Waves and Stability in Continuous Media

Series on Advances in
Mathematics for Applied Sciences — Vol. 4

Proceedings of the Vth International Meeting

WAVES AND STABILITY IN CONTINUOUS MEDIA

Sorrento, Italy 9 – 14 October 1989

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Italia

World Scientific
Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 9128

USA office: 687 Hartwell Street, Teaneck, NJ 07666

UK office: 73 Lynton Mead, Totteridge, London N20 8DH

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ISBN 981-02-0554-6

Printed in Singapore by Utopia Press.

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PREFACE

The 5th Biennial Conference on *"Waves and Stability in Continuous Media"* was held in Sorrento (Italy) from October 9 to 14, 1989. Same as the previous four conferences this was a gathering of researchers who are interested in discussing and presenting current problems on Waves and Stability in Continuous Media. The scientific program included about twenty invited speakers and short contributed papers.

The warm atmosphere created during the Conference has produced a deep interchange of ideas and new common projects in this area.

The success of the Conference was due to the generous financial support of several agencies that are listed in p. viii.

I wish to thank the members of both the Scientific and Organizing Committee as well as the members of the Organizing Secretariat for their help.

Last, but not least, I must acknowledge the participation of so many registrants from different countries. Their collaboration was essential for the success of the Conference.

Salvatore Rionero

CONFERENCE SPONSORS

The organization of the 5th International Conference on Waves and Stability in Continuous Media was made possible by the financial support of:

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ON STABILITY AND LONG TERM BEHAVIOUR FOR RAREFIED GASES

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ABSTRACT

The talk discusses the stability and long term behaviour of rarefied gases modeled by the Boltzmann equation. A review of results for initial values far from equilibrium is given.

A Formal Discussion

Let us consider a rarefied gas described through its one particle density F in phase space. Back in 1872 Boltzmann gave the evolution of such a gas through an equation which expresses the material derivative of F in terms of the gain from molecules colliding into a region by binary collisions minus the loss from those colliding out of the region. In the absence of exterior forces his equation is

$$\partial_t F + v\nabla_x F = \text{gain} - \text{loss} = QF.$$

Here ∇_x is the gradient with respect to the space variable x, v is velocity and t is time. Finally Q is the so called collision operator,

$$QF(x,v_1) = \int_{R^3\times B} (F(x,v_1')F(x,v_2') - F(x,v_1)F(x,v_2))k(v_1,v_2,u)dv_2du.$$

Given two molecules of initial velocities (v_1,v_2) and initially separated in space, (v_1',v_2') denotes the velocities after collision. The details of the collision process are described with the help of the parameter set B, which is often chosen as angular coordinates for a sphere. Finally k is a weight function depending on the interaction between the molecules.

On physical grounds, the gas left to itself should settle down towards some equilibrium, and that in a stable way, i.e. gases with a similar initial preparation should approach similar equilibria in a similar way. And of course, this ought to be mirrored by the solution F of the Boltzmann equation model. Surprisingly , after the equation has been around for almost 120 years, the mathematical understanding of these questions is still far from complete. For the great variety of more complicated phenomena such as shocks, bifurcations and waves, which in many cases are fairly well understood on the gas dynamic level, any more fundamental kinetic studies have hardly even started, and for a reasonable picture above the perturbation level, we can only put our hopes to the future.

Before turning to the mathematical results on equilibrium and stability, let us see what the Boltzmann equation seems to tell us formally about the long time behaviour. Consider the entropy

$$HF(t) = \int F(t)\log F(t)\,dxdv.$$

We have

$$(\partial_t + v\nabla_x)(F\log F) = (\partial_t + v\nabla_x)F + \log F(\partial_t + v\nabla_x)F = QF + \log F\cdot QF,$$

and (by a change of variables argument)

$$\int QF\,dxdv = 0.$$

Hence, under suitable boundary conditions

(1) $\partial_t \int F\log F\, dxdv = \int(\partial_t + v\nabla_x)F\log F dxdv = \int QF dxdv + \int \log F\cdot QF dxdv$

$= \int \log F\cdot QF dxdv$ = (change of variables)

$= \frac{1}{4}\int (F_1'F_2' - F_1F_2)(\log F_1F_2 - \log F_1'F_2')k\, dxdv \leq 0.$

And so the entropy is decreasing. By a similar change of variables, mass, v-moments, and energy can be seen to be conserved. Finally a lemma by Gibbs tells us that the entropy is bounded from below by its value for a certain maxwellian $E_o = a\exp(-b|v|^2 + (c\cdot v))$ depending on the initial value.

Integrating (1) with respect to time we formally get

$$\int E_o \log E_o\, dxdv \leq \int F(t)\log F(t) dxdv = \int F_o \log F_o dxdv - \frac{1}{4}\int_0^t \int (F_1'F_2' - F_1F_2)(\log F_1'F_2' - \log F_1F_2)k\, dxdvds,$$

and so

$$F_1'F_2' - F_1F_2 \to 0 \text{ as } t \to \infty.$$

As mentioned above we also expect F to converge to some equilibrium w, which should then satisfy $w_1'w_2' - w_1w_2 = 0$. This gives for f=logw that

$$f(v_1') + f(v_2') = f(v_1) + f(v_2).$$

Essentially by arguments going back to Cauchy, it follows that

$$f = \alpha + (v_o\cdot v) - \beta|v|^2, \quad w = \exp(f) = a\exp(-b|v|^2 + (v_o\cdot v))$$

with coefficients possibly depending on other variables such as x and t. Hence the equilibrium is a (possibly local) maxwellian.

In the case of the forces between the molecules being inversely proportional to the fifth power of the distance, the v-moments formally satisfy a system of ordinary differential equations which can easily be analysed having exponential convergence to equilibrium and stability.

Strict Results

The formal arguments indicate the presence of convergence to equilibrium, even exponential convergence, and stability. When we next turn to the strict mathematical results and starting far from equilibrium, we will see that the strength of the known results steadily decreases, when progressing from a) the spacehomogeneous case via b) the spacedependent case close to spacehomogeneous to c) the spacedependent case in general.

In the spacehomogeneous case a) the understanding is good for hard forces, i.e. those of inverse power >5 under suitable cutoffs in the u-variable, which means playing down the grazing collisions. Let

$$k(v_1,v_2,u) = h(u)|v_1-v_2|^{\beta},\ 0<\beta\leq 1, \text{ with h bounded,}$$

$$L^1_r = \{\text{measurable functions f with } (1+|v|^r)f\in L^1(R^3)\},$$

$$L^1_{r\gamma} = \{f:R_+ \to L^1_r \text{ with } \sup_{t>0} \exp(\gamma t)\ ||f(t)||_r < \infty\}.$$

Given a maxwellian E_o the following result holds.

<u>Theorem</u>[1] *There exists* $r_o\geq 2+\beta$, *and* $\mu>0$, *such that if* $r_1>r_o$ *and*

$$F_o\in L^1_{r_1},\ \int(1,v,v^2)F_o dv = \int(1,v,v^2)E_o dv,\ F_o \log F_o \in L^1,$$

then

$$||F - E_o||_{r,\gamma} < \infty \quad \textit{for } r<r_1,\ 0\leq\gamma<\mu.$$

Moreover, if $2\leq r\leq r_1-\beta$, *then given* $\varepsilon>0$, *and a bounded set* $B\subset L^1_{r_1}$,

there is a $\delta>0$ *such that (stability)*

$$\sup_{t>0}||F-\hat{F}||_{r,0} < \varepsilon \quad \textit{if} \quad ||F_o - \tilde{F}_o||_r < \delta \textit{ and } \tilde{F}_o \in B.$$

The proof uses the spectral properties of the linearized collision operator and a powerful apriori estimate for Q. It is fairly technical and the interested reader is referred to the article[1].