Rotating fields in general relativity

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Preface

This book introduces the reader to research work on a particular aspect of rotating fields in general relativity. It should be accessible to someone with an elementary knowledge of general relativity, such as that obtained in an undergraduate course on general relativity at a British university. A person with some maturity in mathematical physics may be able to follow it without knowing general relativity, as I have given a brief introduction to the relevant aspects of general relativity in Chapter 1.

My intention has been to write a short book which can provide a relatively quick entry into some research topics. I have therefore made only a brief mention of some topics such as the important group theoretic generation of solutions by Kinnersley and others. A significant part of this book deals with interior solutions, for which these techniques are not yet applicable. I have also not touched upon Petrov classification of solutions as this is marginal to the problems considered in this book. The connecting link of the topics considered here is the Weyl–Lewis–Papapetrou form of the stationary axially symmetric metric, which is derived in detail in Chapters 1 and 2.

A significant part of the book is based on my own work and for this reason the book may be considered as too specialized. However, all research is specialized and I believe it is instructive for the beginning research worker to be shown a piece of work carried out to a certain stage of completion. Besides, I have tried, wherever possible, to bring out points of general interest.

In the earlier parts of the book I have usually carried out calculations explicitly. In the later parts I have left gaps which the reader is urged to fill him or herself. I frequently found myself running out of letters to use as symbols. I have sometimes used the same letters in different parts of the book, the different uses of which should be obvious from the context. I hope the reader will not mind this minor inconvenience.

Most of this book was written and much of the work on which parts of the book are based was done while I was at the City University, London. I am grateful to Prof. M.A. Jaswon and other members of the Mathematics Department there for their support. The book was completed during a visit to the Institute for Advanced Study, Princeton, New Jersey. I am grateful to Prof. F.J. Dyson and Prof. H. Woolf for hospitality there. I thank Alison Buttery, Catharine Rhubart, Ceinwen Sanderson and Veola Williams for typing parts of the manuscript. Lastly, I thank my wife Suraiya and my daughters Sadaf and Nargis for support and encouragement during the period in which this book was written.

Jamal Nazrul Islam

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Introduction

1.1. Newtonian theory of gravitation

In Newton's theory of gravitation any sufficiently small piece of matter attracts any other sufficiently small piece of matter with a force which is inversely proportional to the square of the distance between the two pieces and which is proportional to the product of their masses. The constant of proportionality is G, Newton's gravitational constant. In this book we shall use units such that $G = 1$. From the inverse square law one can deduce that the gravitational field of a distribution of mass is described completely by a single function Φ of position, say of cartesian coordinates (x, y, z) and possibly of time t, which satisfies Poisson's equation inside matter, as follows:

$$\nabla^2\Phi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(x, y, z, t) = -4\pi\varepsilon(x, y, z, t), \tag{1.1}$$

where ε is the density of mass, which is also a function of x, y, z and possibly t. Outside matter, in empty space, Φ satisfies Laplace's equation, as follows:

$$\nabla^2\Phi = 0, \tag{1.2}$$

a solution of which we will refer to as a harmonic function. The function Φ is referred to as the gravitational potential and has the physical significance that a particle of mass m placed in a gravitational field at the point (x, y, z) experiences a force $\mathbf{F}$ given by

$$\mathbf{F} = m\nabla\Phi. \tag{1.3}$$

Equation (1.1) can be solved using the standard integral representation

$$\Phi(x, y, z, t) = \int \frac{\varepsilon(x', y', z', t)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'}{[(x - x')^2 + (y - y')^2 + (z - z)^2]^{1/2}}, \tag{1.4}$$

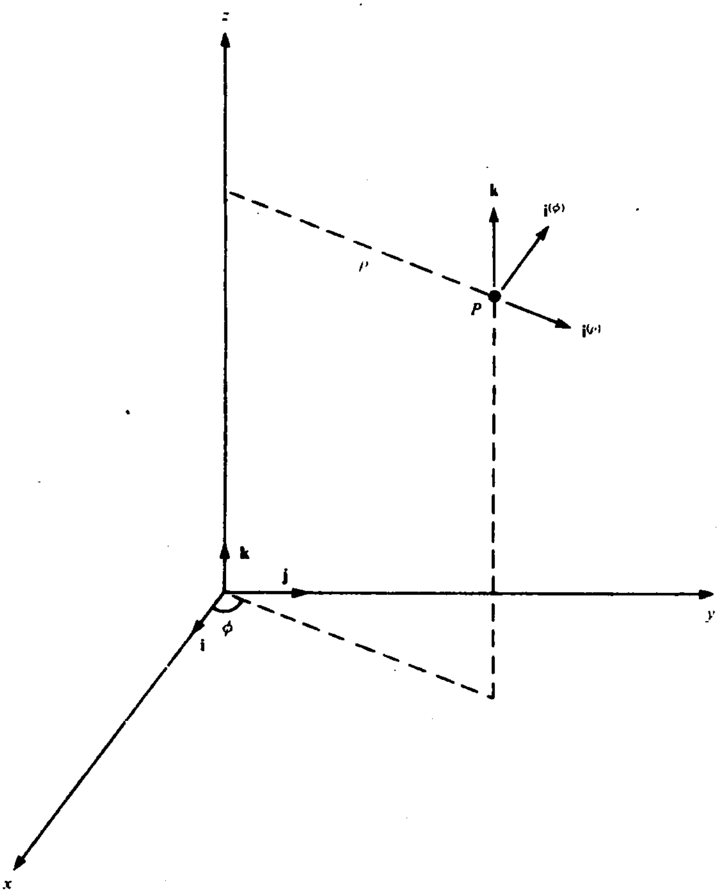
where the integration is over the region in which ε is non-zero. However, (1.4) is usually difficult to evaluate and other simpler methods are used to arrive at the potential Φ, especially in situations involving symmetries.

In this book we shall be concerned with situations of axial or cylindrical symmetry. These symmetries have to be defined rigorously in general relativity, but it is useful to have in mind the simpler definitions of these symmetries in the Newtonian situation. Even here we shall take a somewhat intuitive approach lacking in rigour but it will be simple and adequate for our purpose. We first define cylindrical polar coordinates. A point P has cylindrical polar coordinates (ρ, ϕ, z) related to its cartesian coordinates (x, y, z) as follows:

$$x = \rho \cos \phi, \quad y = \rho \sin \phi. \tag{1.5}$$

Thus ρ is the distance of the point P from the z-axis and ϕ is the angle which

Fig. 1.1. Illustration of cylindrical polar coordinates.



the plane containing the z-axis and the point P makes with the plane $y = 0$ (Fig. 1.1). The angle ϕ is often referred to as the azimuthal angle. A scalar function of position $f(x, y, z)$, is said to be axially symmetric (with the z-axis as the axis of symmetry) if, when expressed in terms of coordinates (ρ, ϕ, z), it is independent of ϕ, that is,

$$f(\rho\cos\phi, \rho\sin\phi, z) = F(\rho, z), \tag{1.6}$$

where $F(\rho, z)$ is a function of ρ and z only. Thus axially symmetric functions have rotational symmetry about the z-axis. There are different ways of interpreting this last statement. Consider a circle passing through P with its centre on the z-axis and its plane parallel to the plane $z = 0$. An axially symmetric function has the same value at all points of this circle, since this circle is described by fixed values of ρ and z. Thus the surfaces $F(\rho, z) =$ constant have the property that they look the same in position and shape if they are rotated by any given angle about the z-axis, that is, they are invariant under a rotation of the coordinate system about the z-axis.

Consider now a three-dimensional vector field $\mathbf{H}(x, y, z)$ with components $H^{(x)}$, $H^{(y)}$, $H^{(z)}$ along the x, y and z-axis respectively, that is,

$$\mathbf{H}(x, y, z) = \mathbf{i}H^{(x)}(x, y, z) + \mathbf{j}H^{(y)}(x, y, z) + \mathbf{k}H^{(z)}(x, y, z), \tag{1.7}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the x, y, z axes respectively. The vector field $\mathbf{H}$ is axially symmetric if its components when expressed in terms of the triad of unit vectors $(\mathbf{i}^{(\rho)}, \mathbf{i}^{(\phi)}, \mathbf{k})$ are independent of ϕ. The unit vectors $\mathbf{i}^{(\rho)}$ and $\mathbf{i}^{(\phi)}$ are defined as follows:

$$\mathbf{i}^{(\rho)} = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi, \quad \mathbf{i}^{(\phi)} = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi. \tag{1.8}$$

The vector $\mathbf{i}^{(\rho)}$ points radially away from the z-axis and is parallel to the plane $z = 0$, while $\mathbf{i}^{(\phi)}$ is also parallel to this plane and is perpendicular to $\mathbf{i}^{(\rho)}$, pointing in the direction of increasing ϕ (Fig. 1.1). Thus $\mathbf{H}$ is axially symmetric if

$$\mathbf{H} = \mathbf{i}^{(\rho)}H^{(\rho)}(\rho, z) + \mathbf{i}^{(\phi)}H^{(\phi)}(\rho, z) + \mathbf{k}H^{(z)}(\rho, z). \tag{1.9}$$

Note that if $\mathbf{H}$ is axially symmetric (so that $H^{(\rho)}, H^{(\phi)}, H^{(z)}$ are independent of ϕ) its components in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ do depend on ϕ in the following simple manner:

$$\mathbf{H} = \mathbf{i}(\cos\phi H^{(\rho)} - \sin\phi H^{(\phi)}) + \mathbf{j}(\sin\phi H^{(\rho)} + \cos\phi H^{(\phi)}) + \mathbf{k}H^{(z)}. \tag{1.10}$$

From (1.9) it is clear that an axially symmetric vector field is in a suitable sense invariant under rotation of the axes about the z-axis by any given angle.

Cylindrical symmetry can be defined in the same manner as above if we make the various functions independent of z in addition to being independent of ϕ. Thus a scalar function $f(x, y)$ (which is independent of z) is cylindrically symmetric (with the z-axis as the axis of symmetry) if, when expressed in terms of coordinates ρ, ϕ, z it is a function of ρ only:

$$f(\rho\cos\phi, \rho\sin\phi) = F(\rho). \tag{1.11}$$

Thus if a and b are constants, the function $a\rho^2$ is cylindrically symmetric but the function $a\rho^2 + bz^2$, although axially symmetric, is not cylindrically symmetric. The vector field **H** is cylindrically symmetric if the components $H^{(\rho)}$, $H^{(\phi)}$ and $H^{(z)}$ are functions of ρ only. Cylindrically symmetric systems are invariant not only under a rotation about the z-axis, but they are also invariant under a translation parallel to the z-axis. These remarks may seem obvious but they help to fix ideas for the more complicated situations encountered later in the book and besides, confusion does arise sometimes between axial and cylindrical symmetry.

As the title of the book implies, we shall be concerned with rotating systems. It is therefore pertinent to consider a simple rotating system in Newtonian theory, namely, the case of uniformly (rigidly) rotating inviscid homogeneous fluid, where the rotation is steady, that is, independent of time. As is well known, the boundary of such a fluid mass is an oblate spheroid, which is an ellipsoid with two equal axes, these equal axes being greater than the third axis, A prolate spheroid is one in which the equal axes are smaller than the third axis. A typical portion of the material of the fluid mass is kept in equilibrium by gravitational, pressure and centrifugal forces. We need not concern ourselves with the equations governing these forces (see, for example, Chandrasekhar, 1969). If we assume the centre of the spheroid to be at the centre of the coordinate system, the gravitational potential inside the matter is given by

$$\Phi(\rho, z) = a\rho^2 + bz^2 + \Phi_0, \tag{1.12}$$

where a, b, Φ_0 are constants, and Φ is given by a more complicated but explicitly known function of ρ and z outside the matter. From (1.1) we see that a, b and ε (which is constant in this case) are related by

$$\nabla^2\Phi \equiv \Phi_{\rho\rho} + \rho^{-1}\Phi_\rho + \Phi_{zz} = 4a + 2b = -4\pi\varepsilon, \tag{1.13}$$

where $\Phi_\rho \equiv \partial\Phi/\partial\rho$, etc. The interior and exterior potentials join smoothly at the boundary of the matter distribution, where the pressure is zero. By joining smoothly we mean that the potential and its partial derivatives with respect to ρ and z are continuous at the boundary.

We will now describe one of the fundamental differences between a Newtonian rotating system and a general relativistic one. Let a test particle of mass m be released from rest at a great distance from the rotating mass considered above in the equatorial plane $z = 0$ (Fig. 1.2). According to (1.3) the force on the test particle is given by

$$\mathbf{F} = m\nabla\Phi = \left\{ m\left(\mathbf{i}^{(\rho)}\frac{\partial}{\partial\rho} + \mathbf{i}^{(\phi)}\rho^{-1}\frac{\partial}{\partial\phi} + \mathbf{k}\frac{\partial}{\partial z}\right)\Phi \right\}_{z=0} \tag{1.14}$$

Since the system is axially symmetric about the z-axis, Φ is independent of ϕ so that the coefficient of $\mathbf{i}^{(\phi)}$ in (1.14) vanishes. This is true for all z and not just for $z = 0$. Thus for any position of the test particle there is no transverse force on the particle in the $\mathbf{i}^{(\phi)}$ direction. Returning to (1.14), the system has reflection symmetry about the plane $z = 0$, so Φ depends on z through z^2 only (recall that (1.12) is just the interior potential; the exterior potential is given by a different function), so that

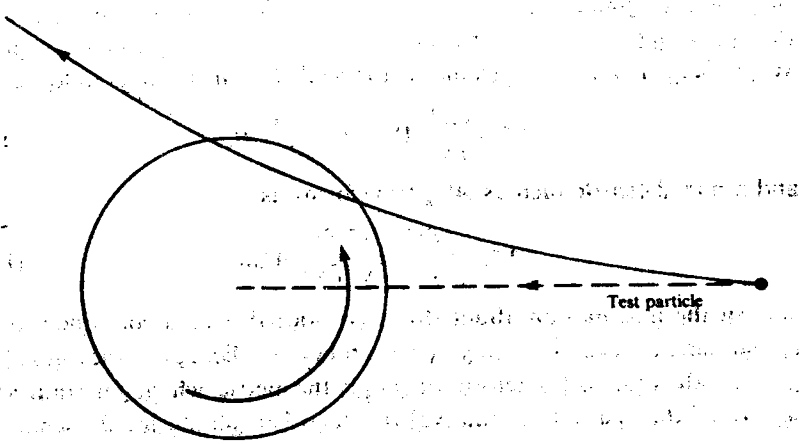
$$\left\{\frac{\partial\Phi}{\partial z}\right\}_{z=0} = \left\{2z\frac{\partial\Phi}{\partial z^2}\right\}_{z=0} = 0. \tag{1.15}$$

Thus

$$\mathbf{F} = m\frac{\partial\Phi}{\partial\rho}(\rho, 0)\mathbf{i}^{(\rho)}, \tag{1.16}$$

so that the force on the test particle is radial and it will follow the dashed straight line through the centre of the mass (Fig. 1.2). Similar considerations also apply for any rotating body which has axial symmetry and reflection symmetry on the plane $z = 0$, but we consider the above example for definiteness. Considering now the corresponding situation in general relativity, that is, the case of a steadily and rigidly rotating homogeneous

Fig. 1.2. Path of a test particle released from rest in the equatorial plane of a rotating mass in Newtonian gravitation (dashed line) and in general relativity (continuous line).



inviscid fluid, the first thing to note is that the surface of the mass is no longer a spheroid. In fact due to the non-Euclidean nature of the geometry it is difficult to characterize the surface in coordinates, but it is still true that the pressure vanishes on this surface. The concept of the equatorial plane and the fact that the system has reflection symmetry about this plane can be taken over to general relativity and one can ask what will happen to a test particle if it is released from rest on the equatorial plane. In this case there will be a transverse force on the test particle because of the rotation of the central mass and the particle will follow the continuous line (Fig. 1.2). This phenomenon is related to what is referred to as 'inertial dragging' and will be discussed later. Thus in general relativity matter in motion exerts a force akin to magnetic forces exerted by electric charges in motion. This is not true in Newtonian gravitation.

1.2. Summary of general relativity

The reader is assumed to be familiar with the elements of general relativity but we shall give here a brief review as a reminder of the main results and to collect together formulae some of which will be useful later in the book.

General relativity is formulated in a four-dimensional Riemannian space in which points are labelled by a general non-inertial coordinate system (x^0, x^1, x^2, x^3), often written as x^μ $(\mu = 0, 1, 2, 3)$ (we use the covention that Greek indices take values 0, 1, 2, 3 and repeated Greek indices are to be summed over these values unless otherwise stated; the meaning of other indices will be specified as they arise). Several coordinate patches may be necessary to cover the whole of space–time. The space has three spatial and one time-like dimension. Under a coordinate transformation from x^μ to x'^μ (in which each x'^μ is in general a function of x^0, x^1, x^2, x^3) a contravariant vector field A^μ and a covariant vector field B_μ transform as follows:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu, \quad B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu, \tag{1.17}$$

and a mixed tensor such as $A^\mu{}_{\nu\lambda}$ transforms as

$$A'^\mu{}_{\nu\lambda} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\tau}{\partial x'^\lambda} A^\rho{}_{\sigma\tau}, \tag{1.18}$$

etc. All the information about the gravitational field is contained in the second rank covariant tensor $g_{\mu\nu}$ (the number of indices gives the rank of the tensor) called the metric tensor, or simply the metric, which determines the square of the space–time interval ds^2 between infinitesimally separated events or points x^μ and $x^\mu + dx^\mu$ as follows ($g_{\mu\nu} = g_{\nu\mu}$):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.19}$$

The contravariant tensor corresponding to $g_{\mu\nu}$ is denoted by $g^{\mu\nu}$ and is defined by

$$g_{\mu\nu} g^{\nu\lambda} = \delta^\lambda_\mu \tag{1.20}$$

where δ^λ_μ is the Kronecker delta, which equals unity if $\lambda = \mu$ (no summation) and zero otherwise. Indices can be raised or lowered by using the metric tensor as follows:

$$A^\mu = g^{\mu\nu} A_\nu, \quad A_\mu = g_{\mu\nu} A^\nu. \tag{1.21}$$

The generalization of ordinary (partial) differentiation to Riemannian space is given by covariant differentiation denoted by a semicolon and defined for a contravariant and a covariant vector as follows:

$$A^\mu{}_{;\nu} = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda} A^\lambda, \tag{1.22a}$$

$$A_{\mu;\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} A_\lambda. \tag{1.22b}$$

Here the $\Gamma^\mu_{\nu\lambda}$ are called Christoffel symbols; they have the property $\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu}$ and are given in terms of the metric tensor as follows:

$$\Gamma^\mu_{\nu\lambda} = \tfrac{1}{2} g^{\mu\sigma}(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\nu\lambda,\sigma}), \tag{1.23}$$

where a comma denotes partial differentiation with respect to the corresponding variable: $g_{\sigma\nu,\lambda} \equiv \partial g_{\sigma\nu}/\partial x^\lambda$. For covariant differentiation of tensors of higher rank, there is a term corresponding to each contravariant index analogous to the second terms in (1.22a) and a term corresponding to each covariant index analogous to the second term in (1.22b) (with a negative sign). Equation (1.23) has the consequence that the covariant derivative of the metric tensor vanishes:

$$g_{\mu\nu;\lambda} = 0, \quad g^{\mu\nu}{}_{;\lambda} = 0. \tag{1.24}$$

Under a coordinate transformation from x^μ to x'^μ the $\Gamma^\mu_{\nu\lambda}$ transform as follows:

$$\Gamma'^\mu_{\nu\lambda} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\tau}{\partial x'^\lambda} \Gamma^\rho_{\sigma\tau} + \frac{\partial^2 x^\sigma}{\partial x'^\nu \partial x'^\lambda} \frac{\partial x'^\mu}{\partial x^\sigma}, \tag{1.25}$$

so that the $\Gamma^\mu_{\nu\lambda}$ do not form components of a tensor since the transformation law (1.25) is different to that of a tensor (see (1.18)).

For any covariant vector A_μ it can be shown that

$$A_{\mu;\nu;\lambda} - A_{\mu;\lambda;\nu} = A_\sigma R^\sigma{}_{\mu\nu\lambda}, \tag{1.26}$$

where $R^{\sigma}{}_{\mu\nu\lambda}$ is the Riemann tensor defined by

$$R^{\sigma}{}_{\mu\nu\lambda}=\Gamma^{\sigma}_{\mu\lambda,\nu}-\Gamma^{\sigma}_{\mu\nu,\lambda}+\Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda}-\Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu}. \tag{1.27}$$

The Riemann tensor has the following symmetry properties:

$$R_{\sigma\mu\nu\lambda}=-R_{\mu\sigma\nu\lambda}=-R_{\sigma\mu\lambda\nu}, \tag{1.28a}$$

$$R_{\sigma\mu\nu\lambda}=R_{\nu\lambda\sigma\mu}, \tag{1.28b}$$

$$R_{\sigma\mu\nu\lambda}+R_{\sigma\lambda\mu\nu}+R_{\sigma\nu\lambda\mu}=0, \tag{1.28c}$$

and satisfies the Bianchi identity:

$$R^{\sigma}{}_{\mu\nu\lambda;\rho}+R^{\sigma}{}_{\mu\rho\nu;\lambda}+R^{\sigma}{}_{\mu\lambda\rho;\nu}=0. \tag{1.29}$$

The Ricci tensor $R_{\mu\nu}$ is defined by

$$R_{\mu\nu}=g^{\lambda\sigma}R_{\lambda\mu\sigma\nu}=R^{\sigma}{}_{\mu\sigma\nu}. \tag{1.30}$$

From (1.27) and (1.30) it follows that $R_{\mu\nu}$ is given by

$$R_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu,\lambda}-\Gamma^{\lambda}_{\mu\lambda,\nu}+\Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma}-\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}. \tag{1.31}$$

Let the determinant of $g_{\mu\nu}$ considered as a matrix be denoted by g. Then another expression for $R_{\mu\nu}$ is given as follows:

$$R_{\mu\nu}=\frac{1}{(-g)^{1/2}}[\Gamma^{\lambda}_{\mu\nu}(-g)^{1/2}]_{,\lambda}-[\log(-g)^{1/2}]_{,\mu\nu}-\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}. \tag{1.31a}$$

This follows from the fact that from (1.23) and the properties of matrices one can show that

$$\Gamma^{\lambda}_{\mu\lambda}=[\log(-g)^{1/2}]_{,\mu}. \tag{1.32}$$

From (1.31a) it follows that $R_{\mu\nu}=R_{\nu\mu}$. There is no agreed convention for the sign of the Riemann and Ricci tensors – some authors define these with opposite sign to (1.27) and (1.31). The Ricci scalar R is defined by

$$R=g^{\mu\nu}R_{\mu\nu}. \tag{1.33}$$

By contracting the Bianchi identity on the pair of indices $\mu\nu$ and $\sigma\rho$ (that is, multiplying it by $g^{\mu\nu}$ and $g^{\sigma\rho}$) one can deduce the identity

$$(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R)_{;\nu}=0. \tag{1.34}$$

The tensor $G^{\mu\nu}\equiv(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)$ is sometimes called the Einstein tensor.

We are now in a position to write down the fundamental equations of general relativity. These are Einstein's equations given by

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=8\pi T_{\mu\nu}, \tag{1.35}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the source producing the gravitational field. In (1.35) and throughout the following we use units such that Newton's gravitational constant and the velocity of light are both equal to unity. For a perfect fluid $T_{\mu\nu}$ takes the following form:

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu}, \tag{1.36}$$

where ε is the mass-energy density (note the contrast to (1.1) where ε is just the mass density), p is the pressure and u^μ is the four-velocity of matter given by

$$u^\mu = \frac{dx^\mu}{ds}, \tag{1.37}$$

where $x^\mu(s)$ describes the world-line of the matter in terms of the proper time s along the world-line. The energy-momentum tensor for the electromagnetic field will be considered in a later chapter. From (1.34) we see that Einstein's equations (1.35) are compatible with the following equation

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{1.38}$$

which is the equation for the conservation of mass-energy and momentum.

The equations of motion of a particle in a gravitational field are given by the geodesic equations:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0. \tag{1.39}$$

Geodesics can also be introduced through the concept of parallel transfer. We will consider this concept and related matters in the next section.

This completes our brief survey of some of the elements of general relativity that we shall assume knowledge of in the following. In the rest of this chapter we shall cover some additional topics which are pertinent to the following chapters.

1.3. Curves in Riemannian space

A curve in Riemannian space is defined by points $x^\mu(\lambda)$ where x^μ are suitably differentiable functions of the real parameter λ, varying over some interval of the real line. Consider a coordinate transformation from x^μ to x'^μ. The set of quantities $dx^\mu/d\lambda$ transform to $dx'^\mu/d\lambda$ given by

$$\frac{dx'^\mu}{d\lambda} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\lambda}, \tag{1.40}$$

that is, $dx^\mu/d\lambda$ are components of a contravariant vector. This is called the tangent vector to the curve. The curve, or a portion of it, is time-like, light-like or space-like according as to whether $g_{\mu\nu}\, dx^\mu/d\lambda\, dx^\nu/d\lambda > 0$, $= 0$, or < 0.

(At any point $g_{\mu\nu}$ can be reduced to the diagonal form $(1, -1, -1, -1)$ by a suitable transformation.) The length of the time-like or space-like curve from $\lambda = \lambda_1$ to $\lambda = \lambda_2$ is given by:

$$L_{12} = \int_{\lambda_1}^{\lambda_2} \left(\left| g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right| \right)^{1/2} d\lambda. \tag{1.41}$$

The length of a light-like curve is zero. For an arbitrary vector field Y^μ its covariant derivative along the curve is the vector (defined along the curve) $Y^\mu{}_{;\nu}(dx^\nu/d\lambda)$. A similar definition can be given for the covariant derivative of an arbitrary tensor field along the curve. The vector field Y^μ is said to be parallelly transported along the curve if

$$\begin{aligned} Y^\mu{}_{;\nu}\frac{dx^\nu}{d\lambda} &= Y^\mu{}_{,\nu}\frac{dx^\nu}{d\lambda} + \Gamma^\mu_{\nu\sigma} Y^\sigma \frac{dx^\nu}{d\lambda} \\ &= \frac{dY^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma} Y^\sigma \frac{dx^\nu}{d\lambda} = 0. \end{aligned} \tag{1.42}$$

A similar definition holds for tensors. Given any curve $x^\mu(\lambda)$ with end points $\lambda = \lambda_1$ and $\lambda = \lambda_2$ the theory of solutions of ordinary differential equations shows that if the $\Gamma^\mu_{\nu\sigma}$ are suitably differentiable functions of the x^μ one obtains a unique tensor at $\lambda = \lambda_2$ by parallelly transferring any given tensor from $\lambda = \lambda_1$ along $x^\mu(\lambda)$, if the latter are differentiable in λ. A particular case is the covariant derivative of the tangent vector itself along $x^\mu(\lambda)$. The curve is said to be a geodesic curve if the tangent vector is transported parallelly along the curve, that is (putting $Y^\mu = dx^\mu/d\lambda$ in (1.42)) if

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{1.43}$$

A geodesic, or a portion of it, can be time-like, light-like or space-like according to the type of curve it is.

Two vector fields V^μ, W^μ are normal or orthogonal to each other if $g_{\mu\nu}V^\mu W^\nu = 0$. If V^μ is time-like and orthogonal to W^μ then the latter is necessarily space-like. A space-like three-surface is a surface defined by $f(x^0, x^1, x^2, x^3) = 0$ such that $g^{\mu\nu}f_{,\mu}f_{,\nu} > 0$ when $f = 0$. The unit normal vector to this surface is given by $n^\mu = (g^{\alpha\beta}f_{,\alpha}f_{,\beta})^{-1/2}\, g^{\mu\nu}f_{,\nu}$.

Given a vector field ξ^μ, one can define a set of curves filling all space such that the tangent vector to any curve of this set at any point coincides with the value of the vector field at that point. This is done by solving the set of first order differential equations.

$$\frac{dx^\mu}{d\lambda} = \xi^\mu(x(\lambda)), \tag{1.44}$$

where on the right hand side we have put x for all four components of the coordinates. This set of curves is referred to as the congruence of curves generated by the given vector field. In general there is a unique member of this congruence passing through any given point. A particular member of the congruence is sometimes referred to as an orbit. Consider now the vector field given by $(\xi^0, \xi^1, \xi^2, \xi^3) = (1, 0, 0, 0)$. From (1.44) we see that the congruence of this vector field is the set of curves given by

$$(x^0 = \lambda, x^1 = \text{constant}, x^2 = \text{constant}, x^3 = \text{constant}). \tag{1.45}$$

This vector field is also referred to as the vector field $\partial/\partial x^0$. One similarly defines the vector fields $\partial/\partial x^1$, $\partial/\partial x^2$, $\partial/\partial x^3$. That is, corresponding to the coordinate system x^μ we have the four contravariant vector fields $\partial/\partial x^\mu$. A general vector field X^μ can be written without components in terms of $\partial/\partial x^\mu$ as follows:

$$\mathbf{X} = X^\mu \frac{\partial}{\partial x^\mu}. \tag{1.46}$$

This is related to the fact that contravariant vectors at any point can be regarded as operators acting on differentiable functions $f(x^0, x^1, x^2, x^3)$; when the vector acts on the function, the result is the derivative of the function in the direction of the vector field, as follows:

$$\mathbf{X}(f) = X^\mu \frac{\partial f}{\partial x^\mu}. \tag{1.47}$$

As is well known, differential geometry and, correspondingly, general relativity can be developed independently of coordinates and components. We shall not be concerned with this approach except incidentally. Whenever we use this approach we shall specify the relevant points as we have done in (1.47).

1.4. Killing vectors

Einstein's exterior equations $R_{\mu\nu} = 0$ (obtained from (1.35) by setting $T_{\mu\nu} = 0$) are a set of coupled non-linear partial differential equations for the ten unknown functions $g_{\mu\nu}$. The interior equations (1.35) may involve other unknown functions such as the mass-energy density and the pressure. Because of the freedom to carry out general coordinate transformations one can in general impose four conditions on the ten functions $g_{\mu\nu}$. Later we will show explicitly how this is done in a case involving symmetries. In most situations of physical interest one has space–time symmetries which reduce further the number of unknown functions. To determine the simplest form