Pseudo-Differential Operators

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Preface

Pseudo-differential operators are a natural extension of linear partial differential operators. The theory of such operators, which grew out of the study of singular integral operators of Giraud, Mikhlin, Calderón, and Zygmund, among others, developed rapidly after 1965 with the systematic studies of Kohn-Nirenberg, Hörmander, and others. (The term pseudo-differential operator first appeared in a paper of Kohn and Nirenberg.)

In 1958 Calderón [1] proved the local uniqueness theorem for the initial-value problem of a linear partial differential equation by expressing a linear partial differential operator by means of singular integral operators. In a sense, this was the birth of pseudo-differential operators. Calderón's idea of converting the theory of linear partial differential equations into an algebraic theory for the characteristic polynomials, or symbols, of the differential operators by means of Fourier transforms, radically changed the general theory of linear partial differential equations.

On the other hand, pseudo-differential operators in the present form have been introduced by Kohn and Nirenberg, Hörmander, and others, in a natural way, by expressing the symbol of a differential operator by means of its asymptotic expansion. This idea gave rise to many systematic studies on the algebra and calculus of pseudo-differential operators.

The theory of pseudo-differential operators has found many fields of application. For example, Atiyah and Singer have used pseudo-differential operators in an essential way to prove that the geometric and analytic indexes coincide.

At present, pseudo-differential operators, as introduced in the present monograph, are used widely in the study of linear partial differential equations. Thus, large parts of classical applied mathematics, including potentials, Green's functions, and fundamental solutions, have been newly reconsidered from the point of view of pseudo-differential operators.

Despite the many applications of pseudo-differential operators, there are still no books introducing this theory, except for several sets of lecture notes. One reason for this may be that the field is still undergoing major development and has not yet reached the perfection of a systematic theory. Another reason is that heretofore a fairly deep knowledge of functional analysis has been required for the understanding of the whole theory.

Recently the author has examined the structure of the theory of pseudo-differential operators and has noticed that the basic part of the theory can be constructed solely by means of elementary calculus and the elementary theory of the Fourier transform. Since, in a sense, the basic theory combines the Leibnitz formula and the Taylor expansion formula by means of the Fourier transform, it can therefore be understood by those undergraduates who have mastered elementary calculus.

This monograph consists of ten chapters, an appendix, and a bibliography. Chapters 1–3, and especially Chapter 2, contain the fundamental theory based on elementary calculus. Chapters 4–10 are devoted to applications. Among comparatively new and basic results which require almost no prerequisite the author selected those which lead to further study.

To render this book accessible to undergraduates in science and engineering, Chapter 1 introduces the reader to those function spaces and Fourier transforms which are used in subsequent chapters. In §6 of the same chapter, it is shown that the fundamental theorems of the theory of integration hold also for the so-called oscillatory integral

$$\mathrm{O_s} - \iint e^{-iy\cdot\eta} a(\eta, y)\,dy\,đ\eta,$$

where $a(\eta, y)$ is a smooth function of polynomial growth. By means of these theorems the reader will be able to carry out the integrations freely.

Chapter 2 lays the basis for the theory of pseudo-differential operators. The author introduces the class of symbols $S^m_{\rho,\delta}$ which originates from the Hörmander class $S^m_{\rho,\delta}(\Omega)$ by taking for Ω the Euclidean space $\mathbf{R}^n$. Then a pseudo-differential operator of class $S^m_{\rho,\delta}$ is defined as a mapping of the Schwartz space $\mathscr{S}$ of rapidly decreasing functions into $\mathscr{S}$. For this class a detailed study of the product of two pseudo-differential operators and of the adjoint of a pseudo-differential operator is undertaken by means of the method of asymptotic expansion of the symbols. In Chapter 3 pseudo-differential operators are extended as operators on Sobolev spaces (which are Hilbert spaces), and also on the spaces of Schwartz distributions. Thus one can apply the theory of functional analysis to pseudo-differential operators. It should be noted that one could make the reading of Chapters 2 and 3 quite easy by considering only the class S^m, defined as $S^m_{1,0}$. There are many applications of operators of class S^m. In fact, in Chapters 4, 6, and 9 one uses only the theory of the class S^m.

In Chapter 4, the theory of pseudo-differential operators is used to give a simple proof of the beautiful theorem obtained by Hörmander [5] on the hypoellipticity of second-order pseudo-differential operators. In Chapter 5 the Noether formula is derived. This formula is basic in the calculation of the index of a Fredholm pseudo-differential operator. In Chapter 6, on the basis of the Dirichlet problem, the general elliptic boundary-value problem for an operator is reduced to the problem of a system of pseudo-differential operators on the boundary.

In Chapter 7, the class of pseudo-differential operators is extended, by means of basic weight functions, to include pseudo-differential operators with multiple symbols, and this yields the formula to represent an operator with multiple symbol by means of a single symbol. As an application the initial-value problems for hyperbolic and parabolic equations are solved. The purpose of Chapter 8 is to construct the complex power of a hypoelliptic operator solely by means of the symbol calculus. In Chapter 9, as an application to geometry, pseudo-differential operators are used to derive the fixed-point formula of Atiyah-Bott-Lefschetz.

In Chapter 10, which has been completely rewritten and updated from the Japanese edition, the author describes the fundamental results on Fourier integral operators considered as a generalization of pseudo-differential operators. By means of Fourier integral operators one can explain quite explicitly the propagation of the singular supports of the solutions of partial differential equations. The complete theory of Fourier integral operators reaches well beyond the scope of the present book. The interested reader is referred, in the bibliography, to the papers of Hörmander, Duistermaat-Hörmander, Èskin, Egorov, Maslov, Arnold, Duistermaat, and many others.

The Appendix updates Chapters 1 to 9 and contains bibliographical notes. The second part of the Bibliography updates the bibliography of the Japanese edition contained in the first part.

Although pseudo-differential operators are now universally known and widely used, it is worth pointing out that in Japan as well as in other countries, following Calderón's work, essentially the same operators have been treated as an extension of singular integral operators, and have led to many new results. A detailed account of those results obtained in Japan is found in the book of S. Mizohata [3]. On the other hand, Sato's theory of hyperfunctions gives another approach to the theory of pseudo-differential operators, as an extension of partial differential operators with analytic coefficients, whereas the approach followed in this monograph is based mainly on C^∞-functions.

Among the many classes of pseudo-differential operators that have been defined, extended, and applied, it is the author's belief that the class $S^m_{\rho,\delta}$ treated here will have lasting value as a standard and handy class in further study and application of pseudo-differential operators.

I should like to thank Professor Hiroshi Fujita of the University of Tokyo, who gave me the chance to write this book; and Mr. Hideo Arai of Iwanami Shoten, who has given me constant help during its publication. My sincere gratitude should also go to my respected teacher, Emeritus Professor Mitio Nagumo of Osaka University, whose invaluable lectures and guidance have led me to study the present subject. During the completion of the manuscript Professor Mitsuru Ikawa, Miss Chisato Tsutsumi, Mr. Kazuo Taniguchi, and other members of the Department of Mathematics of Osaka University have given me valuable suggestions and comments. To all of them I wish to give my hearty thanks. Lastly, my deepest gratitude should go to my friends, Professor Rémi Vaillancourt of the University of Ottawa, for his painstaking translation of the book, and Professor Michihiro Nagase of Osaka University, for reading through the manuscript and making valuable comments regarding the translation during his stay at the University of Ottawa; I also thank Professors Vaillancourt and Nagase for their scrupulous reading of the galleys. I would also like to thank Iwanami Shoten and The MIT Press for the publication of this English edition.

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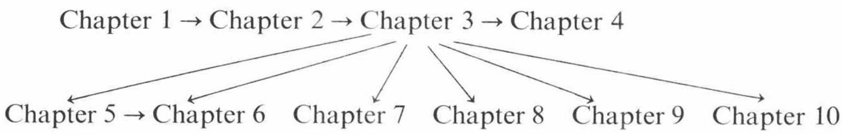
June 1974 and November 1980

Explanatory Notes

1. The present edition consists of ten chapters, an appendix, and a bibliography. Chapters 1 to 3 are devoted to the basic theory, and Chapters 4 to 10 to the applications. Chapter 1 is a prerequisite to the other chapters. Chapters 2 and 3 are devoted to the basic theory of pseudo-differential operators. The logical diagram is roughly as follows:



2. Each chapter is divided into sections (§). Definitions, theorems, propositions, and lemmas in each section, e.g., in §1, are numbered sequentially as Definition 1.1, Proposition 1.2, Lemma 1.3, Theorem 1.4, Proposition 1.5, for easy reference. In referring to them, we write, e.g., Theorem 1.4, if in the same chapter, and Theorem 1.4 of Chapter 1, if in a different chapter.

3. The mathematical notation and terminology is, on the whole, standard, except for those terms appearing for the first time in such a monograph. A Table of Notation follows.

4. The first six sections of the Appendix describe new topics which have been developed since the publication of the Japanese edition. Section VII presents bibliographical notes on recent publications related to Chapters 1 to 10 and to new additional topics.

5. The Bibliography at the end of this work, by no means complete, lists the works referred to in this book. It consists of two parts, the first being the bibliography of the Japanese edition and the second containing more recent titles.

Notation

(numbers refer to pages)

Multi-index Notation

For $x = (x_1, \ldots, x_n) \in \mathbf{R}_x^n$, $\xi = (\xi_1, \ldots, \xi_n)$, $\xi' = (\xi'_1, \ldots, \xi'_n) \in \mathbf{R}_\xi^n$, and multi-indices of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, we use the notation:

$$
\begin{cases}
\langle x \rangle = \sqrt{1 + |x|^2}, \quad \langle \xi \rangle = \sqrt{1 + |\xi|^2}, \quad \langle \xi; \xi' \rangle = \sqrt{1 + |\xi|^2 + |\xi'|^2}, \\
|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \\
\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n), \\
\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad \partial_{x_j} = \dfrac{\partial}{\partial x_j}, \\
D_{x_j} = -i\dfrac{\partial}{\partial x_j} \quad (j = 1, \ldots, n), \\
\alpha \geqq \beta \Leftrightarrow \alpha_j \geqq \beta_j \quad (j = 1, \ldots, n), \quad \dbinom{\alpha}{\beta} = \dfrac{\alpha!}{\beta!(\alpha - \beta)!} \quad \text{for} \quad \alpha \geqq \beta, \\
x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n, \quad đ\xi = (2\pi)^{-n} d\xi = (2\pi)^{-n} d\xi_1 \cdots d\xi_n \\
6, 7, 34, 64
\end{cases}
$$

General Notation

$\Omega' \subset\subset \Omega$ means that the closure $\overline{\Omega'}$ of the set Ω' is compact and is contained in the interior of the set Ω 12

$\Phi \subset\subset \Psi$ in Ω means that $\Phi, \Psi \in C_0^\infty(\Omega)$ and $\Psi(x) = 1$ in a neighborhood of supp Φ; $\{\Phi_j\}_{j=1}^\infty \subset\subset \{\Psi_j\}_{j=1}^\infty$ in $\{\Omega_j\}_{j=1}^\infty \Leftrightarrow \Phi_j \subset\subset \Psi_j$ in Ω_j $(j = 1, 2, \ldots)$ 96

$T_\nu \to T$ in V' weakly means that $\{T_\nu\}_{\nu=1}^\infty$ converges weakly to T in V' 21

$\hat{f}(\xi) = \mathscr{F}[f](\xi) = \int e^{-ix\cdot\xi} f(x)dx$ (the Fourier transform of $f(x)$),

$\overline{\mathscr{F}}[\hat{f}](x) = \int e^{ix\cdot\xi} \hat{f}(\xi) đ\xi$ (the inverse or conjugate Fourier transform of a function $\hat{f}(\xi)$) 34

$$\mathrm{O}_{\mathrm{S}}[e^{-iy\cdot\eta}a] = \mathrm{O}_{\mathrm{S}} - \iint e^{-iy\cdot\eta}a(\eta, y)dyđ\eta$$
$$= \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon\eta, \varepsilon y)a(\eta, y)dyđ\eta$$

(the oscillatory integral of $a\in\mathscr{A}$, where $\chi\in\mathscr{S}$ in $\mathbf{R}^{2n}$, $\chi(0, 0) = 1$) 46f.

$dx''\cdot đ\xi'\cdot dx'\cdot đ\xi$ means that the integrations are done in the order $dx''\to đ\xi'\to dx'\to đ\xi$ 56, 65

sing supp u is the singular support of u, $WF(u)$ is the wave front set of u ($u\in\mathscr{D}'(\Omega)$) 125, 301

$\mathscr{B}_t^m(V)\ni u(t)$ in I, $\mathscr{B}_t(V) = \bigcap_{m=0}^{\infty}\mathscr{B}_t^m(V)$ in I 210

The Main Spaces of Functions

$C^m(\Omega)$ is the set of m-times continuously-differentiable functions in Ω 6

$C(\Omega) = C^0(\Omega)$, $C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$ 6

$\operatorname{supp} f = \overline{\{x\in\Omega;\ f(x)\neq 0\}}\cap\Omega$ ($f\in C(\Omega)$) 7

$C_0^m(\Omega) = \{f\in C^m(\Omega);\ \operatorname{supp} f \subset\subset \Omega\}$, $C_0(\Omega) = C_0^0(\Omega)$,

$C_0^\infty(\Omega) = \bigcap_{m=0}^{\infty} C_0^m(\Omega)$ 7

$\mathscr{E}(\Omega)$ is $C^\infty(\Omega)$ with a topology (a Fréchet space) 12

$\mathscr{D}(\Omega)$ is $C_0^\infty(\Omega)$ with a topology (a locally convex linear topological space) 13

$\mathscr{B}^m(\Omega) = \{f\in C^m(\Omega);\ |f|_{m,\Omega} \equiv \max_{|\alpha|\le m}\sup_{x\in\Omega}|\partial_x^\alpha f(x)| < \infty\}$ (a Banach space) 7, 10

$\mathscr{B}(\Omega) = \bigcap_{m=0}^{\infty}\mathscr{B}^m(\Omega)$ (a Fréchet space) 7, 10

$L_2(\Omega)$ is the set of square integrable functions on Ω (a Hilbert space) 7

$L_1^{\mathrm{loc}}(\Omega) = \left\{f(x);\ \int_K |f(x)|dx < \infty \text{ for any compact subset } K \text{ of } \Omega\right\}$ 29

$\mathscr{S} = \{f\in C^\infty(\mathbf{R}^n);\ |f|_{l,\mathscr{S}} \equiv \max_{l_1+|\alpha|\le l}\sup_{x\in\mathbf{R}^n}\{\langle x\rangle^{l_1}|\partial_x^\alpha f(x)|\} < \infty,\ l = 0, 1, \ldots\}$
(called the space of rapidly decreasing functions; a Fréchet space) 7, 10

$\dot{\mathscr{B}}^m = \{f\in\mathscr{B}^m(\mathbf{R}^n);\ |\partial_x^\alpha f(x)|\to 0\ (|x|\to\infty),\ |\alpha|\le m\}$, $\dot{\mathscr{B}} = \bigcap_{m=0}^{\infty}\dot{\mathscr{B}}^m$ 26

$\mathscr{A}^m_{\delta,\tau} = \{a(\eta, y) \in C^\infty(\mathbf{R}^{2n});\ |a|^{(m)}_l \equiv \max_{|\alpha+\beta| \leqq l} \sup_{(\eta,y)\in\mathbf{R}^{2n}} (|\partial^\alpha_\eta \partial^\beta_y a(\eta, y)| \langle y\rangle^{-\tau} \langle\eta\rangle^{-(m+\delta|\beta|)}) < \infty, l = 0, 1, \ldots\}$
(the class of amplitude functions; a Fréchet space);

$\mathscr{A} = \bigcup_{0\leqq\delta<1} \bigcup_{-\infty<m<\infty} \bigcup_{0\leqq\tau} \mathscr{A}^m_{\delta,\tau}$ 46

V' is the dual space of a linear topological space V, $\mathscr{D}'(\Omega)$, $\mathscr{E}'(\Omega)$, $\mathscr{S}'$, etc. 20

For M a C^∞-manifold and E a C^∞ vector bundle (on M), $C^\infty(M)$, $C_0^\infty(M)$, $\mathscr{D}'(M)$, $\mathscr{E}'(M)$, $C^\infty(E)$, $C_0^\infty(E)$, $\mathscr{D}'(E)$, $\mathscr{E}'(E)$, etc. 95, 106, 116, 117

Sobolev Spaces

$H_s = \{u \in \mathscr{S}';\ \langle D_x\rangle^s u \in L_2(\mathbf{R}^n)\}$, $H_{\lambda,s} = \{u \in \mathscr{S}';\ \lambda(D_x)^s u \in L_2(\mathbf{R}^n)\}$ 118, 224

$H_k(\Omega) = \{u \in \mathscr{D}'(\Omega);\ \partial^\alpha_x u \in L_2(\mathbf{R}^n), |\alpha| \leqq k\}$, $\mathring{H}_k(\Omega) = \overline{C_0^\infty(\Omega)}$ in $H_k(\Omega)$ 196

$H_s(M)$, $H_s(E)$ 150, 153

$\Lambda_{s,M}$, $\Lambda_{s,E}$ 115, 149, 153

Classes of Pseudo-Differential Operators

$S^m_{\rho,\delta} = \{p(x, \xi) \in C^\infty(\mathbf{R}^{2n});\ \forall\alpha, \beta, |p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|} \text{ in } \mathbf{R}^{2n}\}$ 54

$P = p(X, D_x) \in S^m_{\rho,\delta}$ means that P is a pseudo-differential operator with symbol $\sigma(P)(x, \xi) = p(x, \xi)$ 54

$S^m = S^m_{1,0}$, $S^{-\infty} = \bigcap_m S^m (= \bigcap_m S^m_{\rho,\delta})$, $S^\infty_{\rho,\delta} = \bigcup_m S^m_{\rho,\delta}$ 54

$S^{m,m'}_{\rho,\delta}$ is the class of double symbols

$\mathcal{S}^{m,m'}_{\rho,\delta}$ is the class of pseudo-differential operators with double symbols 64, 65

$\Lambda^s = \langle D_x\rangle^s (\in \mathcal{S}^s)$ is the operator with symbol $\langle\xi\rangle^s$ 55, 123

$p_L(x, \xi)$ is the simplified symbol of the double symbol $p(x, \xi, x', \xi')$ 73

$p_1 \circ p_2(x, \xi)$ is the simplified symbol for the product $P_1 P_2$ of $P_j = p_j(X, D_x)$ $(j = 1, 2)$ 60

$\dot{S}^m_{\rho,\delta}$, $\tilde{S}^m_{\rho,\delta}$ are classes of slowly varying symbols 144

$S^m_{\rho,\delta}(M)$, $S^m_{\rho,\delta}(E)$, $S^m_{\rho,\delta}(E_1, E_2)$ 96, 109

$S^m_{\lambda,\rho,\delta}$ is the class of symbols defined by means of a basic weight function $\lambda(\xi)$ 222

$S_{\lambda,\rho,\delta}^{\tilde{m}_\nu;\tilde{m}'_\nu}$ is the class of multiple symbols defined by means of $\lambda(\xi)$ 226
$S_\rho^m = S_{\rho,1-\rho}^m$ with $1/2 \leqq \rho \leqq 1$ 278
$S_\rho^m((k)) = \{p(x, \xi) \in S_\rho^m; p_{(\beta)}^{(\alpha)}(x, \xi) \in S_\rho^{m-|\alpha|}$ when $|\alpha + \beta| \leqq k\}$ 278
$\mathscr{P}_\rho(\tau)$, $\mathscr{P}_\rho(\tau, l)$ are classes of phase functions 279
$\phi_1 \# \cdots \# \phi_{\nu+1}$ is the multi-product of the $\nu + 1$ phase functions $\phi_1, \ldots, \phi_{\nu+1}$ 332, 342
$S_{\rho,\phi}^m(S_{\rho,\phi^*}^m)$ is the class of (conjugate) Fourier integral operators with phase function $\phi(x, \xi)$ 280

Contents

Pseudo-Differential Operators

Introduction

What is a pseudo-differential operator?

A very rough answer to this question will be given in this introduction. Consider a linear partial differential operator of order m defined on the n-dimensional Euclidean space $\mathbf{R}_x^n$:

$$Pu(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha u(x), \quad x = (x_1, \ldots, x_n) \in \mathbf{R}_x^n$$

$$(\alpha = (\alpha_1, \ldots, \alpha_n), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n},$$

$$D_{x_j} = -i \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n).$$

To the operator P we associate the polynomial $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$ ($\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$) in $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}_\xi^n$; the polynomial $p(x, \xi)$ is called the *symbol* of P.

As is well known, the Fourier transforms of $D_{x_j} u(x)$, $x_j u(x)$ are $\xi_j \hat{u}(\xi)$, $-D_{\xi_j} \hat{u}(\xi)$, respectively, where $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$. Therefore, when P is an operator with constant coefficients, its symbol is a polynomial in ξ, $p(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$, with the constant coefficients a_α; and we see that the Fourier transform $\widehat{Pu}(\xi)$ of $Pu(x)$ is equal to $p(\xi) \hat{u}(\xi)$. Hence the differential operator P is converted into an algebraic operation: multiplication by the polynomial $p(\xi)$. This is why the Fourier transform is an effective tool in the study of linear partial differential equations with constant coefficients. But for differential equations with variable coefficients, the situation is not so simple. For example, consider the operator P given by $Pu(x) = D_x^2 u(x) + x^2 u(x)$, whose symbol has polynomial coefficients in $\mathbf{R}_x^1$. Then the Fourier transform of $Pu(x)$, i.e., $\widehat{Pu}(\xi) = \xi^2 \hat{u}(\xi) + D_\xi^2 \hat{u}(\xi)$ in $\mathbf{R}_\xi^1$, is of the same form. Therefore in this case the situation is not simplified by using the Fourier transform. We say, roughly, that the aim of the theory of pseudo-differential operators is to convert the theory of linear partial differential equations with variable coefficients into an algebraic theory of the corresponding symbols through the theory of Fourier transforms.

More concretely, since $\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi)$, then by the Fourier inversion formula we can write

$$D_x^\alpha u(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} \xi^\alpha \hat{u}(\xi) d\xi \quad (x \cdot \xi = x_1 \xi_1 + \ldots + x_n \xi_n).$$

Therefore, using the symbol $p(x, \xi)$, the operator $Pu(x)$ can be written in the form