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Nobuaki Obata

White Noise Calculus and Fock Space

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Introduction

The white noise calculus (or analysis) was launched out by Hida [1] in 1975 with his lecture notes on generalized Brownian functionals. This new approach toward an infinite dimensional analysis was deeply motivated by Lévy [1] who considerably developed functional analysis on $L^2(0,1)$ and actually analysis of Brownian functionals. The root of white noise calculus is to switch a functional of Brownian motion $f(B(t); t \in \mathbb{R})$ with one of white noise $\phi(\dot{B}(t); t \in \mathbb{R})$, where $\dot{B}(t)$ is a time derivative of a Brownian motion $B(t)$. Although each Brownian path $B(t)$ is not smooth enough, $\dot{B}(t)$ is thought of as a generalized stochastic process and ϕ is realized as a generalized white noise functional in our language. We may thereby regard $\{\dot{B}(t)\}$ as a collection of infinitely many independent random variables and hence a coordinate system of an infinite dimensional space.

The mathematical framework of the white noise calculus is based upon an infinite dimensional analogue of the Schwartz distribution theory, where the role of the Lebesgue measure on $\mathbb{R}^n$ is played by the Gaussian measure μ on the dual of a certain nuclear space E. In the classical case where $\dot{B}(t)$ is formulated, we take $E = \mathcal{S}(\mathbb{R})$ and the Gaussian measure μ on E^* defined by the characteristic functional:

$$\exp\left(-\frac{|\xi|^2}{2}\right) = \int_{E^*} e^{i\langle x,\xi\rangle}\,\mu(dx), \qquad \xi \in E,$$

where $|\xi|$ is the usual L^2-norm of ξ. Then the Hilbert space $(L^2) = L^2(E^*, \mu)$ is canonically isomorphic to the (Boson) Fock space over $L^2(\mathbb{R})$ through the Wiener-Itô-Segal isomorphism and links the test and generalized functionals. Namely, in a specific way (called *standard construction*) we construct a nuclear Fréchet space (E) densely and continuously imbedded in (L^2), and by duality we obtain a Gelfand triple:

$$(E) \subset (L^2) = L^2(E^*, \mu) \subset (E)^*.$$

An element in (E) is a *test white noise functional* and hence an element in $(E)^*$ is a *generalized white noise functional.* The above picture is easily understood as a direct analogy of $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ which is a frame of the Schwartz distribution theory. Then, $\dot{B}(t) = x(t)$, $x \in E^*$, gives us a realization of the time derivative of a Brownian motion and, in fact, $x \mapsto x(t)$ becomes a generalized white noise functional for each fixed $t \in \mathbb{R}$.

In our actual discussion we do not restrict ourselves to the case of $E = \mathcal{S}(\mathbb{R})$ and $H = L^2(\mathbb{R})$ but deal with a more general function space on a topological space T. Typically T is a time-parameter space and is often taken to be a more general topological space where quantum field theory may be formulated. Again $\{x(t); t \in T\}$ is considered as a coordinate system of E^* intuitively. In fact, within our framework we may discuss not only functionals in $\{x(t); t \in T\}$ but also operators derived from this coordinate system. The coordinate differential operator $\partial_t = \partial/\partial x(t)$ is well defined as a continuous derivation on (E). We have also multiplication operators by coordinate functions $x(t)$, which are, in fact, operators from (E) into $(E)^*$. Furthermore, ∂_t^* is a continuous linear operator on $(E)^*$. The operators ∂_t and ∂_t^* correspond respectively to an *annihilation operator* and a *creation operator* at a point $t \in T$ and they satisfy the so-called canonical commutation relation in a generalized sense. The above mentioned formulation was consolidated in the basic works of Kubo and Takenaka [1]-[4] and has been widely accepted.

The main purpose of these lecture notes is to develop operator theory on white noise functionals as well as to offer a systematic introduction to white noise calculus. From that point of view it is most remarkable that we are free from smeared creation and annihilation operators. In other words, ∂_t and ∂_t^* are not operator-valued distributions but usual operators for themselves. This leads us to an integral kernel operator:

$$\Xi_{l,m}(\kappa) = \int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m)\partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

where κ is a *distribution* in $l+m$ variables. The use of distributions as integral kernels allows us to discuss a large class of operators on Fock space. In fact, *every* continuous operator Ξ from (E) into $(E)^*$ admits a unique decomposition into a sum of integral kernel operators:

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E),$$

where the series converges in $(E)^*$. Moreover, if Ξ is a continuous operator from (E) into itself, the series converges in (E). In the process we investigate precise norm estimates of such operators and obtain a method of reconstructing an operator from its symbol. The above expression is called *Fock expansion* and will play a key role in our discussion.

Although applications of white noise calculus are widely spreading, the present lecture notes are strongly oriented toward *infinite dimensional harmonic analysis*. The clue to go on is found in the following three topics: (i) infinite dimensional rotation group; (ii) Laplacians; (iii) Fourier transform. Being almost as new as the white noise calculus, they have been so far discussed somehow separately. Since the very beginning of the development Hida has emphasized the importance of the infinite dimensional rotation group $O(E; H)$, that is, the group of automorphisms of the Gelfand triple $E \subset H \subset E^*$. In fact, it played an interesting role in the study of symmetry of Brownian motion and Gaussian random fields. There are various candidates for infinite dimensional Laplacians which possess some typical properties of a finite dimensional Laplacian. So far the Gross Laplacian Δ_G, the number operator N and the Lévy Laplacian Δ_L have been found to be important in white noise calculus, though the Lévy Laplacian is not discussed in these lecture notes. As for Fourier transform, among some candidates that have been discussed Kuo's Fourier transform (simply called the *Fourier transform* hereafter) has been found well suited to white noise calculus.

In these lecture notes the above listed three subjects are treated systematically by means of our operator calculus and are found closely related to each other. For example, the Gross Laplacian Δ_G and the number operator N are characterized by their rotation-invariance. The Fourier transform intertwines the coordinate differential operators and coordinate multiplication oprators just as in the case of finite dimension and, this property actually characterizes the Fourier transform. Moreover, the Fourier transform is imbedded in a one-parameter transformation group of the generalized white noise functionals (called the *Fourier-Mehler transform*) and its infinitesimal generator is expressed with Δ_G and N. These results would suggest a fruitful application of white noise calculus to infinite dimensional harmonic analysis. It is also expected that our operator calculus is useful in some problems in quantum field theory and quantum probability.

As is well known, a lot of efforts to develop distribution theories on an infinite dimensional space equipped with Gaussian measure have been made by many authors. In fact, mathematical study of Brownian motion or equivalently of white noise is now one of the most important and vital fields of mathematics toward infinite dimensional analysis.

Since the main purpose is to develop an operator theory on white noise functionals, the present lecture notes are mostly based on a functional analytic point of view rather than probability theory or stochastic analysis. In Chapter 1 we survey some fundamentals in functional analysis required during the main discussion and propose a notion of a standard countably Hilbert space which makes the discussion clearer. The purpose of Chapter 2 is to establish the well-known Wiener-Itô-Segal isomorphism between $L^2(E^*, \mu)$ and the Fock space. Chapter 3 is devoted to a study of generalized white noise functionals. In Chapter 4 we develop an operator theory on white noise functionals, or equivalently on Fock space, in terms of Hida's differential operators ∂_t and their duals ∂_t^*. By means of the operator theory we discuss in Chapter 5 a few topics toward harmonic analysis including first order differential operators, the number operator, the Gross Laplacians, infinite dimensional rotation group, Fourier transform and certain one-parameter transformation groups. Chapter 6 is added after finishing the first draft of these lecture notes. We discuss integral-sum kernel operators, the finite dimensional calculus derived from our framework and a generalization to cover vector-valued white noise functionals. These topics are expected to open a new area in infinite dimensional analysis.

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My basic references have been among others the works of Hida-Potthoff [1], Kuo [7], [9], Lee [3] and Yan [4] which I appreciated highly. The readers are recommended to consult the recently published monograph Hida-Kuo-Potthoff-Streit [1] which contains different topics and various applications. It will complement our discussion certainly.

Finally I am extremely grateful to Professor T. Hida for his constant encouragement. He initiated the white noise calculus around 1975 and remains always a fount of knowledge.

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Chapter 1

Prerequisites

1.1 Locally convex spaces in general

We first agree that all vector spaces under consideration are over the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. A topological vector space $\mathfrak{X}$ is called *locally convex* if the topology of $\mathfrak{X}$ is Hausdorff and given by a family of seminorms $\{\|\cdot\|_\alpha\}_{\alpha\in A}$. Then the seminorms are called *defining seminorms* for $\mathfrak{X}$. Without changing the topology we may choose a *directed* family of defining seminorms for $\mathfrak{X}$, which means that for any $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\|\xi\|_\alpha \le \|\xi\|_\gamma$ and $\|\xi\|_\beta \le \|\xi\|_\gamma$ for all $\xi \in \mathfrak{X}$. In that case A becomes a directed set naturally. Unless otherwise stated, $\mathfrak{X} \cong \mathfrak{Y}$ means that two locally convex spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic as topological vector spaces.

For a systematic study of locally convex spaces we introduce general notion of projective and inductive systems and their limits. Let $\{\mathfrak{X}_\alpha\}_{\alpha\in A}$ be a family of locally convex spaces. The *direct product*

$$\prod_{\alpha\in A} \mathfrak{X}_\alpha = \{(\xi_\alpha)_{\alpha\in A}\,;\, \xi_\alpha \in \mathfrak{X}_\alpha\}$$

is always equipped with the weakest locally convex topology such that the canonical projection $p_\beta : \prod_{\alpha\in A} \mathfrak{X}_\alpha \to \mathfrak{X}_\beta$ is continuous for all $\beta \in A$. The *direct sum*

$$\bigoplus_{\alpha\in A} \mathfrak{X}_\alpha = \left\{(\xi_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} \mathfrak{X}_\alpha\,;\, \xi_\alpha = 0 \text{ except finitely many } \alpha \in A\right\}$$

is equipped with the strongest locally convex topology such that the canonical injection $i_\beta : \mathfrak{X}_\beta \to \bigoplus_{\alpha\in A} \mathfrak{X}_\alpha$ is continuous for all $\beta \in A$.

Let $\{\mathfrak{X}_\alpha\}_{\alpha\in A}$ be a family of locally convex spaces, with A being a directed set. Suppose that we are given a continuous linear map $f_{\alpha,\beta} : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$ for any pair $\alpha, \beta \in A$ with $\alpha \le \beta$. Then $\{\mathfrak{X}_\alpha, f_{\alpha,\beta}\}$ is called a *projective system* of locally convex spaces if (i) $f_{\alpha,\alpha} = \text{id.}$; and (ii) $f_{\alpha,\gamma} = f_{\alpha,\beta} f_{\beta,\gamma}$ whenever $\alpha \le \beta \le \gamma$. Then

$$\operatorname*{proj\,lim}_{\alpha\in A} \mathfrak{X}_\alpha = \left\{(\xi_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} \mathfrak{X}_\alpha\,;\, f_{\alpha,\beta}(\xi_\beta) = \xi_\alpha \text{ whenever } \alpha \le \beta\right\}$$

with the relative topology induced from $\prod_{\alpha\in A} \mathfrak{X}_\alpha$ is called the *projective limit* of $\{\mathfrak{X}_\alpha, f_{\alpha,\beta}\}$. So far as the projective limit is under consideration, it suffices to consider a *reduced* projective system; namely, every canonical projection $p_\beta : \operatorname{proj\,lim}_{\alpha\in A} \mathfrak{X}_\alpha \to \mathfrak{X}_\beta$ has a dense image.

We now introduce a dual object. Let $\{\mathfrak{X}_\alpha\}_{\alpha\in A}$ be the same as above and suppose that we are given a continuous linear operator $g_{\alpha,\beta} : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$ for all pair $\alpha, \beta \in A$ with $\alpha \ge \beta$. Then $\{\mathfrak{X}_\alpha, g_{\alpha,\beta}\}$ is called an *inductive system* of locally convex spaces if (i) $g_{\alpha,\alpha} = \text{id.}$; and (ii) $g_{\alpha,\gamma} = g_{\alpha,\beta}g_{\beta,\gamma}$ whenever $\alpha \ge \beta \ge \gamma$. Consider $\sum_{\alpha\ge\beta} \operatorname{Ran}(i_\beta - i_\alpha g_{\alpha,\beta})$ which is a subspace of $\bigoplus_{\alpha\in A} \mathfrak{X}_\alpha$ generated by the ranges of the linear maps $i_\beta - i_\alpha g_{\alpha,\beta}$, where α, β run over all pairs with $\alpha \ge \beta$. If $\sum_{\alpha\ge\beta} \operatorname{Ran}(i_\beta - i_\alpha g_{\alpha,\beta})$ is closed, the quotient space

$$\operatorname*{ind\,lim}_{\alpha\in A} \mathfrak{X}_\alpha = \bigoplus_{\alpha\in A} \mathfrak{X}_\alpha \Big/ \sum_{\alpha\ge\beta} \operatorname{Ran}(i_\beta - i_\alpha g_{\alpha,\beta})$$

equipped with the quotient topology is called the *inductive limit* of $\{\mathfrak{X}_\alpha, g_{\alpha,\beta}\}$.

If $\|\cdot\|$ is a seminorm on a vector space $\mathfrak{X}$, then $\mathfrak{N} = \{\xi \in \mathfrak{X}; \|\xi\| = 0\}$ becomes a subspace of $\mathfrak{X}$ and the quotient space $\mathfrak{X}/\mathfrak{N}$ admits a natural *norm* which is denoted by the same symbol. The completion of $\mathfrak{X}/\mathfrak{N}$ with respect to this norm $\|\cdot\|$ is called the *Banach space associated with the seminorm* $\|\cdot\|$. Now consider two seminorms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ satisfying $\|\xi\|_\alpha \le C\,\|\xi\|_\beta$, $\xi \in \mathfrak{X}$, for some $C \ge 0$. Note that $\mathfrak{N}_\beta = \{\xi \in \mathfrak{X}; \|\xi\|_\beta = 0\} \subset \mathfrak{N}_\alpha = \{\xi \in \mathfrak{X}; \|\xi\|_\alpha = 0\}$. Let $\mathfrak{X}_\alpha$ and $\mathfrak{X}_\beta$ be the Banach spaces associated with $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$, respectively. Then, the canonical map from $\mathfrak{X}/\mathfrak{N}_\beta$ onto $\mathfrak{X}/\mathfrak{N}_\alpha$ extends to a continuous linear map $f_{\alpha,\beta} : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$.

Proposition 1.1.1 *Let $\mathfrak{X}$ be a locally convex space with a directed family of defining seminorms $\{\|\cdot\|_\alpha\}_{\alpha\in A}$. Then, notations being as above, $\{\mathfrak{X}_\alpha, f_{\alpha,\beta}\}$ becomes a reduced projective system of Banach spaces. If in addition $\mathfrak{X}$ is complete, $\mathfrak{X} \cong \operatorname{proj\,lim}_{\alpha\in A} \mathfrak{X}_\alpha$.*

Let $\mathfrak{X}$ be a locally convex space with defining seminorms $\{\|\cdot\|_\alpha\}_{\alpha\in A}$. A subset $S \subset \mathfrak{X}$ is called *bounded* if $\sup_{\xi\in S} \|\xi\|_\alpha < \infty$ for all $\alpha \in A$. Let $\mathfrak{X}^*$ be the *dual space* of $\mathfrak{X}$, i.e., the space of continuous linear functionals on $\mathfrak{X}$ and we denote the canonical bilinear form on $\mathfrak{X}^* \times \mathfrak{X}$ by $\langle\cdot\,,\cdot\rangle$ or similar symbols. Unless otherwise stated, $\mathfrak{X}^*$ always carries the *strong dual topology* or the *topology of bounded convergence.* This topology is defined by the seminorms:

$$\|x\|_S = \sup_{\xi\in S} |\langle x\,,\xi\rangle|, \qquad x \in \mathfrak{X}^*,$$

where S runs over the bounded subsets of $\mathfrak{X}$. In that case $\mathfrak{X}^*$ is called the *strong dual space* as well. For a continuous linear operator T from a locally convex space $\mathfrak{X}$ into another $\mathfrak{Y}$ its adjoint T^* is defined by $\langle T^*y\,,\xi\rangle = \langle y\,,T\xi\rangle$, $y \in \mathfrak{Y}^*$, $\xi \in \mathfrak{X}$. Then T^* becomes a continuous linear operator from $\mathfrak{Y}^*$ into $\mathfrak{X}^*$.

In accord with Proposition 1.1.1 we can discuss the dual space of a locally convex space. We keep the notations there. Since the canonical map $p_\alpha : \mathfrak{X} \to \mathfrak{X}_\alpha$ has a dense image, its adjoint map $p_\alpha^* : \mathfrak{X}_\alpha^* \to \mathfrak{X}^*$ is injective and thereby $\mathfrak{X}_\alpha^*$ is regarded as a subspace of $\mathfrak{X}^*$. In that case $\mathfrak{X}_\alpha^*$ consists of linear functionals on $\mathfrak{X}$ which are continuous with respect to $\|\cdot\|_\alpha$. Therefore,

$$\mathfrak{X}^* = \bigcup_{\alpha\in A} \mathfrak{X}_\alpha^* \qquad \text{as vector spaces.}$$

Note also that $\mathfrak{X}_\alpha^* \subset \mathfrak{X}_\beta^*$ for $\alpha \leq \beta$. Namely, in a purely algebraic sense $\mathfrak{X}^*$ is the inductive limit of $\{\mathfrak{X}_\alpha^*\}$. In general, if $\{\mathfrak{X}_\alpha, f_{\alpha,\beta}\}$ is a projective system of locally convex spaces, $\{\mathfrak{X}_\alpha^*, f_{\alpha,\beta}^*\}$ becomes an inductive system of locally convex spaces in an obvious way. Unfortunately, with respect to the strong dual topology $\mathfrak{X}^* \cong \operatorname{ind\,lim}_{\alpha\in A} \mathfrak{X}_\alpha^*$ does not hold in general. While, it is true whenever $\mathfrak{X}^*$ and $\mathfrak{X}_\alpha^*$ are equipped with the Mackey topologies $\tau(\mathfrak{X}^*, \mathfrak{X})$ and $\tau(\mathfrak{X}_\alpha^*, \mathfrak{X}_\alpha)$, respectively. Instead of going into a detailed topological argument we note a class of locally convex spaces $\mathfrak{X}$ for which the strong dual topology coincides with the Mackey topology $\tau(\mathfrak{X}^*, \mathfrak{X})$.

A locally convex space is called *Fréchet* if it is metrizable and complete. Recall that a locally convex space is metrizable if and only if it admits a countable set of defining seminorms. A locally convex space $\mathfrak{X}$ is called *reflexive* if the canonical injection $\mathfrak{X} \to \mathfrak{X}^{**}$ is a topological isomorphism, where $\mathfrak{X}^{**}$ is the strong bidual of $\mathfrak{X}$. It is known that for a reflexive Fréchet space $\mathfrak{X}$ the strong dual topology on $\mathfrak{X}^*$ coincides with the Mackey topology $\tau(\mathfrak{X}^*, \mathfrak{X})$. Since the projective limit of a sequence of reflexive Fréchet spaces is again a reflexive Fréchet space, we have the following

Proposition 1.1.2 *Let $\{\mathfrak{X}_n\}_{n=1}^\infty$ be a reduced projective sequence of reflexive Fréchet spaces. Then,*

$$\left(\operatorname*{proj\,lim}_{n\to\infty} \mathfrak{X}_n\right)^* \cong \operatorname*{ind\,lim}_{n\to\infty} \mathfrak{X}_n^*,$$

where the strong dual topologies are taken into consideration.

We note another important property of a Fréchet space (in fact, a characteristic property of a barreled topological vector space).

Proposition 1.1.3 *Let $\mathfrak{X}$ be a Fréchet space. Then for a subset $S \subset \mathfrak{X}^*$ the following four properties are equivalent:*

(i) *S is equicontinuous, i.e., if $\{\|\cdot\|_\alpha\}_{\alpha\in A}$ is a directed family of defining seminorms for $\mathfrak{X}$, one may find $C \geq 0$ and $\alpha \in A$ such that $|\langle x, \xi\rangle| \leq C\,\|\xi\|_\alpha$ for all $\xi \in \mathfrak{X}$ and $x \in S$;*

(ii) *S is (strongly) bounded;*

(iii) *S is weakly bounded;*

(iv) *S is relatively weakly compact.*

1.2 Countably Hilbert spaces

A seminorm $\|\cdot\|$ on a vector space $\mathfrak{X}$ over $\mathbb{R}$ (resp. $\mathbb{C}$) is called *Hilbertian* if it is induced by some non-negative, symmetric bilinear (resp. hermitian sesquilinear) form $(\cdot,\cdot)$ on $\mathfrak{X} \times \mathfrak{X}$, namely if $\|\xi\|^2 = (\xi, \xi)$ for all $\xi \in \mathfrak{X}$. Here it is not assumed that $(\xi,\xi) = 0$ implies $\xi = 0$. We further agree that a hermitian sesquilinear form is linear on the right and antilinear on the left. The Banach space associated with a Hilbertian seminorm becomes a Hilbert space in an obvious way.

A complete locally convex space $\mathfrak{X}$ is called a *countably Hilbert space* or a *CH-space* for brevity if it admits a countable set of defining Hilbertian seminorms. We first note the following

Proposition 1.2.1 *Any CH-space is a projective limit of a reduced projective sequence of Hilbert spaces, and therefore, is a reflexive Fréchet space.*

Then, in view of Proposition 1.1.2 we have

Proposition 1.2.2 *Let $\mathfrak{X}$ be a CH-space and let $\{H_n, f_{m,n}\}$ be a reduced projective sequence of Hilbert spaces such that $\mathfrak{X} \cong \text{proj lim}_{n\to\infty} H_n$. Then, $\{H_n^*, f_{m,n}^*\}$ becomes an inductive sequence of Hilbert spaces and $\mathfrak{X}^* \cong \text{ind lim}_{n\to\infty} H_n^*$. Moreover, $\{H_n^*\}_{n=0}^{\infty}$ is regarded as an increasing family of subspaces of $\mathfrak{X}^*$ and $\mathfrak{X}^* = \bigcup_{n=0}^{\infty} H_n^*$ as vector spaces.*

We shall be mostly concerned with a particular class (or construction) of CH-spaces. The following general result will be useful.

Lemma 1.2.3 *Let A be a positive linear operator in a complex Hilbert space $\mathfrak{H}$. Then A is selfadjoint if and only if $(1 + A)\,\text{Dom}(A) = \mathfrak{H}$.*

We first consider the complex case. Let $\mathfrak{H}$ be a complex Hilbert space with norm $\|\cdot\|_0$ and let A be a selfadjoint operator in $\mathfrak{H}$ with (dense) domain $\text{Dom}(A) \subset \mathfrak{H}$. Suppose $\inf \text{Spec}(A) > 0$ and put

$$\rho = (\inf \text{Spec}(A))^{-1}. \tag{1.1}$$

According to the spectral theory we may define a (positive) selfadjoint operator A^p for all $p \in \mathbb{R}$ with (maximal) domain $\text{Dom}(A^p) \subset \mathfrak{H}$. For the moment suppose $p \geq 0$. Since $0 \notin \text{Spec}(A^p)$, by definition A^p admits a dense range and bounded inverse. In fact, we see from Lemma 1.2.3 that the range of A^p coincides with the whole $\mathfrak{H}$ because we have $\inf \text{Spec}(A^p) > 0$ by assumption. Therefore $(A^p)^{-1}$ is everywhere defined bounded operator on $\mathfrak{H}$. In that case $(A^p)^{-1} = A^{-p}$, $p \geq 0$, in particular, $\text{Dom}(A^{-p}) = \mathfrak{H}$, and

$$\|A^{-p}\|_{\text{op}} = \rho^p, \qquad p \geq 0. \tag{1.2}$$

Note also that

$$A^{-p}A^{-q} = A^{-(p+q)}, \qquad A^pA^q \subset A^{p+q}, \qquad p, q \geq 0.$$

It is known that the closure of A^pA^q coincides with A^{p+q}, $p, q \geq 0$.

We now introduce a family of Hilbertian norms:

$$\|\xi\|_p = \|A^p\xi\|_0, \qquad \xi \in \text{Dom}(A^p), \quad p \in \mathbb{R}. \tag{1.3}$$

Note that $\text{Dom}(A^q) \subset \text{Dom}(A^p)$ whenever $q \geq p \geq 0$. In fact, by (1.2) we have

$$\|\xi\|_p = \|A^{-(q-p)}A^q\xi\|_0 \leq \rho^{q-p}\|\xi\|_q, \qquad \xi \in \text{Dom}(A^q). \tag{1.4}$$

Equipped with the norm $\|\cdot\|_p$ the vector space $\text{Dom}(A^p)$ becomes a Hilbert space which we shall denote by $\mathfrak{E}_p$. Then, the inclusion $\text{Dom}(A^q) \subset \text{Dom}(A^p)$, $q \geq p \geq 0$, gives rise to a continuous injection $f_{p,q} : \mathfrak{E}_q \to \mathfrak{E}_p$ and $\{\mathfrak{E}_p, f_{p,q}\}$ becomes a projective system of Hilbert spaces. In this case we have also a chain of Hilbert spaces:

$$\cdots \subset \mathfrak{E}_q \subset \cdots \subset \mathfrak{E}_p \subset \cdots \subset \mathfrak{E}_0 = \mathfrak{H}, \qquad q \geq p \geq 0. \tag{1.5}$$

Lemma 1.2.4 *For any $q \geq p \geq 0$ the vector subspece $\mathfrak{E}_q$ is dense in $\mathfrak{E}_p$. In particular, the projective system $\{\mathfrak{E}_p, f_{p,q}\}$ is reduced.*

PROOF. Obviously, $\mathfrak{E}_{q-p} = \mathrm{Dom}(A^{q-p})$ is a dense subspace of $\mathfrak{H}$. Note also from definition (1.3) that A^p is an isometric isomorphism from $\mathfrak{E}_p$ onto $\mathfrak{H}$. Hence the inverse image of $\mathfrak{E}_{q-p}$ is a dense subspace of $\mathfrak{E}_p$. On the other hand,

$$(A^p)^{-1}(\mathfrak{E}_{q-p}) = A^{-p}(A^{-(q-p)}\mathfrak{H}) = A^{-p-(q-p)}\mathfrak{H} = A^{-q}\mathfrak{H} = \mathfrak{E}_q.$$

Consequently, $\mathfrak{E}_q$ is dense in $\mathfrak{E}_p$. qed

By virtue of (1.5) a subspace of $\mathfrak{H}$ defined by

$$\mathfrak{E} = \bigcap_{p\geq 0} \mathfrak{E}_p \tag{1.6}$$

becomes a CH-space equipped with the Hilbertian seminorms $\{\|\cdot\|_p\}_{p\geq 0}$. Obviously $\mathfrak{E}$ is isomorphic to the projective limit:

$$\mathfrak{E} \cong \operatorname*{proj\,lim}_{p\to\infty} \mathfrak{E}_p.$$

To sum up, given a pair $(\mathfrak{H}, A)$ where A is a selfadjoint operator in a complex Hilbert space $\mathfrak{H}$ with $\inf \mathrm{Spec}(A) > 0$, we have constructed a CH-space $\mathfrak{E}$.

Definition 1.2.5 The above $\mathfrak{E}$ is called a *standard CH-space constructed from* $(\mathfrak{H}, A)$.

As may be proved easily from definition, we have

Lemma 1.2.6 *Let A be a positive selfadjoint operator in $\mathfrak{H}$ with* $\inf \mathrm{Spec}(A) > 0$. *Then, the standard CH-spaces constructed from $(\mathfrak{H}, A)$ and $(\mathfrak{H}, A^s)$ are isomorphic for any $s > 0$.*

As for the dual space of $\mathfrak{E}$, it follows from Proposition 1.2.2 that

$$\mathfrak{E}^* \cong \operatorname*{ind\,lim}_{p\to\infty} \mathfrak{E}_p^* \qquad \text{and} \qquad \mathfrak{E}^* = \bigcup_{p\geq 0} \mathfrak{E}_p^* \qquad \text{as vector spaces.}$$

Recall that $\mathfrak{E}_p^*$ is identified with the space of linear functionals on $\mathfrak{E}$ which are continuous with respect to $\|\cdot\|_p$. With this identification the canonical bilinear forms on $\mathfrak{E}^* \times \mathfrak{E}$ and on $\mathfrak{E}_p^* \times \mathfrak{E}_p$, $p \geq 0$, are denoted by the same symbol $\langle \cdot\,, \cdot \rangle$.

By virtue of our particular construction $\mathfrak{E}_p^*$ and $\mathfrak{E}^*$ can be described more explicitly. We have already defined in (1.3) a Hilbertian norm $\|\cdot\|_{-p}$ on $\mathfrak{H}$ for $p \geq 0$. Let $\mathfrak{E}_{-p}$ be the completion of $\mathfrak{H}$ with respect to $\|\cdot\|_{-p}$. Then the identity map from $\mathfrak{H}$ onto itself extends to a continuous injection $f_{-q,-p} : \mathfrak{E}_{-p} \to \mathfrak{E}_{-q}$ whenever $q \geq p \geq 0$, and thereby $\{\mathfrak{E}_{-p}, f_{-q,-p}\}$ becomes an inductive system of Hilbert spaces. Furthermore, there is a natural inclusion relation:

$$\mathfrak{H} = \mathfrak{E}_0 \subset \cdots \subset \mathfrak{E}_{-p} \subset \cdots \subset \mathfrak{E}_{-q} \subset \cdots \qquad q \geq p \geq 0. \tag{1.7}$$

Recall that $A^{-p} : \mathfrak{H} \to \mathfrak{E}_p$ is a bounded operator and by definition it satisfies

$$\|A^{-p}\xi\|_0 = \|\xi\|_{-p}, \qquad \xi \in \mathfrak{H}.$$

Therefore A^{-p} extends to an isometric isomorphism $\widetilde{A^{-p}}$ from $\mathfrak{E}_{-p}$ onto $\mathfrak{H}$.

The inner product $(\cdot,\cdot)_p$ of $\mathfrak{E}_p$ is by definition given as

$$(\xi,\eta)_p = (A^p\xi, A^p\eta)_0, \qquad \xi,\eta \in \mathfrak{E}_p. \tag{1.8}$$

It follows from Riesz' theorem that there exists an isometric anti-isomorphism $R_p : \mathfrak{E}_p^* \to \mathfrak{E}_p$ such that

$$\langle x^*, \xi\rangle = (R_p(x^*), \xi)_p, \qquad x^* \in \mathfrak{E}_p^*, \quad \xi \in \mathfrak{E}_p.$$

On the other hand, in view of (1.8) we have

$$(R_p(x^*), \xi)_p = (A^p R_p(x^*), A^p\xi)_0 = \left(\widetilde{A^{-p}}(\widetilde{A^{-p}})^{-1} A^p R_p(x^*), A^p\xi\right)_0.$$

Thus, $h_p = (\widetilde{A^{-p}})^{-1} \circ A^p \circ R_p : \mathfrak{E}_p^* \longrightarrow \mathfrak{E}_{-p}$ becomes an isometric anti-isomorphism such that

$$\langle x^*, \xi\rangle = \left(\widetilde{A^{-p}} h_p(x^*), A^p\xi\right)_0, \qquad x^* \in \mathfrak{E}_p^*, \quad \xi \in \mathfrak{E}_p. \tag{1.9}$$

Moreover, using $A^{-(q-p)}\widetilde{A^{-p}} = \widetilde{A^{-q}} f_{-q,-p}$, one may prove easily that $f_{-q,-p} \circ h_p = h_q \circ f_{p,q}^*$ for any $0 \le p \le q$. Consequently,

Lemma 1.2.7 *Two inductive systems $\{\mathfrak{E}_p^*, f_{p,q}^*\}$ and $\{\mathfrak{E}_{-p}, f_{-q,-p}\}$ of Hilbert spaces are anti-isomorphic under the isometric anti-isomorphisms $\{h_p\}$. Therefore, $\mathfrak{E}^*$ is anti-isomorphic to* $\operatorname{ind\,lim}_{p\to\infty} \mathfrak{E}_{-p}$.

To be sure we shall give the inverse h_p^{-1} more explicitly. Let $x \in \mathfrak{E}_{-p}$, $p \ge 0$. Then $\widetilde{A^{-p}}x \in \mathfrak{H}$ and we obtain a continuous linear function $\xi \mapsto (\widetilde{A^{-p}}x, A^p\xi)_0$, $\xi \in \mathfrak{E}$. In fact,

$$|(\widetilde{A^{-p}}x, A^p\xi)_0| \le \|\widetilde{A^{-p}}x\|_0 \, \|A^p\xi\|_0 = \|x\|_{-p} \, \|\xi\|_p.$$

Therefore there exists $x^* \in \mathfrak{E}_p^*$ such that

$$\langle x^*, \xi\rangle = (\widetilde{A^{-p}}x, A^p\xi)_0, \qquad \xi \in \mathfrak{E}_p.$$

Thus (1.9) is reproduced and, as is easily verified, $x^* = h_p^{-1}(x)$. The correspondence $x \mapsto x^*$ yields an anti-linear isomorphism from $\bigcup_{p\ge 0} \mathfrak{E}_{-p}$ onto $\mathfrak{E}^*$. In that case, identifying x with x^*, we come to

$$\mathfrak{E}^* = \bigcup_{p\ge 0} \mathfrak{E}_{-p}, \tag{1.10}$$

namely, the union of the increasing chain of Hilbert spaces (1.7). This is a counterpart of (1.5) and (1.6).

Lemma 1.2.8 *Let $\{e_j\}_{j=0}^{\infty}$ be a complete orthonormal basis of $\mathfrak{H}$. Under the identification (1.10) we have*

$$\|x\|_{-p}^2 = \sum_{j=0}^{\infty} \left|\left\langle x, A^{-p}e_j \right\rangle\right|^2, \qquad x \in \mathfrak{E}^*, \quad p \geq 0.$$

PROOF. In fact, identifying x with x^* we see that

$$\begin{aligned} \|x\|_{-p}^2 &= \|\widetilde{A^{-p}}x\|_0^2 = \sum_{j=0}^{\infty} \left|\left(\widetilde{A^{-p}}x, e_j\right)_0\right|^2 \\ &= \sum_{j=0}^{\infty} \left|\left(\widetilde{A^{-p}}x, A^p A^{-p} e_j\right)_0\right|^2 = \sum_{j=0}^{\infty} \left|\left\langle x, A^{-p}e_j \right\rangle\right|^2, \end{aligned}$$

where (1.9) is taken into consideration. qed

From the universal property of an inductive limit we may deduce the following

Proposition 1.2.9 *A linear operator T from $\mathfrak{E}^*$ into a locally convex space $\mathfrak{X}$ is continuous if and only if the restriction of T to $\mathfrak{E}_{-p}$ is continuous for all $p \geq 0$.*

We are now in a position to discuss the real case. Let $\mathfrak{H}$ be a real Hilbert space and its complexification is denoted by $\mathfrak{H}_{\mathbb{C}}$. A densely defined operator A in $\mathfrak{H}$ admits a unique extension to a densely defined operator $A_{\mathbb{C}}$ in $\mathfrak{H}_{\mathbb{C}}$. If $A_{\mathbb{C}}$ is selfadjoint with $\inf \mathrm{Spec}(A_{\mathbb{C}}) > 1$, we say simply that A is a selfadjoint operator in $\mathfrak{H}$ with $\inf \mathrm{Spec}(A) > 1$. Taking the real part of the complex CH-space constructed from $(\mathfrak{H}_{\mathbb{C}}, A_{\mathbb{C}})$, we obtain a real CH-space $\mathfrak{E}$ imbedded in $\mathfrak{H}$. This $\mathfrak{E}$ is called a CH-space constructed from $(\mathfrak{H}, A)$. The above discussion for complex spaces are also valid for real spaces with obvious modification.

1.3 Nuclear spaces and kernel theorem

We begin with the following

Definition 1.3.1 A locally convex space $\mathfrak{X}$ equipped with defining Hilbertian seminorms $\{\|\cdot\|_\alpha\}_{\alpha \in A}$ is called *nuclear* if for any $\alpha \in A$ there is $\beta \in A$ with $\alpha \leq \beta$ such that the canonical map $f_{\alpha,\beta} : \mathfrak{X}_\beta \to \mathfrak{X}_\alpha$ is of Hilbert-Schmidt type.

By definition a nuclear Fréchet space is a CH-space. As for structural characterization of a nuclear Fréchet space we mention the following

Proposition 1.3.2 *A nuclear Fréchet space $\mathfrak{E}$ admits a sequence of defining Hilbertian seminorms $\{|\cdot|_n\}_{n=0}^{\infty}$ such that*

(i) $|\xi|_n \leq C_n |\xi|_{n+1}$, $\xi \in \mathfrak{E}$, *with some $C_n \geq 0$;*

(ii) $f_{n,n+1} : H_{n+1} \to H_n$ *is of Hilbert-Schmidt type, where H_n is the Hilbert space associated with $|\cdot|_n$;*

(iii) $\{H_n, f_{m,n}\}$ *is a reduced projective sequence of Hilbert spaces;*

(iv) $\mathfrak{E} \cong \text{proj}\lim_{n\to\infty} H_n$.
Conversely, if $\{H_n, f_{m,n}\}$ *is a (reduced) projective sequence of Hilbert spaces with* $f_{n,n+1}$ *being of Hilbert-Schmidt type, then* $\text{proj}\lim_{n\to\infty} H_n$ *becomes a nuclear Fréchet space.*

Proposition 1.3.3 *A Fréchet space* $\mathfrak{X}$ *is nuclear if and only if so is* $\mathfrak{X}^*$.

Proposition 1.3.4 *A standard CH-space* $\mathfrak{E}$ *constructed from* $(\mathfrak{H}, A)$ *is nuclear if and only if* A^{-r} *is of Hilbert-Schmidt type for some* $r > 0$.

PROOF. Let $\|\cdot\|_p$ be the defining seminorms of $\mathfrak{E}$ given as $\|\xi\|_p = \|A^p\xi\|_0$ and denote by $\mathfrak{E}_p$ the associated Hilbert space.

Suppose first that A^{-r} is of Hilbert-Schmidt type with $r > 0$. Then, there exists a complete orthonormal basis $\{e_j\}_{j=0}^{\infty}$ for $\mathfrak{H}$ contained in $\text{Dom}(A)$ such that $Ae_j = \lambda_j e_j$ with $\lambda_j > 0$ satisfying $\sum_{j=0}^{\infty} \lambda_j^{-2r} < \infty$. Note that $\{\lambda_j^{-(p+r)} e_j\}_{j=0}^{\infty}$ is a complete orthonormal basis for $\mathfrak{E}_{p+r}$ and

$$\sum_{j=0}^{\infty} \left\| \lambda_j^{-(p+r)} e_j \right\|_p^2 = \sum_{j=0}^{\infty} \lambda_j^{-2r} < \infty.$$

Hence the canonical map $f_{p,p+r} : \mathfrak{E}_{p+r} \to \mathfrak{E}_p$ is of Hilbert-Schmidt type for all $p \geq 0$. Therefore, $\mathfrak{E}$ is nuclear.

Conversely, suppose that $\mathfrak{E}$ is nuclear. Let $\{|\cdot|_n\}_{n=0}^{\infty}$ be a sequence of Hilbertian seminorms described as in Proposition 1.3.2. Since $\mathfrak{E} \cong \text{proj}\lim_{p\to\infty} \mathfrak{E}_p$ as well, we may find $n \geq 0$ and $r \geq 0$ such that

$$\|\xi\|_0 \leq C\,|\xi|_n\,, \qquad |\xi|_{n+1} \leq C'\,\|\xi\|_r\,, \qquad \xi \in \mathfrak{E},$$

with some $C, C' \geq 0$. Then we have a chain of canonical maps:

$$\mathfrak{E}_0 = \mathfrak{H} \longleftarrow H_n \overset{f_{n,n+1}}{\longleftarrow} H_{n+1} \longleftarrow \mathfrak{E}_r.$$

Since $f_{n,n+1}$ is of Hilbert-Schmidt type, so is the composion of the three which is nothing but the canonical map $\mathfrak{E}_r \to \mathfrak{E}_0 = \mathfrak{H}$. Let $\{e_j\}_{j=0}^{\infty}$ be a complete orthonormal basis of $\mathfrak{H}$. The obvious relation

$$(A^{-r}e_i\,, A^{-r}e_j)_r = (A^r A^{-r}e_i\,, A^r A^{-r}e_j)_0 = (e_i\,, e_j)_0 = \delta_{ij}$$

means that $\{A^{-r}e_j\}_{j=0}^{\infty}$ is an orthonormal sequence in $\mathfrak{E}_r$. Since the canonical map $\mathfrak{E}_r \to \mathfrak{E}_0 = \mathfrak{H}$ is of Hilbert-Schmidt type, $\sum_{j=0}^{\infty} \|A^{-r}e_j\|_0^2 < \infty$, that is, A^{-r} is of Hilbert-Schmidt type. qed

In particular,

Corollary 1.3.5 *A standard CH-space constructed from* $(\mathfrak{H}, A)$ *is nuclear if* A *is a positive selfadjoint operator with Hilbert-Schmidt inverse.*