# CHAOTIC DYNAMICS

*From the*
*One-Dimensional Endomorphism*
*to the*
*Two-Dimensional Diffeomorphism*

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World Scientific
Singapore • New Jersey • Hong Kong

*Published by*

**World Scientific Publishing Co. Pte. Ltd.**
P.O. Box 128, Farrer Road, Singapore 9128

*U.S.A. office*: World Scientific Publishing Co., Inc.
687 Hartwell Street, Teaneck NJ 07666, USA

Library of Congress Cataloging-in-Publication data is available.

**CHAOTIC DYNAMICS**



**ISBN 9971-50-324-7**

**Printed in Singapore** by Fong and Sons Printers Pte. Ltd.

# PREFACE

The subject matter concerns the study of complex motions of dynamical systems, via the basic mathematical tool of maps (equivalent denomination recurrences; a non-invertible map being an endomorphism, an invertible and differentiable one being a diffeomorphism). More precisely, the book treats the properties of a class of determinist mathematical models, the solutions of which have not the periodical regularity. Nowadays, depending on the context, such solutions are called "chaotic". Since few years, chaotic dynamics has become a favour choice for the most part of disciplines dealing with evolution phenomena. It results a considerable increase of the number of publications.

Due to its sudden and "explosive" growth, the subject matter has been developed independently of fundamental results obtained before. Then many present results are rediscoveries, or variants of older ones.

From this point of view the book will appear non-conventional, because its basic references are not those of the literature now considered as classical. Some readers will no doubt find occasions to feel irritated by it. More particularly, the assertion that so popular present notions as those of "invariant coordinates", "kneading invariant",and others , are variant of Myrberg's results will appear disappointing for some. Nevertheless one of the purposes of this publication is to give to the reader an information about generally unknown works of Hadamard, Lattès, Cigala, Myrberg, Neimark, Leonov, Pulkin, ..., and thus to restore certain anteriorities. However, it is worth of note that this choice does by no means constitute a negative judgment of the works of the literature now called classical. Generally written according to the present standards of the "abstract theory of dynamical systems", they are excellent and necessary from the point of view of formal exposition and rigour. With these criterions, such works conserve wholly their own merits, and must be considered as productions of quality.

The debate which might result from the choices of this book would not be a new one. So, in his paper, "George David Birkhoff and his mathematical work" (Bull. Amer. Math. Soc., May 1946, vol. 52, n° 5, part 1, pp. 357-391), Morse wrote :

*"Birkhoff was uncompromising in his appraisal of mathematics, by the test of originality and relevance. For him the systematic organization or exposition of a mathematical theory was always secondary in importance to its discovery. I recall his remarks on a mathematical treatise that had come to his attention and eventually had a wide circulation but which he did not regard as original. Birkhoff said : "I read this book through in half hour"... Some of the current mathematical theories were regarded by Birkhoff as no more than relatively obvious elaborations of concrete exemples".*

Besides before Birkhoff, the famous mathematician Halphen is also known to have often complained that non-essential generalizations are overcrowding the publication media. In connection with this, it is well known that the majority of scientists were not led to their discoveries by a process of deduction from general postulates or general principles, but rather by a thorought examination of properly chosen particular cases. The generalizations have come later, because it is far easier to generalize an established result than to discover a new line of argument |G23|.

The book may also appear non-conventional from another point of view, which incidentally is a consequence of the first one. Indeed, another possible reason of irritation for certain abstractly inclined readers might be related to the terminology adopted from the original works and the absence of the previously mentioned ars-pro-artis generalizations so popular in contemporary mathematics. The resulting formulation of problems may therefore appear primitive, sometimes even simplistic. The style may appear less formal and rigorous than that of contemporary papers about the abstract theory of dynamical systems. The answer to this eventual remark consits in stating precisely that *the book does not pretend to be a mathematical treatise on dynamical systems* (it is published in a physics collection), because it is not presented in a sufficiently elaborate and rigorous form. Its objective is far more modest and consists of :

-(a)- giving among the basic tools of dynamical systems theory, a large part of the most efficient ones for solving practical problems (cf. such problems for example in |G 35||G 18||N 5|).

-(b)- understanding dominating internal mechanisms of evolution processes, and thus preserving a phenomenological transparency through the simplest possible concrete examples (in the sense of the above mentioned Birkhoff's citation), from which perhaps "standard" mathematical theories might be elaborated. Indeed, it can be considered that the field of "*concrete dynamic systems*" is constituted by two sets of results. The first one is related to the study of problems directly suggested by practice (Physics, Engineering,...). The second one concerns the study of equations, not directly tied with practice, but having the lowest dimension, and the simplest structure, which permit to isolate in the purest form a "mathematical phenomenon", by eliminating "parasitic effects" of a more complicated structure. The bringing to light of the phenomenon of Myrberg's period doubling cascades, and their accumulations, from the very simple discrete dynamic system $x_{n+1} = x_n^2 - \lambda$ (x being a real variable, $\lambda$ a real parameter, n = 0, 1, 2, ...) is of such a type, by reason of its many practical implications. The Smale's horseshoe is also of this type. Consequently, the approach which is adopted is a geometric one, based on an important number of figures.

-(c)- stimulating some ways of research from the choices of examples.

Is is worth of note that the choice of this less formal style has again the advantage of optimizing the interdisciplinary communication, even if it has to appear old-fashioned to abstractly inclined readers.

Beyond as systematic as possible a study of critical cases in the Ljapunov'sense, and of the crossing through the corresponding bifurcations, it is attempted to realize the mentioned objective by showing that *two basic fractal bifurcations structures* play a fundamental role in the understanding of the dominating internal mechanisms in a wide class of chaotic behaviours. They were called :

- "*box-within-a-box*" (or "*embedded boxes*" in |G 14|) *bifurcations structure* ("structure de bifurcations boites-emboitées" in French |G 32| |M 25|, 1975), which corresponds to an ordering of the Myrberg'spectra (often called in the contemporary literature Feigenbaum's cascades of bifurcations by period doubling).

- "*boxes in files*" *bifurcations structure* ("structure de bifurcation boites en files" |M 27|, 1978) which corresponds to an ordering of the Farey's sequences of fractions in their lowest terms, and which was elaborated from an adaptation of the Leonov's results |L 14| about piecewise linear maps bifurcations structure.

From the knowledge of these structures, it resulted the elaboration of a *particular symbolism* associated with their fractal properties, and the introduction of new notions such as :

. *fuzzy*, or *chaotic*, *basin boundary* ("frontière floue" |M 23|, 1975)
. *average value cycle* ("cycle en valeur moyenne" |M 22|, 1976) known now under the denomination *phenomenon of intermittency*
. *average value cyclic chaotic segment* ("segment stochastique cyclique en valeur moyenne" |M 22|, 1976) related to what it is now called *attractor in crisis*.
. *adjoint cycles*, and *self adjoint cycle* |K 3||F 7| (1985) which extend the possibilities of the symbolic dynamics, and which permits an understanding of the complex communications between sheets of a foliated box-within-a-box bifurcation structure |F 7||F 8||F 9|.

. the notions related to various non-classical global bifurcations, a part of them characterizing the transition order $\rightleftarrows$ chaos.

A non negligible part of the book constitutes a synopsis of slowly accumulated results obtained by the research group of Toulouse which works about non linear dynamical systems since 1962. The existence of this group would not be possible without my meeting I. GUMOWSKI in 1958.

From this personal event resulted my "initiation" into the famous (but in this time rather little known in western countries) works of the Andronov's school |B 14|. Since that time, a long friendly collaboration has permited a continuous and fruitfull action in this field, and many common publications using Andronov ideas. I. Gumowski is gratefully acknowledged for his invaluable contribution to the researches of the Toulouse group, which thus has profited of his interdisciplinary scientific knowledges of exceptional extent. All the researchers who worked, and who presently are working in this group, are also acknowledged. A particular mention is made to H. Kawakami, who efficiency collaborated at this task as a visiting Professor in 1984-1985. His presence gave a new and important impulsion to the studies related to the two-dimensional diffeomorphism, when he introduced the above mentioned fruitful notion of "adjoint cycles" and "self adjoint cycle".

Since 1978, R. Thom has accepted to present some papers of the group in "Comptes Rendus de l'Académie des Sciences de Paris". He is warmly thanked for his interest to the corresponding work, and for some fruitfull discussions which resulted.

Voluntarily certain subjects are not treated (renormalization, fractal dimension, applications to practical dynamical problems, ...). For a usefull complementary information on them, and others developed in this book, among the numerous available publications, it is suggested to consult :

* The Neimark's book |N 5| (1972) devoted to the method of point mappings and their applications to the theory of automatic control and the theory of dynamical systems. Scientist of the Andronov's school, Neimark has brought one of the most important contributions to this question.

* The Guckenheimer and Holmes book |G 18| (1983) dealing with non linear oscillations, dynamical systems and bifurcations, which is one of the most interesting publications recently published.

* The Gumowski and Mira's book |G 35| (1980) about chaotic dynamics, based on the point mappings method, with applications to engineering problems. The book |G 36| (1980) of the same authors completes the theoretical aspects of |G 35|.

* The book "Chaos" by Hao Bai-Lin (World Scientific Singapore, 1984), which is a selection of the classical literature on chaos with 41 papers among the most known.

* The Holmes and Whitley's paper |H 10| (1984) which constitutes the most complete study on homoclinic tangency in two-dimensional diffeomorphisms.The original study of |H11| (1985) by Holmes and Williams.

* The series of papers by Grebogi, Mc Donald, Kostelich, Ott, and Yorke |G 6|-|G 10|, |M 3||M 4| (1983-1986) representing a very important contribution to the questions of fractal and fuzzy basin boundaries,

chaotic transients.

* The book "Circuits non linéaires" by M. Hasler and J. Neirynck (Presses Polytechniques Romandes, 1985) which gives examples of electrical circuits with chaotic behaviours.

* A recentsystematic study of chaotic dynamics in an electrical circuit can also be found in the paper "The double scroll" by L.O. Chua, T. Matsumoto and M. Komuro, I.E.E.E. Trans. on Circ. and Syst., vol CAS 32, n° 8 (1985), 798-813. About such studies, it is worth of note the important contribution of Hayashi and Kawakami since 1972 (cf. some references in |K 2| and |M 48|).

* The very recent book "Difference equations and their applications" by A.N. Sharkovsky, Yu. L. Maistrenko, E. Yu. Romanenko (Kiev Naukova Dumka, 1986). Here, new results of the authors, concerning methods of qualitative study of different classes of difference equations, and differential-difference equations, are laid.

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Not having a usual practice of english, the quality of the english of this book is certainly affected. The reader may excuse this fact.

A part of the research of Chapter 6 was executed under contract DRET n° 85-1303 (French Ministry of Defense).

The most part of the figures were drawn by G. Roussel, and a part of the text was typed by Mrs C. Grima. These contributions are gratefully acknowledged.

Toulouse, April 1987

Christian MIRA

# CONTENTS

# Chapter 1
# DYNAMIC SYSTEMS AND RECURRENCES. GENERALITIES

## 1.1. CONTINUOUS DYNAMIC SYSTEMS AND DISCRETE DYNAMIC SYSTEMS

The content of this monograph concerns evolution processes, or equivalently dynamic systems. Till now, a generally accepted, and unambiguous, definition of this notion seems not yet exist. In a concrete context, that of the experimental physics, for example, a "dynamic system" is a limited configuration of objects, or again a more or less subjective description of this configuration in a verbal, or mathematical, form. Because it is impossible to study a limited configuration when it is imbedded into the whole universe, it is necessary to separate, or isolate, this configuration via an appropriate artifice consisting at the final step in choosing a certain number of relevant variables, and parameters. This indispensable process of separation leads to what is called a constraint of separation, which can have different forms according to the considered limited configuration, the desidered mathematical formulation, and the nature of the sought solution. In an abstract context, it is supposed that the constraint of separation is known, and the expression "dynamic system" simply designates the equation of the motion |G 33|.

A first possible formulation of motion equations is that ordinary differential equations with real variables, the time t being the independant variable, in one of the two explicit forms :

(1.1) $dx_i/dt = f_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, m$, $x_i(t_o) = x_{i_o}$

(1.2) $dx_i/dt = f_i(x_1, x_2, \ldots, x_n, t)$, $i = 1,2,\ldots,m$, $x_i(t_o) = x_{i_o}$

The first one is called a m-dimensional *autonomous equation* and the second one a m-dimensional *non-autonomous equation*. When the existence and uniqueness conditions are satisfied for the initial condition $x_i(t_o)$, the solution is a continuous function of t, and corresponds to what is called a *phase trajectory*, or a motion, in the *phase space* of coordinates $x_1, \ldots, x_m$. The equations (1.1), (1.2) also can be represented in a vectorial form $X^\bullet = F(X)$, $X^\bullet = F(X, t)$, $X^\bullet = dX/dt$.

A second possible formulation is that of recurrences, or equivalently *point-mappings, maps, iterations* (these forms are indifferently used in this book), for which the time t is no longer a continuous independant variable but a sequence of integers, n = 0, 1, 2, ... (discrete time), in one of the two explicit vectorial forms :

(1.3) $X_{n+1} = F(X_n)$, or $X_{n+1} = T\, X_n$, $X\,(n=0) = X_o$

(1.4) $X_{n+1} = F(X_n, n)$, or $X_{n+1} = T_n\, X_n$, $X(n=0) = X_o$

X being a real vector with components $(x_1, \ldots, x_p)$, F a single valued vectorial, function of X. In (1.3) (*autonomous* equation), T is the p-dimensional point-mapping which gives the point $X_{n+1}$ from the point $X_n$, and which is independant of n. In (1.4), (*non autonomous* equation), the point-mapping depends on n. When existence and uniqueness conditions are satisfied for the initial condition $X_o$, the solution $X_n = X(n, X_o)$ is a sequence of points (*iterated sequence*, or *orbit*, or *discrete phase trajectory*) in the p-dimensional *discrete phase plane*. The recurrences (1.3), (1.4) can be *directly* the mathematical models of a class of dynamic systems which are by nature of the discrete type, i.e. the available information about their states is only obtained in a sampled form. However, the autonomous recurrence (1.3) plays a fundamental role, when it is associated with an autonomous differential equation (1.1), the dimension p of (1.3) being not higher than (m-1), or with (1.2) when the $f_i$ are periodical functions of t with a fixed period. In this case (1.3) is the result of the application of the *Poincaré's method of section*. The reduction of the effective dimension of the differential equation, which is so obtained, makes easier its study. Details about this method will be considered in an other paragraph. It should be stressed that the inverse process, consisting in the construction of a differential equation from a recurrence (1.3), is generally not possible. The functional space of recurrences is thus much richer than that of differential equations, and this richness induces some far-reaching theoretical and practical properties |G 35| |G 36|.

## 1.2. BIRKHOFF CLASSIFICATION OF DYNAMIC MOTIONS. IMPREDICTIBILITY OF CHAOTIC MOTIONS

In his book "Dynamical Systems" |B 9| Birkhoff wrote : *"the final aim of the theory of dynamical systems must be directed towards the qualitative determination of all possible types of motions and the interrelations of these motions"*. Going towards this purpose, by using ideas developped by Poincaré, as a first step Birkhoff proposed en 1927 a classification of dynamic motions, which has been refined by Andronov in 1933 with the diagram of fig. 1.1. In this diagram, starting from the lower part up to the top, an increase of the structural complexity of motions manifests itself as a gradual transition from orderly to chaotic (erratic, or stochastic) motions.

It is recalled that steady states of *quasi periodic* type correspond to oscillations containing a finite number M of mutually incommensurable frequencies. Steady states of almost periodic type correspond to oscillations containing infinitely many mutually incommensurable frequencies ($M \to \infty$). Let us denote by $X(t, X_o)$ a dynamical system defined in some metric space R, the trajectory of which passes through the point $X_o$.

The motion $X(t, X_o)$ is called *recurrent*, if for any $\varepsilon > 0$ there can be found a $\tau(\varepsilon) > 0$ such that any arc of the phase trajectory of time length $\tau$ approximates the entire trajectory with a precision to within $\varepsilon$. In other words, whatever may be the numbers $t_1$, $t_2$, there can be found a number $t_3$ such that $t_2 < t_3 < t_2+\tau$, and $\rho[X(t_1,X_o), X(t_3,X_o)] < \varepsilon$, where $\rho(\alpha,\beta)$ is the distance between $\alpha$ and $\beta$ |N 8|.