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Nanhua Xi

Representations of Affine Hecke Algebras

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Introduction

Let K be a p-adic field with finite residue class field of q elements. Let $\mathcal{G}$ be a connected split reductive group over K with connected center. Let $\mathcal{I}$ be an Iwahori subgroup of $\mathcal{G}$ and $\mathcal{T}$ the 'diagonal' subgroup of $\mathcal{I}$ (in a suitable sense). The group $N_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$ (here $N_{\mathcal{G}}(\mathcal{T})$ is the normalizer of $\mathcal{T}$ in $\mathcal{G}$) is an extended affine Weyl group W (i.e. $W = \Omega \ltimes W'$ for certain abelian group Ω and for certain affine Weyl group W'). It is known that $\mathcal{G} = \bigcup_{w \in W} \mathcal{I}w\mathcal{I}$ and one can define an interesting associative ring structure on the free abelian group H_q with basis $\mathcal{I}w\mathcal{I}$, $w \in W$ (see [IM]). The ring H_q is an affine Hecke ring and we call $\mathbf{H}_q = H_q \otimes \mathbb{C}$ an affine Hecke algebra. According to Borel [Bo1] and Matsumoto [M], the category of admissible complex representations of G which have nonzero vectors fixed by $\mathcal{I}$ is equivalent to the category of finite dimensional representations (over $\mathbb{C}$) of $\mathbf{H}_q$. Thus an interesting part of the study of representations of p-adic groups can be reduced to that of affine Hecke algebras.

According to a conjecture of Langlands (see [La]) the irreducible complex representations of $\mathcal{G}$ should be essentially parametrized by the representations of the Galois group $\mathrm{Gal}(\bar{K}/K)$ into the complex dual group $\mathcal{G}^*(\mathbb{C})$ of $\mathcal{G}$ (in the sense of [La]): $\mathrm{Gal}(\bar{K}/K) \to \mathcal{G}^*(\mathbb{C})$.

Let Γ be the quotient group of $\mathrm{Gal}(\bar{K}/K)$ corresponding to the maximal tamely ramified extension of K. The group Γ has the generators F (Frobenius) and M (Monodromy), subject to the relation $FMF^{-1} = M^q$. According to the conjecture, the irreducible complex representations of $\mathcal{G}$ which have nonzero vectors fixed by the Iwahori group $\mathcal{I}$ should be essentially parametrized by the homomorphisms $\Gamma \to \mathcal{G}^*(\mathbb{C})$. More exactly, Langlands' original conjecture says that the representations should roughly be parametrized by the conjugacy classes of semisimple elements in $\mathcal{G}^*(\mathbb{C})$. A later refinement of the conjecture, due independently to Deligne and Langlands, adds nilpotent elements in the picture. Thus the representations considered should be essentially parametrized by the conjugacy classes of the pair (s, N) such that $\mathrm{Ad}(s)N = qN$, where s is a semisimple element of $\mathcal{G}^*(\mathbb{C})$, N is a nilpotent element in the Lie algebra $\mathbf{g}$ of $\mathcal{G}^*(\mathbb{C})$, and we say two pairs (s, N), (s', N') are conjugate if $s' = gsg^{-1}$, $N' = \mathrm{Ad}(g)N$ for some $g \in G$. For group $GL_n(K)$ this was proved by Berstein and Zelevinsky [BZ], [Z]. For general case, Lusztig (see [L4]) added a third ingredient to (s, N), namely an irreducible representation ρ of the group $A(s,N) = C_G(s) \cap C_G(N)/(C_G(s) \cap C_G(N))^0$ (here $G = \mathcal{G}^*(\mathbb{C})$ and $C_G(\cdot)$ denotes the centralizer in G) appearing in the representation of the group $A(s, N)$ on the total complex coefficient homology group of $\mathcal{B}_N^s$, here $\mathcal{B}_N^s$ is the variety of Borel subalgebras of $\mathbf{g}$ containing N and fixed by $\mathrm{Ad}(s)$.

Now the category of admissible complex representations of $\mathcal{G}$ which have nonzero vectors fixed by $\mathcal{I}$ is equivalent to the category of finite dimensional representations (over $\mathbb{C}$) of the Hecke algebra $\mathbf{H}_q$ with respect to the Iwahori group $\mathcal{I}$ (see [Bo1, M]). Therefore the conjecture can be stated as

(∗) The irreducible representations of $\mathbf{H}_q$ are naturally 1-1 correspondence with the conjugacy classes of triples (s, N, ρ) as above.

The conjecture (*) was proved by Kazhdan and Lusztig in [KL4]. Actually they proved that (*) is true when q is not a root of 1 (one can define $\mathbf{H}_q$ for arbitrary $q \in \mathbb{C}^*$). In [G1] Ginsburg also announced a proof, but the proof contains some errors since the main result is not correct as stated, see [KL4, p.155]. However the work [G1] contains some very interesting ideas. Combining [KL4] and [G1] we can prove that (*) is true if the order of q is not too small (see chapter 6, Theorem 6.6, actually we get more). In chapter 7 we shall show that (*) is not true if q is a root of 1 of certain orders. It is expected that (*) is not true only when q is one of those roots of 1 (see [L17]).

In this book we also show that cells in affine Weyl groups are interesting to understand representations of affine Hecke algebras.

Now we explain some details of the book.

In chapter 1 we give the definitions of Coxeter groups and of Hecke algebras. We also recollect some definitions and results in [KL1, L6], which will be needed later. In chapter 2 we give the definitions of extended affine Weyl groups and of affine Hecke algebras, and recall some results on cells in affine Weyl groups. Following Berstein, the center of an affine Hecke algebra is explicitly described. In chapter 3 we describe the lowest generalized two-sided cell of an affine Weyl group (Theorem 3.22). Naturally in chapter 4 we generalize Kato's result on q-analogue of weight multiplicity (see [Ka2]).

In chapter 5 we recall some work on Deligne-Langlands conjecture for Hecke algebras by Ginsburg [G1-G2], Kazhdan and Lusztig [KL4]. We give some discussions to the standard modules (in the sense of [KL4]). For type A_n it is not difficult to determine the dimensions of standard modules. We also state two conjectures, one is concerned with the based rings of cells in affine Weyl groups, and another is for simple modules of affine Hecke algebras with two parameters, which is an analogue of the conjecture (*). In chapter 6 we introduce an equivalence relations in $T \times \mathbb{C}^*$, where T is a maximal torus of a connected reductive group over $\mathbb{C}$. Combining some properties of the equivalence relation, results of Ginzburg and of Kazhdan & Lusztig in chapter 5, we prove that (*) is true when the order of q is not too small (Theorem 6.6). In chapter 7 we show that if q is a root of 1 of certain orders the conjecture (*) is not true by using some results in [Ka2] and in chapter 6.

In chapter 8 we unify the definitions of principal series representations in [M, 4.1.5; L2, 8.11] by means of two-sided cells of an affine Weyl group and also give some discussions to the representations. In chapter 9 we are interested in relations among affine Hecke algebras of the same root system. In chapter 10 we give some discussions to certain remarkable quotient algebras of $\mathbf{H}_q$.

The chapters 11 and 12 are based on preprints "The based rings of cells in affine Weyl groups of type $\tilde{G}_2$, $\tilde{B}_2$" and "Some simple modules of affine Hecke algebras" respectively. In chapter 11 we verify the conjecture in [L14] for cells in affine Weyl groups of type $\tilde{G}_2$, $\tilde{B}_2$. In chapetr 12 we show that the conjecture in [L14] is true for the second highest two-sided cell in an affine Weyl group. Once we know the structures of the based rings we can know the structures of the corresponding standard $\mathbf{H}_q$-modules. The explicit knowledge of based rings provides a way to compute the dimensions of simple $\mathbf{H}_q$-modules and their multiplicities in standard modules, also can be used to classify the simple $\mathbf{H}_q$-modules even though q is a root of 1. In chapter 11 we work out the dimensions of simple $\mathbf{H}_q$-modules for type $\tilde{A}_2$. An immediate consequence is that for type A_2 we see $\mathbf{H}_q \not\simeq \mathbf{H}_1 = \mathbb{C}[W]$ whenever q is not equal to 1. This leads to several questions.

I would like to thank Professor T.A. Springer for some helpful conversations. I am grateful to the referees for helpful comments. I wish to thank Professor J. Shi for several helpful comments. I am indebted to Ms. D. Baeumer for the helps in preparing the $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX file of the book. Particular thanks are due to the series editors of the Lecture Notes in Mathematics for kind and helpful correspondences. To my pleasure, after the acceptance Professor F. Hirzebruch kindly permits to publish the book as one volume of the Bonn subseries of Lecture Notes in Mathematics.

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Contents

1. Hecke Algebras

In this chapter we give the definitions of extended Coxeter groups and of their Hecke algebras and provide a few examples. Some definitions (such as these of Kazhdan-Lusztig polynomials and cells) and results in [KL1, L6] are recalled. We also show how to apply the definitions in [L6], which are generalizations of those in [KL1]. Several questions are proposed. We refer to [B, Hu] for more details about Coxeter groups and their Hecke algebras.

1.1. Basic definitions. A Coxeter group is a group W' which possesses a set $S = \{s_i\}_{i \in I}$ of generators subject to the relations

$$s_i^2 = 1, \qquad (s_is_j)^{m_{ij}} = 1 \ \ (i \neq j),$$

where $m_{ij} \in \{2, 3, 4, ..., \infty\}$. We also write m_{st} for m_{ij} when $s = s_i$ and $t = s_j$.

We call (W', S) a Coxeter system and S the set of distinguished generators or the set of simple reflections. Let l be the length function of W' and $\leq$ denote the usual partial order in W'.

In Lie theory we often need to consider extended Coxeter groups. If a group Ω acts on a Coxeter system (W', S), we then define a group structure on $W = \Omega \ltimes W'$ by $(\omega_1, w_1)(\omega_2, w_2) = (\omega_1\omega_2, \omega_2^{-1}(w_1)w_2)$. The group W is called an extended Coxeter group. For convenience we also call (W, S) an extended Coxeter group and (W', S) a Coxeter group. The length function l can be extended to W by defining $l(\omega w) = l(w)$, and the partial order $\leq$ can be extended to W by defining $\omega w \leq \omega' u$ if and only if $\omega = \omega'$, $w \leq u$, where $\omega', \omega \in \Omega$ and $w, u \in W'$. We denote the extensions again by l and $\leq$ respectively.

Let $\mathbf{q}_s^{\frac{1}{2}}$, $s \in S$ be indeterminates. We assume that $\mathbf{q}_s^{\frac{1}{2}} = \mathbf{q}_t^{\frac{1}{2}}$ if and only if s, t are conjugate in W. Let $\mathcal{A} = \mathbb{Z}[\mathbf{q}_s^{\frac{1}{2}}, \mathbf{q}_s^{-\frac{1}{2}}]_{s \in S}$ be the ring of all Laurant polynomials in $\mathbf{q}_s^{\frac{1}{2}}$, $s \in S$ with integer coefficients. The (generic) Hecke algebra $\mathcal{H}$ (over $\mathcal{A}$) of W is an associative $\mathcal{A}$-algebra. As an $\mathcal{A}$-module, $\mathcal{H}$ is free with a basis T_w, $w \in W$, and multiplication laws are

(1.1.1) $(T_s - \mathbf{q}_s)(T_s + 1) = 0$, if $s \in S$; $\qquad T_wT_u = T_{wu}$, if $l(wu) = l(w) + l(u)$.

The generic Hecke algebra of W actually can be defined over $\mathbb{Z}[\mathbf{q}_s]_{s \in S}$, but it is convenient to define it over $\mathcal{A}$ for introducing Kazhdan-Lusztig polynomials and for defining cells in W.

Let $\mathcal{H}'$ be the subalgebra of $\mathcal{H}$ generated by T_s, $s \in S$. Then the algebra $\mathcal{H}$ is isomorphic to the "twisted" tensor product $\mathbb{Z}[\Omega] \otimes_{\mathbb{Z}} \mathcal{H}'$ by assigning $T_{\omega w} \to \omega \otimes T_w$, where $\mathbb{Z}[\Omega]$ is the group algebra of Ω over $\mathbb{Z}$, and the multiplication in $\mathbb{Z}[\Omega] \otimes_{\mathbb{Z}} \mathcal{H}'$ is given by

$$(\omega \otimes T_w)(\omega' \otimes T_u) = \omega\omega' \otimes T_{\omega'^{-1}(w)}T_u.$$

Note that $s, t \in S$ may be conjugate in W but not conjugate in W', thus $\mathcal{H}'$ may not be the generic Hecke algebra of W' in the previous sense.

For an arbitrary $\mathcal{A}$-algebra $\mathcal{A}'$, the $\mathcal{A}'$-algebra $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{A}'$ is called a Hecke algebra.

Convention: For each element w in W we shall denote the image in $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{A}'$ of T_w by the same notation.

1.2. Two special choices of $\mathcal{A}'$ are of particular interests.

(a). Let $\mathbf{q}^{\frac{1}{2}}$ be an indeterminate and let $A = \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$ be the ring of all Laurant polynomials in $\mathbf{q}^{\frac{1}{2}}$ with integer coefficients. Choose integers c_s, $s \in S$ such that $c_s = c_t$ whenever s and t are conjugate in W. There is a unique ring homomorphism from $\mathcal{A}$ to A such that $\mathbf{q}_s^{\frac{1}{2}}$ maps to $\mathbf{q}^{\frac{c_s}{2}}$ for every $s \in S$. Thus A is an $\mathcal{A}$-algebra. The multiplication laws in the Hecke algebra $\mathcal{H} \otimes_{\mathcal{A}} A$ are (recall the convention at the end of 1.1)

(1.2.1) $(T_s - \mathbf{q}^{c_s})(T_s + 1) = 0, \quad \text{if } s \in S; \qquad T_w T_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u).$

(b). When all integers c_s $(s \in S)$ are 1, we denote the Hecke algebra $\mathcal{H} \otimes_{\mathcal{A}} A$ by H. The multiplication laws in H are

(1.2.2) $(T_s - \mathbf{q})(T_s + 1) = 0, \quad \text{if } s \in S; \qquad T_w T_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u).$

Sometimes H is also called the generic Hecke algebra of W (with one parameter). By now the Hecke algebra H and its various specializations $H \otimes_A A'$ are the most extensively studied Hecke algebras.

There is also a slight generalization of the Hecke algebra $\mathcal{H}$. Let R be a commutative ring with 1. For every s in S, choose u_s, $v_s \in R$ such that $u_s = u_t$, $v_s = v_t$ whenever s, t are conjugate in W. Then there exists a unique associative R-algebra $\tilde{\mathcal{H}}$, which is a free R-module with a basis T'_w, $w \in W$ and multiplication is given by

(1.2.3) $\quad T_s'^2 = u_s T'_s + v_s, \quad \text{if } s \in S; \qquad T'_w T'_u = T'_{wu}, \quad \text{if } l(wu) = l(w) + l(u).$

(see, e.g. [Hu]). It is often that the R-algebra $\tilde{\mathcal{H}}$ is actually a Hecke algebra. Suppose that v_s has a square root $v_s^{\frac{1}{2}}$ in R and $v_s^{\frac{1}{2}}$ is invertible in R. Further we assume that there exists an invertible element $u_s'^{\frac{1}{2}} \in R$ such that

(1.2.4)
$$u_s'^{\frac{1}{2}} - u_s'^{-\frac{1}{2}} = u_s v_s^{-\frac{1}{2}}.$$

Set $T''_s := u_s'^{\frac{1}{2}} v_s^{-\frac{1}{2}} T'_s$, then $T_s''^2 = (u'_s - 1)T''_s + u'_s$. In this case the algebra $\tilde{\mathcal{H}}$ is a Hecke algebra in the sense of 1.1.

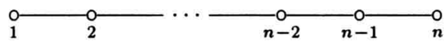
In Lie theory there are also other interesting algebras of Hecke type, see for example, [BM, Ca, MS].

1.3. Examples of Coxeter groups. It is convenient to represent a Coxeter system (W', S) by a graph Σ, usually called the Coxeter graph of (W', S). The vertex set of Σ is one to one correspondence with S; a pair of vertices corresponding to s_i, s_j are jointed with an edge whenever $m_{ij} \geq 3$, and label such an edge with m_{ij} when $m_{ij} \geq 4$. Thus the graph Σ determines (W', S) up to an isomorphism.

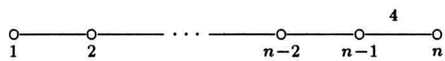
A Coxeter system (W', S) is called irreducible if for any $s, t \in S$ we can find a sequence $s = t_0, t_1, \cdots, t_k = t$ in S such that $m_{t_i, t_{i+1}} \geq 3$ (i.e. $t_i t_{i+1}$ is not equal to $t_{i+1} t_i$) for $i = 0, 1, ..., k-1$. We also call W' an irreducible Coxeter group when (W', S) is irreducible. Obviously every Coxeter group is a direct product of some irreducible Coxeter groups.

The most important Coxeter groups in Lie theory are Weyl groups and affine Weyl groups. They are classified. The Coxeter graphs of irreducible Weyl groups and irreducible affine Weyl groups are as follows.

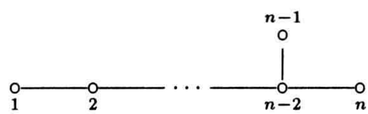
Type A_n $(n \geq 1)$.



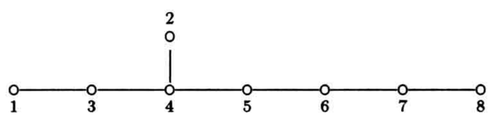
Type B_n $(n \geq 2)$.



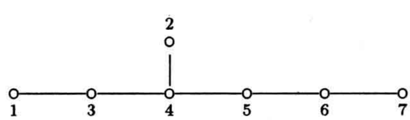
Type D_n $(n \geq 4)$.



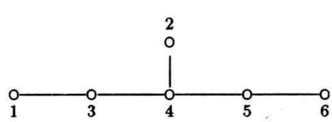
Type E_8.



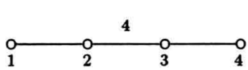
Type E_7.



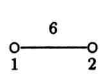
Type E_6.



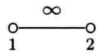
Type F_4.



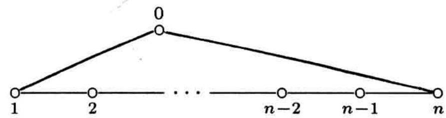
Type G_2.



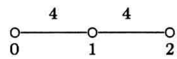
Type $\tilde{A}_1$.



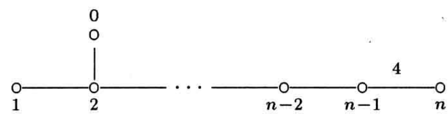
Type $\tilde{A}_n$ $(n \geq 2)$.



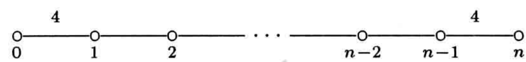
Type $\tilde{B}_2 = \tilde{C}_2$.



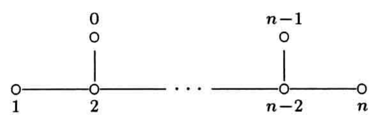
Type $\tilde{B}_n$ $(n \geq 3)$.



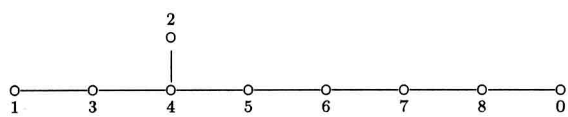
Type $\tilde{C}_n$ $(n \geq 3)$.



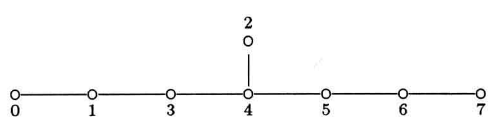
Type $\tilde{D}_n$ $(n \geq 4)$.



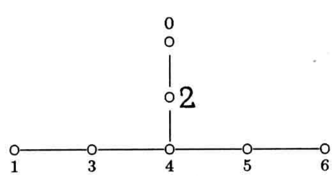
Type $\tilde{E}_8$.



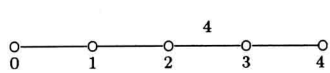
Type $\tilde{E}_7$.

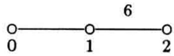


Type $\tilde{E}_6$.



Type $\tilde{F}_4$.

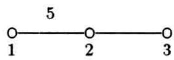


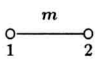
Type $\tilde{G}_2$. 

1.4. The Weyl group of type A_n is just the symmetric group $\mathfrak{S}_{n+1}$ of degree $n+1$. One may choose $\{(12),\ (23),\ \cdots, (n, n+1)\}$ as the set of simple reflections of $\mathfrak{S}_{n+1}$.

Except Weyl groups, the other irreducible finite Coxeter groups are dihedral groups $I_2(m)$ ($m = 5$ or $m > 6$, when $m = 3, 4, 6$, $I_2(m)$ are Weyl groups) and Coxeter groups of type H_3 or H_4. Their Coxeter graphs are as follows.

Type H_4. 

Type H_3. 

Type $I_2(m)$. 

When (W', S) is crystallographic (i.e., $m_{ij} = 2, 3, 4, 6, \infty$ for arbitrary s_i, s_j in S), W' can be realized as the Weyl group of certain Kac-Moody algebra (see [K]). Thus we have a Schubert variety $\mathcal{B}_w$ for each element $w \in W'$. This is a key to apply the powerful intersection cohomology theory to the Kazhdan-Lusztig theory.

1.5. Examples of Hecke algebras. (a). Let G be a Chevalley group over a finite field of q elements. Let B be a Borel subgroup of G and T the maximal torus in B. Then the group $W_0 = N_G(T)/T$ is a Weyl group. We have $G = \bigcup_{w \in W_0} BwB$. Let H be the free $\mathbb{Z}$-module generated by the double cosets BwB, $w \in W_0$. We denote T_w the double coset BwB when it is regarded as an element in H. Define the multiplication in H by

(1.5.1) $$T_w T_u = \sum_v m_{w,u,v} T_v,$$

where the structure constants $m_{w,u,v}$ are defined as the number of cosets of the form Bx in the set $Bw^{-1}Bv \cap BuB$:

$$m_{w,u,v} = |Bw^{-1}Bv \cap BuB/B|.$$

Then H is an associative ring with unit T_e, where e is the neutral element in W_0. Moreover we have

(1.5.2) $(T_s - q)(T_s + 1) = 0$, if $s \in S_0$; $\quad T_w T_u = T_{wu}$, if $l(wu) = l(w) + l(u)$,

where S_0 is the set of simple reflections in W_0. (See [I]).

It is well known that $H \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathrm{End} 1_B^G$, where 1_B^G stands for the induced representation of the unit representation 1_B (over $\mathbb{C}$) of B (see [I, C2, Cu]). Thus part of the study of 1_B^G can be reduced to that of $H \otimes_{\mathbb{Z}} \mathbb{C}$.

(b). Let K be a p-adic field such that its residue field k contains q elements. Let G be a Chevalley group over the field K. Let B be an Iwahori subgroup of G and T the 'diagonal' subgroup of B (in a suitable sense). Then the group $W = N_G(T)/T$ is an extended affine Weyl group (i.e., there is a commutative group Ω which acts on an affine Weyl group (W', S) such that $W \simeq \Omega \ltimes W'$, see 2.1 for definition). As the above example we have $G = \bigcup_{w\in W} BwB$. Let H be the free $\mathbb{Z}$-module generated by the double cosets BwB, $w \in W$. We denote T_w the double coset BwB when it is regarded as an element in H. Define the multiplication in H by

$$(1.5.3) \qquad T_wT_u = \sum_v m_{w,u,v}T_v,$$

where the structure constants $m_{w,u,v}$ are defined as the number of cosets of the form Bx in the set $Bw^{-1}Bv \cap BuB$:

$$m_{w,u,v} = |Bw^{-1}Bv \cap BuB/B|.$$

Then H is an associative ring with unit T_e, where e is the neutral element in W. Moreover we have

$$(1.5.4) \quad (T_s - q)(T_s + 1) = 0, \quad \text{if } s \in S; \qquad T_wT_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u),$$

where S is the set of simple reflections in W. (See [IM, p.44]).

It is known that the category of admissible complex representations of G which have nonzero vectors fixed by B is equivalent to the category of finite dimensional representations (over $\mathbb{C}$) of $H\otimes_{\mathbb{Z}}\mathbb{C}$ (see [Bo1, M]). Thus the representations of $H\otimes_{\mathbb{Z}}\mathbb{C}$ may be ragarded as an interesting part of the representation theory of p-adic groups.

1.6. Kazhdan-Lusztig polynomials. The work [KL1] stimulates a lot of work and deeply increased our understanding of Coxeter groups and of Hecke algebras. The key role is the Kazhdan-Lusztig polynomials. In this section we recall some definitions and results in [KL1].

We keep the notations in 1.1 and in 1.2 (b). Thus (W', S) is a Coxeter system, $W = \Omega \ltimes W'$ is an extended Coxeter group and H is the generic Hecke algebra of W over $A = \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$.

Let $a \to \bar{a}$ be the involution of the ring A defined by $\bar{\mathbf{q}}^{\frac{1}{2}} = \mathbf{q}^{-\frac{1}{2}}$. This extends to an involution $h \to \bar{h}$ of the ring H defined by

$$\overline{\sum a_wT_w} = \sum \bar{a}_wT^{-1}_{w^{-1}}, \quad a_w \in A.$$

Note that T_w is invertible for any $w \in W$ since $T_s^{-1} = \mathbf{q}^{-1}T_s + (\mathbf{q}^{-1} - 1)$ for $s \in S$ and $T_\omega^{-1} = T_{\omega^{-1}}$ for $\omega \in \Omega$. Then (see [KL1, (1.1.c)]):

(a) For each $w \in W$, there is a unique element $C_w \in H$ such that

$$\bar{C}_w = C_w,$$

$$C_w = \mathbf{q}^{-\frac{l(w)}{2}} \sum_{y \le w} P_{y,w} T_y,$$

where $P_{y,w} \in A$ is a polynomial in $\mathbf{q}$ of degree $\le \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w} = 1$.

The assertion (a) is equivalent to the following result.

(b) For each $w \in W$, there is a unique element $C'_w \in H$ such that $\bar{C}'_w = C'_w$ and $C'_w = \sum_{y \le w} (-1)^{l(w)-l(y)} \mathbf{q}^{\frac{l(w)}{2}} \mathbf{q}^{-l(y)} \bar{P}_{y,w} T_y$, where $P_{y,w} \in A$ is a polynomial in $\mathbf{q}$ of degree $\le \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w} = 1$.

Note that our notations C_w and C'_w exchange these in [KL1] since we shall mainly use the elements C_w.

Obviously the elements C_w, $w \in W$ form an A-basis of H and the elements C'_w, $w \in W$ also form an A-basis of H. They are related by three involutions.

(c) Let j be the involution of the ring H given by

$$j(\sum a_w T_w) = \sum \bar{a}_w (-\mathbf{q})^{-l(w)} T_w,$$

then $C'_w = (-1)^{l(w)} j(C_w)$. (See [KL1]).

(d) Let Φ be the involution of the ring H defined by

$$\Phi(\mathbf{q}^{\frac{1}{2}}) = -\mathbf{q}^{\frac{1}{2}}, \ \Phi(T_w) = (-\mathbf{q})^{l(w)} T_{w^{-1}}^{-1},$$

then $C'_w = \Phi(C_w)$. (See [L11, 3.2, p.259]).

(e) Let k be the involution of the A-algebra H given by

$$k(\sum a_w T_w) = \sum a_w (-\mathbf{q})^{l(w)} T_{w^{-1}}^{-1},$$

then $C'_w = (-1)^{l(w)} k(C_w)$.

We give a proof of (e). It is easy to see that

$$k(\sum a_w T_w) = j(\overline{\sum a_w T_w}).$$

That is, k is the composition of j and $\bar{\cdot}$. Since $\bar{C}_w = C_w$, according to (c) we get $k(C_w) = j(\bar{C}_w) = j(C_w) = (-1)^{l(w)} C'_w$.

Note that k is an involution of A-algebra, this fact is useful in transferring some properties of C'_w to C_w.

The polynomials $P_{y,w}$ are called Kazhdan-Lusztig polynomials. For $y < w$ we have $P_{y,w} = \mu(y,w)\mathbf{q}^{\frac{1}{2}(l(w)-l(y)-1)}$+lower degree terms. We say that $y \prec w$ if $\mu(y,w) \ne 0$, we then set $\mu(w,y) = \mu(y,w)$.

1.7. Motivated by his definition of canonical bases of quantum groups (see [L19]), Lusztig gave another construction of the elements C_w, C'_w. Consider the $\mathbb{Z}[\mathbf{q}^{-\frac{1}{2}}]$-submodule $\mathcal{L}$ of H spanned by $\tilde{T}_w = \mathbf{q}^{-\frac{l(w)}{2}} T_w$, $w \in W$ and the $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$-submodule $\mathcal{L}'$ of H spanned by $\tilde{T}_w$, $w \in W$, then (see [L20])

(a) The projection $\pi : \mathcal{L} \to \mathcal{L}/\mathbf{q}^{-\frac{1}{2}}\mathcal{L}$ gives rise to an isomorphism of $\mathbb{Z}$-module $\pi_1 : \mathcal{L} \cap \bar{\mathcal{L}} \tilde{\to} \mathcal{L}/\mathbf{q}^{-\frac{1}{2}}\mathcal{L}$ and $\pi_1^{-1}(\pi(\tilde{T}_w)) = C_w$.

(b) The projection $\pi' : \mathcal{L}' \to \mathcal{L}'/\mathbf{q}^{\frac{1}{2}}\mathcal{L}'$ gives rise to an isomorphism of $\mathbb{Z}$-module $\pi_1 : \mathcal{L}' \cap \bar{\mathcal{L}}' \tilde{\to} \mathcal{L}'/\mathbf{q}^{\frac{1}{2}}\mathcal{L}'$ and $\pi_1'^{-1}(\pi'(\tilde{T}_w)) = C'_w$.

1.8. The elements C_w have the following properties (see [KL1]):

(a) For $s \in S$ we have

$$C_s C_w = \begin{cases} (\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}})C_w, & \text{if } sw \leq w \\ C_{sw} + \sum\limits_{\substack{y \prec w \\ sy \leq y}} \mu(y,w) C_y, & \text{if } sw \geq w. \end{cases}$$

$$C_w C_s = \begin{cases} (\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}})C_w, & \text{if } ws \leq w \\ C_{ws} + \sum\limits_{\substack{y \prec w \\ ys \leq y}} \mu(y,w) C_y, & \text{if } ws \geq w. \end{cases}$$

They are equivalent to the following recursion formulas of the Kazhdan-Lusztig polynomials.

(b) Assume that for $s, t \in S$ we have $sw > w$ and $wt > w$, then

$$P_{y,sw} = \mathbf{q}^{1-a} P_{sy,w} + \mathbf{q}^a P_{y,w} - \sum_{\substack{z \\ y \leq z \prec w \\ sz < z}} \mu(z,w) \mathbf{q}^{\frac{l(w)-l(z)+1}{2}} P_{y,z}, \quad (y \leq sw)$$

where $a = 1$ if $sy < y$, $a = 0$ if $sy > y$; and $P_{y,sw} = P_{sy,sw}$.

$$P_{y,wt} = \mathbf{q}^{1-a} P_{yt,w} + \mathbf{q}^a P_{y,w} - \sum_{\substack{z \\ y \leq z \prec w \\ zt < z}} \mu(z,w) \mathbf{q}^{\frac{l(w)-l(z)+1}{2}} P_{y,z}, \quad (y \leq wt)$$

where $a = 1$ if $yt < y$, $a = 0$ if $yt > y$; and $P_{y,wt} = P_{yt,wt}$.

1.9. When (W', S) is a finite Coxeter group or a crystallographic group, it is known that the coefficients of $P_{y,w}$ are non-negative. This is proved in [KL2, L11] when (W', S) is crystallographic. For H_3, H_4 it was proved by Goresky [Go] and Alvis [A]. For dihedral groups I_m it is trivial since $P_{y,w} = 1$ for any $y \leq w$. It was conjectured in [KL1] that for an arbitrary Coxeter group the Kazhdan-Lusztig polynomials have non-negative coefficients.

1.10. Question. (i). *It is known that the Kazhdan-Lusztig polynomials of crystallographic Coxeter groups are related to middle intersection cohomology groups* *of Schubert varieties. Now what polynomials are related to other intersection cohomology groups of Schubert varieties?*

(ii). *If we loose the restriction on the degree of $P_{y,w}$ to* $\deg P_{y,w} \leq (l(w) - l(y))$, *what happen for the Kazhdan-Lusztig polynomials and the elements C_w.*

1.11. Cell For each element w in W we set

$$L(w) := \{s \in S \mid sw < w\},$$

$$R(w) := \{s \in S \mid ws < w\}.$$

Let w and u be elements in W', we say that $w \underset{L}{\leq} u$ (resp. $w \underset{R}{\leq} u$; $w \underset{LR}{\leq} u$) if there exists a sequence $w = w_0, w_1, ..., w_k = u$ in W' such that for $i = 1, 2, ...k$ we have $\mu(w_{i-1}, w_i) \neq 0$ and $L(w_{i-1}) \not\subseteq L(w_i)$ (resp. $R(w_{i-1}) \not\subseteq R(w_i)$; $L(w_{i-1}) \not\subseteq L(w_i)$ or $R(w_{i-1}) \not\subseteq R(w_i)$). Then for any $\omega, \omega' \in \Omega$ we say that $\omega w \underset{L}{\leq} \omega' u$ (resp. $w\omega \underset{R}{\leq} u\omega'$; $\omega w \underset{LR}{\leq} \omega' u$) if $w \underset{L}{\leq} u$ (resp. $w \underset{R}{\leq} u$; $w \underset{LR}{\leq} u$).

For elements x and y in W we write that $x \underset{L}{\sim} y$ (resp. $x \underset{R}{\sim} y$; $x \underset{LR}{\sim} y$) if $x \underset{L}{\leq} y \underset{L}{\leq} x$ (resp. $x \underset{R}{\leq} y \underset{R}{\leq} x$; $x \underset{LR}{\leq} y \underset{LR}{\leq} x$). The relations $\underset{L}{\leq}$, $\underset{R}{\leq}$, $\underset{LR}{\leq}$ are preorders in W. And the relations $\underset{L}{\sim}$, $\underset{R}{\sim}$, $\underset{LR}{\sim}$ are equivalence relations in W, the corresponding equivalence classes are called left cells, right cells, two-sided cells of W, respectively. The preorder $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}$; $\underset{LR}{\leq}$) induces a partial order on the set of left (resp. right; two-sided) cells of W, we denote it again by $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}$; $\underset{LR}{\leq}$).

When $W = W'$ is a Weyl group, the definitions of left cell and two-sided cell coincide with the definitions given by Joseph [J1-J2]. The cells in Weyl groups were extensively investigated by Barbasch, Lusztig, Joseph, Vogan, etc., and play an important role in the representation theory of finite groups of Lie type (see [L7]) and in the theory of primitive ideals of universal enveloping algebras of semisimple Lie algebras.

For affine Weyl groups, the structure of left cells and two-sided cells are determined for type $\tilde{A}_n$ (see [Sh1, L8]), rank 2, 3 (see [L11, Bé1, D]). Recently Shi found an algorithm, then he and his students determined the structure of cells in affine Weyl groups of type $\tilde{B}_4$, $\tilde{C}_4$, $\tilde{D}_4$ (see [Sh4, Sh5]). For type $\tilde{D}_4$, see also [Ch]. In [L11-L14] Lusztig obtained a series of important results concerned with cells in affine Weyl groups.

1.12. a-function For an extended Coxeter group W the function $a: W \to \mathbb{N}$ was introduced in [L11] and is a useful tool in cell theory and related topics.

Given $w, u \in W$, we write

$$C_w C_u = \sum_{v \in W} h_{w,u,v} C_v, \qquad h_{w,u,v} \in A.$$

For every $v \in W$, we define $a(v)$=the minimal non-negative integer i such that $\mathbf{q}^{\frac{i}{2}} h_{w,u,v}$ is in $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$ for any $w, u \in W$. If such i doesnot exist, we set $a(v) = \infty$.

For a finite Coxeter group, the function a is always bounded. A non-trivial fact is that a is bounded for an affine Weyl group (see [L11]). In [L12] Lusztig obtained some interesting results provided that a is bounded and W' is crystallographic.

Assume that (W', S) is a crystallographic group, then all $h_{w,u,v}$ are Laurant polynomials in $\mathbf{q}^{\frac{1}{2}}$ with the same purity and have non-negative coefficients (see [L11]). It seems natural to hope such property holds for arbitrary Coxeter groups.

Here are four questions.

1.13. Question. (i). *Find out all Coxeter groups whose a-functions are bounded.*

(ii). *Assume that the a-function of a Coxeter group W is bounded and let a_0 be the maximal value of a on W. Is the set $\{w \in W \mid a(w) = a_0\}$ a two-sided cell of W?*

(iii). *Find out a Coxeter group W' such that there exists some $w \in W'$ with $a(w) = \infty$.*

(iv). *Maybe the a-function of a Coxeter group W is always bounded and the maximal value is equal to the length of the longest elements of certain finite Coxeter (or parabolic) subgroups of W.*

Generalized Cells

1.14. Lusztig generalized the definition of cells in [KL1] to the cases of simple reflections being given different weights (see [L6]). Strangely the interesting generalization is less developed. In the rest of the chapter we shall give some discussions to the generalization. We first recall the definition, then show how to apply the definition.

Let (W', S) be a Coxeter system and $W = \Omega \ltimes W'$ be an extended Coxeter group. Let $\varphi: W \to \Gamma$ be a map from W into an abelian group Γ such that $\varphi(\omega s_1 s_2 \cdots s_k \omega') = \varphi(s_1)\varphi(s_2)\cdots\varphi(s_k)$ for any reduced expression $s_1 s_2 \cdots s_k$ in W' and ω, ω' in Ω. Note that $\varphi(w) = \varphi(w')$ whenever w, w' are conjugate in W. For each w in W we shall write $\mathbf{q}_w^{\frac{1}{2}}$ for $\varphi(w)$. Let H_φ be the Hecke algebra of W with respect to φ; this is an associative algebra over the group ring $\mathbb{Z}[\Gamma]$. As a $\mathbb{Z}[\Gamma]$-module, it is free with a basis T_w, $w \in W$. The multiplication is defined by

$$(1.14.1)\quad (T_s - \mathbf{q}_s)(T_s + 1) = 0, \quad \text{if } s \in S; \qquad T_w T_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u).$$

When $\mathbf{q}_s^{\frac{1}{2}} = \mathbf{q}_t^{\frac{1}{2}}$ if and only if s, t are conjugate in W and Γ is a free abelian group with a basis $\mathbf{q}_s^{\frac{1}{2}}$, $s \in S$, the algebra H_φ is canonically isomorphic to the algebra $\mathcal{H}$ in 1.1 if we identify $\mathbb{Z}[\Gamma]$ with $\mathcal{A}$. When $\mathbf{q}_s^{\frac{1}{2}} = \mathbf{q}_t^{\frac{1}{2}}$ for any $s, t \in S$, and Γ is a free abelian group generated by $\mathbf{q}_s^{\frac{1}{2}}$, the algebra H_φ is canonically isomorphic to the