$ and $L_{21} = L(C_{L_2}(t_1))$. Suppose that $L_1 \hookrightarrow L_2$ and $L_2 \hookrightarrow L_1$. Then $L_1 = L_{21}O(L_1)$. Then $L_1 \cap L_2 = L_{21}(O(L_1) \cap L_2)$ since $L_{21} \subseteq L_1 \cap L_2$. But $L_{21} \lhd C_{L_2}(t_1)$. So $L_{21} = L(L_1 \cap L_2)$. Likewise $L_{12} = L(L_1 \cap L_2)$. Hence $L_{12} = L_{21}$. In this case, we say that L_1 and L_2 are *adjacent*. We say that an element L of $\mathcal{L}(\mathcal{I}(X))$ is *equivalent* to an element M if there exists a sequence $L = L_1,L_2,\ldots,L_n$ with $L_n = M$ such that L_i is adjacent to L_{i+1} for $i = 1,2,\ldots,n-1$, and $L_i \in \mathcal{L}(\mathcal{I}(X))$.

Let $\mathcal{L}$ be a subset of $\mathcal{L}(\mathcal{I}(X))$. We say that an element M of $\mathcal{L}$ *dominates* an element L of $\mathcal{L}$ in $\mathcal{L}$ if there exists a sequence of elements of $\mathcal{L}$

(1.3) $$L = L_1,L_2,\ldots,L_n = M$$

with $L_i \in \mathcal{L}(C_X(t_i)) \cap \mathcal{L}$, $t_i \in \mathcal{I}(X)$, $i = 1,2,\ldots,n$, and $[t_i,t_{i+1}] = 1$ and with either $L_i \hookrightarrow L_{i+1}$ or $L_i \downarrow L_{i+1}L_{i+1}^{t_i}$, $i = 1,2,\ldots,n-1$. We say that $L \in \mathcal{L}$ is *maximal with respect to domination in* $\mathcal{L}$ if for every sequence (1.3) of dominating elements obtained from $\mathcal{L}$, L_i is either adjacent to L_{i+1} or $L_i \downarrow L_{i+1}L_{i+1}^{t_i}$, $i = 1,2,\dots,n-1$. If $\mathcal{L}$ consists of quasisimple elements, every element of $\mathcal{L}$ of maximal order is maximal with respect to domination in $\mathcal{L}$. When $\mathcal{L} = \mathcal{L}(\mathcal{I}(X))$, M is said to *dominate* L if it dominates L in $\mathcal{L}$. A subset $\mathcal{L}$ of $\mathcal{L}(\mathcal{I}(X))$ is said to be *closed with respect to domination* if the elements of $\mathcal{L}$ are dominated in $\mathcal{L}(\mathcal{I}(X))$ only by elements of $\mathcal{L}$. If no reference to the set from which the dominating elements are obtained, we mean domination with respect to $\mathcal{L}(\mathcal{I}(G))$.

Michael Aschbacher has defined a *standard subgroup* L of a group X to be a quasisimple subgroup such that $C_X(L)$ is tightly embedded in X, $N_X(C_X(L)) = N_X(L)$, and $[L,L^x] \neq 1$ for all $x \in X$; a subgroup K of X is defined to be *tightly embedded* <u>in</u> X if $|K \cap K^x|$ is odd for all $x \in X \setminus N_X(K)$. A standard subgroup of $C_G(t)$ for some $t \in \mathcal{I}(G)$ is said to be a *standard element* of $\mathcal{L}(\mathcal{I}(X))$. Aschbacher's Component Theorem [2] leads to the following result.

PROPOSITION 1.2. *Let G be a finite group with $L(G)$ simple and $C_G(L(G)) = 1$. Let $\mathcal{L}$ be a subset of quasisimple elements of $\mathcal{L}(\mathcal{I}(G))$ which is closed with respect to domination such that no element of $\mathcal{L}$ is intrinsic of type SL(2,q), q odd, or $\hat{A}_7$.*

(i) An element of $\mathcal{L}$ which is maximal with respect to domination is standard.

(ii) An element $L \in \mathcal{L}$ is standard if and only if $L \in \mathcal{L}(C_G(t))$ for all $t \in \mathcal{L}(C_G(L)$.

<u>Proof</u>. (i) Let L be an element of $\mathcal{L}$ which is maximal with respect to domination. Then L is maximal in the ordering on $\mathcal{L}(\mathcal{I}(G))$ given in [2]. Because L(G) is not an element of $\mathcal{L}$ and because $\mathcal{L}(\mathcal{I}(G))$ contains no intrinsic element of type SL(2,q), q odd, or type A_7, Condition (1) of Theorem 1 of [2] holds. We remark that it suffices to assume that $\mathcal{L}$ consists of quasisimple elements rather than $\mathcal{L}(\mathcal{I}(G))$ consists of quasisimple elements by the argument of [2].

Now (1) of Theorem 1 of [2] implies that L belongs to $\mathcal{L}(C_G(t))$ for all $t \in \mathcal{I}(K)$ where $K = C_G(L)$ and that $[L,L^x] \neq 1$ for $x \in G$. It remains to show that K is tightly embedded in G. Suppose that $|K \cap K^x|$ is even for some $x \in G$.. Let $t \in \mathcal{I}(K \cap K^x)$. Then L and L^x belong to $\mathcal{L}(C_G(t))$. But by (1) of Theorem 1 of [2], $L = L^x$; so $x \in N_G(L)$.

(ii) When $L \in \mathcal{L}(C_G(t))$ for all $t \in \mathcal{I}(C_G(L))$, L is clearly maximal with respect to domination. Hence L is standard in $\mathcal{L}(\mathcal{I}(G))$ by (i). So suppose that L is a standard element of $\mathcal{L}(\mathcal{I}(G))$ and that $t \in K$ where

$K = C_G(L)$. Let $x \in C_G(t)$. Then $|K \cap K^x|$ is even. So $x \in N_G(L)$. Hence $L \triangleleft C_G(t)$. This proves the proposition.

As a first step in determining the structure of $C_G(L)$ when L is a standard element of $\mathcal{L}(\mathcal{J}(G))$ for some group G, we note a direct consequence of the results of Aschbacher and Seitz [7], which applies to the cases we will consider.

Let G be a group with $L(G)$ simple and $O_{2',2}(G) = 1$. Let L be a standard subgroup of type Chev(p), p odd, or type A_n. Suppose that $C_G(L)$ has 2-rank at least 2. Then $C_G(L)$ is a four group and $L \cong PSL(2,5)$ with $G \cong J_2$ or M_{12} or $L \cong A_n$ and $G \cong A_{n+4}$.

Thus in studying standard subgroups L, we are reduced to the case where an S_2-subgroup of $C_G(L)$ has 2-rank 1. If this subgroup is generalized quaternion, [3] applies to give a characterization of G. To handle the case where $C_G(L)$ has a cyclic S_2-subgroup of order at least 4, we have the following lemma. It is related to a similar result of John Thompson which is stated for a set $\mathcal{T}$ of disjoint nonabelian subgroups of a p-group which is closed under conjugation. Of course, only the case $p = 2$ has applications here.

LEMMA 1.3. *Let $\mathcal{T}$ be a family of nonelementary abelian subgroups of a p-group P. Assume that $T_1 \cap T_2 = 1$ and that $T_1^x \in \mathcal{T}$ for all distinct $T_1, T_2 \in \mathcal{T}$ and $x \in P$. Then the elements of $\mathcal{T}$ commute.*

Proof. We argue by induction on $|P|$. Let T_1 and T_2 be distinct elements of $\mathcal{T}$. Then

$$T_1^{T_2} = T_1[T_1, T_2] \quad \text{and} \quad T_2^{T_1} = T_2[T_2, T_1].$$

Thus by induction $P = \langle T_1, T_2 \rangle = T_1T_2[T_1, T_2] = T_1T_2$ and $\mathcal{T} = T_1^P \cup T_2^P$. Then $\langle T_i^P \rangle \neq P$, $i = 1,2$. So by induction T_1^P and T_2^P are abelian. Hence $[P,P] = [T_1, T_2] \subseteq Z(P)$. Consequently,

$$(1.4) \qquad [T_1^p, T_2] = [T_1, T_2^p] = [T_1, T_2]^p.$$

Suppose, say, $T_1^{p^2} \neq 1$. Then T_1^p is nonelementary. By induction, the lemma applies to the pair $(\langle T_1^p, T_2 \rangle, (T_1^p)^P \cup T_2^P)$. So $[T_1^p, T_2] = 1$. Hence $[T_1, T_2] \subseteq T_1$ since $T_1^p \subseteq T_1^t$ for all $t \in T_2$. By hypothesis, $T_2^p \neq 1$, and by (1.4), $[T_1, T_2^p] = [T_1^p, T_2] = 1$. Thus $[T_1, T_2] \subseteq T_1 \cap T_2 = 1$ as desired.

Thus we conclude that $T_1^{p^2} = T_2^{p^2} = 1$. Then

(1.5) $$[T_1^p, T_2^p] = [T_1, T_2^{p^2}] = 1.$$

Hence $[T_1, T_2^p] \subseteq T_1$ and $[T_1^p, T_2] \subseteq T_2$. By (1.4), $[T_1, T_2^p] = [T_1^p, T_2]$. As $T_1 \cap T_2 = 1$, $[T_1, T_2^p] = [T_1^p, T_2] = 1$. Again this implies $[T_1, T_2] = 1$ as desired.

COROLLARY 1.4. *Let H be a group with a tightly embedded subgroup K which has a nonelementary abelian S_2-subgroup T. Then the weak closure W of T in an S_2-subgroup of H is abelian. Furthermore, if $T^h \subseteq W$ for some $h \in H$, then $T^h = T^n$ for some $n \in N_H(W)$.*

Proof. Apply Lemma 1.3 with P be an S_2-subgroup of H and $\mathcal{T} = T^P$ in order to obtain the first statement. So suppose $T^h \subseteq W$ for some $h \in H$. Then $C_H(T) \supseteq \langle W, W^{h^{-1}} \rangle$. Hence there exists $y \in C_H(T)$ such that $W^{h^{-1}y} \subseteq P$. Then $h^{-1}y \in N_H(W)$. So we may set $n = h^{-1}y$ to obtain the last statement.

The *sectional 2-rank* $r(H)$ of a group H has been defined as

(1.6) $$r(H) = \max\{m(X/\Phi(X)) \mid X \text{ a 2-subgroup of } H\}.$$

LEMMA 1.5. *Let H be a finite group with normal subgroup K. Then*

(1.7) $$r(H) \leq r(H/K) + r(K).$$

Proof. It suffices to assume that H is a 2-group with $m(H/\Phi(H)) = r(H)$ since when $H_o \subseteq H$, $r(H_o/(H_o \cap K)) \leq r(H/K)$ and $r(H_o \cap K) \leq r(K)$. Then

$$\begin{aligned} m(H/\Phi(H)) &= m(H/K\Phi(H)) + m(K\Phi(H)/\Phi(H)) \\ &= m(H/K)/K\Phi(H)/K)) + m(K/K \cap \Phi(H)) \\ &\leq r(H/K) + r(K). \end{aligned}$$

In dealing with subgroups of $P\Gamma L(2,q) = \mathrm{Aut}\, PSL(2,q)$, we need some notation. There exists $u \in \mathcal{I}(P\Gamma L(2,q))$ which acts on $PSL(2,q))$ as a diagonal automorphism, and $PGL(2,q) = PSL(2,q)\langle u\rangle$. When $q = r^2$, there exists $t \in \mathcal{I}(P\Gamma L(2,q))$ such that t acts as a field automorphism on $PGL(2,q)$ and $C_{PSL(2,q)}(t) \cong PGL(2,r)$. In this case, choose u and t to belong to the same S_2-subgroup S of $P\Gamma L(2,q)$. Set $S_o = C_S(t)$. Then S_o is the dihedral S_2-subgroup $S \cap PSL(2,q)$ of $PGL(2,r)$, and $Z(S_o) = \langle [t,u] \rangle$. Set $PAL(2,q) = PSL(2,q)\langle t,u\rangle$ and $P\Sigma L(2,q) = PSL(2,q)\langle t\rangle$. A subgroup of $P\Gamma L(2,q)$ containing $P\Sigma L(2,q)$ but not $PGL(2,q)$ will be said to have *$P\Sigma L(2,q)$-type*. If it contains $PAL(2,q)$, it will be said to have *PAL(2,q)-type*.

Finally we take note of some definitions used throughout this paper. We have defined a quasisimple group to be a perfect group L such that $L/Z(L)$ is simple. Define a *2-quasisimple* group to be a group L such that $L/O(L)$ is quasisimple. Thus all 2-components are 2-quasisimple. A *semisimple* group is a direct product of nonabelian simple groups. A *quasi-semisimple* group is a perfect group L such that $L/Z(L)$ is semisimple. A *2-quasisemisimple* group is a group L such that $L/O(L)$ is quasisemi-simple. Thus for any group X, $L(X)$ is 2-quasisemisimple when $L(X) \neq 1$.

2. 2-Components of type PSL(2,q) and A_7. In this section, we extend the argument of [66] in order to obtain a characterization of groups in which $\mathcal{L}(\mathcal{I}(G))$ contains elements of type PSL(2,q), q odd, or type A_7. The properties of these groups which we use are well-known. We refer the reader to [31] and to Propositions 2.2 and 2.4 of Part II for more details. The characterization we need is given in Theorem 2.2 which is obtained from the following proposition, which is also used in Part II. Set $\mathcal{L}_2(q) = \{PSL(2,q)\}$ and set $\mathcal{L}_2(q)^* = \mathcal{L}_2(q) \cup \{A_7\}$. For $t \in \mathcal{I}(G)$, designate by $\mathcal{L}(C_G(t);\mathcal{L}_2(q))$ and $\mathcal{L}(C_G(t);\mathcal{L}_2(q)^*)$ the subsets of $\mathcal{L}(C_G(t))$ consisting of elements of type $\mathcal{L}_2(q)$ and $\mathcal{L}_2(q)^*$ respectively, and set

$$\mathcal{L}(\mathcal{I}(G);\mathcal{L}_2(q)^*) = \cup\{\mathcal{L}(C_G(t);\mathcal{L}_2(q)^*) \mid t \in \mathcal{I}(G)\}.$$

THEOREM 2.1. *Let G be a group with $L(G)$ simple and $C_G(L(G)) = 1$. Assume that $m(C_G(L_t\langle t\rangle/O(L_t)) = 1$ whenever $L_t \in \mathcal{L}(C_G(t));\mathcal{L}_2(q)^*)$ for some $t \in \mathcal{I}(G)$. Furthermore, when $L_t \in \mathcal{L}(C_G(t);\mathcal{L}_2(q))$ and $L(C_{L_t}(s)) \neq 1$ for some $s \in \mathcal{I}(C_G(t))$, set $L_s = [L(C_G(s)),L(C_{L_t}(s))]$ and assume that L_s is 2-quasisimple of type PSL(2,q) or PSL(2,r) where $r^2 = q$. Then $L(G)$ has type $\mathscr{C}$ when $\mathcal{L}(\mathcal{I}(G));\mathcal{L}_2(q)^*) \neq \emptyset$.*

Proof. Take G to be a minmal counterexample, and let L_t be an element of $\mathcal{L}(C_G(t));\mathcal{L}_2(q)^*)$ for $t \in \mathcal{I}(G)$. Then $G = L(G)\langle t\rangle$. Let U_t be an S_2-subgroup of $C_G(L_t/O(L_t))$. By hypothesis, $m(U_t) = 1$. By virtue of Aschbacher's characterization of Chevalley groups [3], U_t is cyclic. Let $U_t = \langle u_t\rangle$. Let S_t be an S_2-subgroup of $C_G(t)$ containing U_t. Set $C_t = C_G(t)$ and $\tilde{C}_t = C_t/U_tO(C_t)$. Denote images in $\tilde{C}_t$ by tildes. Then C_t is isomorphic to a subgroup of Aut L_t. Let M_t be maximal among the subgroups of C_t which contain a dihedral S_2-subgroup. Then M_t has type PGL(2,q), PSL(2,q), or A_7, and $M_t \lhd C_t$. Set $S_{M_t} = S_t \cap M_t$. Therefore $U_t \lhd S_{M_t}$ and S_{M_t}/U_t is dihedral. Set $S_{L_t} = S_t \cap L_t$. Then S_{L_t} is an S_2-subgroup of L_t and $S_{M_t} = S_{L_t}U_t\langle d_t\rangle$ where d_t acts on $L_t/O(L_t)$ as a diagonal outer automorphism when $L_t/O(L_t) \cong PSL(2,q)$ and $d_t = 1$ when $L_t/O(L_t) \cong A_7$. Then $S_t = S_{M_t}\langle s_o\rangle$ where s_o acts on $L_t/O(L_t)$ as a field automorphism or as a transposition according as $L_t/O(L_t) \cong PSL(2,q)$ or A_7. When $S_t \neq S_{M_t}$, $|C_{S_{L_t}\langle s_o\rangle}(S_{L_t})S_{L_t}:S_{L_t}| = 2$. Thus we may take s_o so that $C_{\langle s_o\rangle}(S_{L_t}) = \langle s\rangle U_t$ where s induces an involutory automorphism on $L_t/O(L_t)$. Then $s^2 \in U_t$. Without loss of generality, we may suppose that s_o has minimal order among the elements in the coset s_oU_t. Set $\langle z\rangle = Z(S_t) \cap S_{L_t}$ when $|S_{M_t}| \geq 8$. If $S_t \supset S_{M_t}$, then $|S_{L_t}| \geq 8$ since q is a square when $L_t/O(L_t) \cong PSL(2,q)$; consequently $Z(S_{L_t}) = \langle z\rangle$. If $S_t \supset S_{M_t} \supset S_{L_t}U_t$, then $[s,d_t] = z$ or tz by the structure of $P\Gamma L(2,q)$.

The first reduction in the proof is to show that

(2.1) $$U_t = \langle t\rangle.$$

Suppose that this is not the case. Since $G = L(G)\langle t\rangle = L(G)U_t$, $G = L(G)$. Let S be an S_2-subgroup of G containing S_t. Let W be the weak closure of U_t in S. Because U_t is cyclic, $C_G(L_t\langle t\rangle/O(L_t))$ is tightly embedded in G, and U_t is normal in every 2-subgroup of $C_G(t)$ which contains it. Corollary 1.4 then implies that W is abelian. Therefore $W \subseteq C_S(t) = S_t$. By the result of Gorenstein and Harada [29], $r(G) > 4$ since $G = L(G)$ is a simple group which does not have type $\mathscr{C}$. Because $r(S_t) \leq r(S_{M_t}/U_t) + r(U_t) + 1 \leq 4$ by virtue of Lemma 1.5, $r(S) > r(S_t)$. Hence $S \neq S_t$ and $\langle t\rangle$ is not normal in S. Therefore $W \neq U_t$ and $W \supseteq U_t^x \neq U_t$ for $x \in S \setminus S_t$.

With this choice of x, we now argue that

(2.2) $$t^x = z.$$

Indeed, because $C_{S_t}(S_{L_t}) = U_t\langle s\rangle \not\subseteq U_t^x$ and because both S_{L_t} and U_t^x are normal in S_t,

(2.3) $$1 \neq [S_{L_t}, U] \subseteq S_{L_t} \cap U_t^x.$$

Thus $t^x \in S_{L_t}$. This implies that the elements of U_t do not induce field automorphisms on L_t. Thus they induce inner automorphisms, and t^x is a square in $C_{S_t}(S_{L_t}) = S_{L_t}U_t\langle s\rangle$. This implies (2.2). Also it follows that $|S_{L_t}| \geq 8$ inasmuch as $|U_t| \geq 4$.

Now (2.2) also implies that $t^S = \{t, z\}$. So $|S:S_t| = 2$. We now argue that $r(S) = 4$ by determining the structure of S_t. Let $R = U_tS_{L_t}\langle s\rangle$ and set $T = U_t\langle s\rangle$. Let a be an involution in S_{L_t} distinct from z, and set $S_a = C_{S_t}(a)$. Then $S_a = \langle a,z\rangle \times T$. Let $b = z^x$, and set $S_b = C_{S_t}(b)$. then $S_b = \langle b,t\rangle \times T^x$. As $t \notin \Phi(S_b)$, $S_b \cap U_t = \langle t\rangle$. Thus b inverts U_t. But $z \in T^x$; so $z \in \Phi(S_b)$. This means either $[b, S_{L_t}] = 1$ or b induces a field automorphism on L_t which centralizes S_{L_t}. Thus $S_{L_t}^x = \langle a^x \mid a \in \mathscr{I}(S_{L_t}), a \neq z\rangle \subseteq C_{S_t}(S_{L_t}) = U_t\langle s,z\rangle$. Replacing s by sz if necessary, we may assume that $S_{L_t}^x$ is a dihedral subgroup of T. As $T = U_t\langle s\rangle$, s must invert U_t. Then T is a dihedral