Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Matrix Methods in Analysis

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AMS Subject Classification (1980): Primary: 46-0, Secondary: 28C20, 40A05, 40C05, 46A15, 46G10

ISBN 3-540-15185-0 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-15185-0 Springer-Verlag New York Heidelberg Berlin Tokyo

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

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1. Introduction

In this set of lecture notes, we present a culmination of results on infinite matrices which were evolved by the members of the Katowice Branch of the Mathematics Institute of the Polish Academy of Sciences. In the early history of functional analysis "sliding hump" methods were used extensively to establish some of the early abstract results in functional analysis. For example, the first proofs of versions of the Uniform Boundedness Principle by Hahn and Banach and Hildebrand utilized sliding hump methods ([18], [39], [42], [35]). Since Banach and Steinhaus gave a proof of the Uniform Boundedness Principle based on the Baire Category Theorem, category methods have proven to be very popular in treating various topics in functional analysis [19]. In recent times, there has been a return to "sliding hump" methods in treating various topics in functional analysis and measure theory. For example, in [34] Diestel and Uhl use a lemma of Rosenthal ([64]) as an abstract sliding hump method to treat a variety of topics in vector measure theory.

In a somewhat similar fashion, the Antosik-Mikusinski Diagonal Theorem ([53], [2], [3], [9]) can be considered to be an abstract sliding hump method and has been employed to treat a wide variety of topics in functional analysis and measure theory ([4], [5], [6], [9], [12], [53], [54], [56], [57]). The Antosik-Mikusinski Diagonal Theorem is a result concerning infinite matrices and has proven to be quite effective in treating various topics that were previously treated by Baire category methods (see in particular the texts [12], [56]). These notes present a result concerning infinite matrices which is of an even simpler and more elementary character than the Diagonal Theorem, and which can still be used to treat a wide variety of topics in functional analysis and measure theory ([16]).

In section 2, we present the two basic matrix results evolved by P. Antosik in references [6] - [11], and then in subsequent sections we present various applications of the matrix results to topics in functional analysis and measure theory. After the basic material has been presented in sections 2 and 3, there has been an attempt to make the subsequent chapters on applications independent of one another. Thus, there is some repetition in some of the chapters; for example, summability is mentioned in both sections 5 and 8 and other topics are repeated.

In section 3, we introduce and study the notions of $\mathcal{K}$ convergence and $\mathcal{K}$ boundedness which were also discovered and studied by the Katowice mathematicians ([6] - [11]). An equivalent form of $\mathcal{K}$ convergence was introduced by S. Mazur and W. Orlicz in [52] and also studied by A. Alexiewicz in [1]. The idea was rediscovered in the seminar of P. Antosik and J. Mikusinski. In subsequent sections the notions of $\mathcal{K}$ convergence and $\mathcal{K}$ boundedness will be shown to be effective substitutes for completeness and barrelledness assumptions in many of the classical results of functional analysis. For example, in section 4, we treat the Uniform Boundedness Principle. The classical Uniform Boundedness Principle is well-known to be false in the absence of completeness or barrelledness assumptions, but we present a version of the Uniform Boundedness Principle in Theorem 4.2 which is valid in the absence of any completeness assumption and which contains the classical Uniform Boundedness Theorem for F-spaces as a special case. To illustrate the utility of our general Uniform Boundedness Principle in the absence of completeness, we give a derivation of the Nikodym Boundedness Theorem based on the general Uniform Boundedness Principle.

In section 5, we discuss a classical result on the convergence of operators which is sometimes attributed to Banach and Steinhaus. This result, like the Uniform Boundedness Principle, is known to be false without completeness or barrelledness assumptions. Nevertheless, using the notion of $\mathcal{K}$ convergence, we present a version of this theorem which is valid without any completeness assumptions. As an application of the general result in the absence of completeness, we use it to derive the Nikodym Convergence Theorem, the Brooks-Jewett Theorem, and a result of Hahn, Schur and Toeplitz on summability.

In section 6, we treat bilinear maps using our matrix methods. We derive the classical result of Mazur and Orlicz on the joint continuity of separately continuous bilinear maps and also, using the notion of $\mathcal{K}$ convergence, present several hypocontinuity type of results which are valid without completeness assumptions. Our hypocontinuity results generalize results of Bourbaki.

In section 7, we treat various Orlicz-Pettis type results on subseries convergent series by matrix methods. We derive the classical Orlicz-Pettis Theorem as well as Orlicz-Pettis results for compact operators and the topology of pointwise convergence on certain well-known function spaces.

In section 8, we give generalizations of the classical lemmas of Schur and Phillips to the group-valued case. We show that these general results contain the classical lemmas of Schur and Phillips as special cases. A result of Hahn and Schur on summability is also obtained from the general results.

In section 9, we present a version of the Schur lemma for bounded multiplier convergent series in a metric linear space. This version for bounded multiplier convergent series is motivated by a sharper conclusion of the classical Schur lemma for B-spaces which is obtained in Corollary 8.4. Some general remarks on the vector versions of the summability results of Schur and Hahn are also included.

In section 10, we consider the problem of imbedding c_0 and ℓ^∞ into a B-space. Using the basic matrix lemma of section 2, we obtain the classical results of Bessaga-Pelczynski and Diestel-Faires on imbedding c_0 and ℓ^∞ into B-spaces. We also indicate applications to a large number of well-known results in Banach space theory. The results and method of proof are very analogous to those of Diestel and Uhl ([34] I.4) except that the basic matrix lemma is employed instead of the Rosenthal lemma.

There are two themes which prevail throughout these notes. The first is that the matrix results presented here, although being very elementary in character, are extremely effective in treating various topics in measure theory and functional analysis which have been traditionally treated by Baire category methods. The other theme is that the idea of $\mathcal{K}$ convergence can be used as an effective substitute for completeness assumptions in many classical results in functional analysis. For example, we present versions of the Uniform Boundedness Principle, the Banach-Steinhaus Theorem and classical hypocontinuity results which are valid with no completeness assumptions whatever being present. Applications of these general results in the absence of completeness are indicated.

Many of the topics treated in these notes are standard topics in functional analysis which are treated in a great number of the functional analysis texts by various means including the popular Baire category methods. The matrix methods employed in these notes are of a very elementary character and can be presented without requiring a great deal of mathematical background on the part of the reader. For this reason these matrix methods would seem to be quite appropriate for presentation of some of the classical functional analysis topics to readers with modest mathematical backgrounds. It is the authors' hope that the matrix methods presented here will find their way into the future functional analysis texts.

We conclude this introduction by fixing the notation which will be used in the sequel.

Throughout the remainder of these notes, unless explicitly stated otherwise, E, F and G will denote normed groups. That is, E is assumed to be an Abelian topological group whose topology is generated by a quasi-norm $| \; | : E \to \mathbb{R}_+$. ($| \; |$ is a quasi-norm if $|0| = 0$, $|-x| = |x|$ and $|x+y| \leqslant |x| + |y|$; a quasi-norm generates a metric topology on E via the translation invariant metric $d(x,y) = |x-y|$.)

Recall that the topology of any topological group is always generated by a family of quasi-norms ([27]). Thus, many of the results are actually valid for arbitrary topological groups. We present the results for normed groups only for the sake of simplicity of exposition.

Similarly, X, Y and Z will denote metric linear spaces whose topologies are generated by a quasi-norm $| \; |$. (For convenience, all vector spaces will be assumed to be real; most of the results are valid for complex vector spaces with obvious modifications.) If it is further assumed that X is a normed space, we write $\| \; \|$ for the norm on X.

The space of all continuous linear operators from X into Y will be denoted by L(X,Y). If X and Y are normed spaces, the operator norm of an element $T \in L(X,Y)$ is defined by $\|T\| = \sup\{\|Tx\| : \|x\| \leqslant 1\}$.

If X and Y are two vector spaces in duality with one another by the bilinear pairing $\langle , \rangle$, the weakest topology on X such that the linear maps $x \to \langle x, y\rangle$ are continuous for all $y \in Y$ is denoted by $\sigma(X,Y)$. $\sigma(X,Y)$ is referred to as the weak topology on X induced by Y ([79] 8.2).

Other notations and terminology employed in the notes is standard. Specifically, we follow [38] for the most part.

Finally, for later use, we record a lemma of Drewnowski ([36]) which will be used at several junctures in the text.

Let Σ be an algebra of subsets of a set S. If $\mu : \Sigma \to G$ is a finitely additive set function, then μ is said to be <u>strongly additive</u> (<u>exhaustive</u> or <u>strongly bounded</u>) if $\lim \mu(E_i) = 0$ for each disjoint sequence $\{E_i\}$ from Σ. We have the following result due to Drewnowski.

<u>Lemma 1</u>. Let Σ be a σ-algebra. If $\mu_i : \Sigma \to G$ is a sequence of strongly additive set functions and $\{E_j\}$ is a disjoint sequence from Σ, then there is a subsequence $\{E_{k_j}\}$ of $\{E_j\}$ such that μ_i is countably additive on the σ-algebra generated by $\{E_{k_j}\}$.

Drewnowski states this result for a single strongly additive measure in [36] (see also Diestel and Uhl [34] I.6), but the lemma above can be derived from Drewnowski's result in the following way: let $G^{\mathbb{N}}$ be the space of all G-valued sequences. Equip $G^{\mathbb{N}}$ with the quasi-norm $|\ |$ defined by

$$|g| = \sum_{i=1}^{\infty} |g_i|/(1 + |g_i|)2^i$$

where $g = (g_1, g_2, \ldots)$ and $|g_i|$ is the "norm" of g_i in G. Now define $\mu : \Sigma \to G^{\mathbb{N}}$ by $\mu(E) = (\mu_1(E), \mu_2(E), \ldots)$. Then μ is strongly additive so by Drewnowski's lemma, there is a subsequence $\{E_{k_j}\}$ of $\{E_j\}$ such that μ is countably additive on the σ-algebra, Σ_0, generated by $\{E_{k_j}\}$. Then each μ_i is clearly countably additive on the σ-algebra Σ_0.

2. Basic Matrix Results

In this section we establish the two basic results on infinite matrices which will be used throughout the sequel. The first result is a very simple and elementary result on matrices of non-negative real numbers. This result is then used to establish a convergence type result for matrices with elements in a topological group. Both results are of an elementary character and require only elementary techniques in their proofs.

<u>Lemma 1</u>. Let $x_{ij} \geq 0$ and $\epsilon_{ij} > 0$ for $i,j \in \mathbb{N}$. If $\lim_i x_{ij} = 0$ for each j and $\lim_j x_{ij} = 0$ for each i, then there is a subsequence $\{m_i\}$ of positive integers such that $x_{m_i m_j} < \epsilon_{ij}$ for $i \neq j$.

<u>Proof</u>: Put $m_1 = 1$. There is an $m_2 > m_1$ such that $x_{m_1 m} < \epsilon_{12}$ and $x_{mm_1} < \epsilon_{21}$ for $m \geq m_2$. Then there is an $m_3 > m_2$ such that $x_{m_1 m} \leq \epsilon_{13}$, $x_{m_2 m} < \epsilon_{23}$, $x_{mm_1} < \epsilon_{31}$ and $x_{mm_2} < \epsilon_{32}$ for $m \geq m_3$. An easy induction completes the proof.

Lemma 1 will be used directly in several later results but the principle application of Lemma 1 will be to establish the basic matrix convergence result below.

<u>Basic Matrix Theorem 2</u>. Let E be a normed group and $x_{ij} \in E$ for $i,j \in \mathbb{N}$. Suppose

(I) $\lim_i x_{ij} = x_j$ exists for each j and

(II) for each subsequence $\{m_j\}$ there is a subsequence $\{n_j\}$ of $\{m_j\}$ such that $\{\sum_{j=1}^{\infty} x_{in_j}\}$ is Cauchy.

Then $\lim_i x_{ij} = x_j$ uniformly with respect to j.

In particular, $\lim_i x_{ii} = 0$.

Proof: If the conclusion fails, there is a $\delta > 0$ and a subsequence $\{k_i\}$ such that $\sup_j |x_{k_i j} - x_j| > \delta$. For notational convenience assume $k_i = i$. Set $i_1 = 1$ and pick j_1 such that $|x_{i_1 j_1} - x_{j_1}| > \delta$. By (I) there is $i_2 > i_1$ such that $|x_{i_1 j_1} - x_{i_2 j_1}| > \delta$ and $|x_{ij} - x_j| < \delta$ for $i > i_2$ and $1 \leqslant j \leqslant j_1$. Now pick j_2 such that $|x_{i_2 j_2} - x_{j_2}| > \delta$ and note that $j_2 > j_1$. Continuing by induction, we obtain subsequences $\{i_k\}$ and $\{j_k\}$ such that $| x_{i_k j_k} - x_{i_{k+1} j_k} | > \delta$. Set $Z_{k\ell} = x_{i_k j_\ell} - x_{i_{k+1} j_\ell}$ and note

(1) $$| Z_{kk} | > \delta.$$

Consider the matrix $[|Z_{k\ell}|] = Z$. By (I), the columns of this matrix converge to 0. By (II), the rows of the matrix $[x_{ij}]$ converge to 0 so the same holds for the matrix Z. Let $\epsilon_{ij} > 0$ be such that $\sum_{ij} \epsilon_{ij} < \infty$. By Lemma 1, there is a subsequence $\{m_k\}$ such that $|Z_{m_k m_\ell}| < \epsilon_{k\ell}$ for $k \neq \ell$.

By (II) there is a subsequence $\{n_k\}$ of $\{m_k\}$ such that

(2) $$\lim \sum_{\ell=1}^{\infty} Z_{n_k n_\ell} = 0.$$

Then

(3) $$| Z_{n_k n_k} | \leqslant | \sum_{\ell \neq k} Z_{n_k n_\ell} | + | \sum_{\ell=1}^{\infty} Z_{n_k n_\ell} | < \sum_{\ell \neq k} \epsilon_{n_k n_\ell} +$$

$$|\sum_{\ell=1}^{\infty} Z_{n_k n_\ell}|.$$

Now the first term on the right hand side of (3) goes to 0 as $k \to \infty$ by the convergence of the series $\Sigma\, \epsilon_{k\ell}$ and the second term goes to 0 by (2). But this contradicts (1) and establishes the first part of the conclusion.

The uniform convergence of the limit, $\lim_i x_{ij} = x_j$ and the fact that $\lim_j x_{ij} = 0$ for each i implies that the double limit $\lim_{ij} x_{ij}$ exists and is equal to 0. In particular, this implies $\lim_i x_{ii} = 0$.

This matrix result will be the basic tool used throughout the sequel. A matrix $[x_{ij}]$ which satisfies conditions (I) and (II) of Theorem 2 will be called a $\mathcal{K}$ matrix (the reason for the use of this terminology will be indicated in the next section).

The Basic Matrix Theorem 2 has a very different character than the Antosik-Mikusinski Diagonal Theorem in that the hypothesis and the conclusions have very different forms ([2], [53]). However, Theorem 2 can also be viewed as a diagonal theorem in the sense that the hypotheses of Theorem 2 imply that the diagonal sequence $\{x_{ii}\}$ converges to zero. In fact, if one first shows only that the diagonal sequence converges to zero, then it is not difficult to use this to show that in fact the columns of the matrix are uniformly convergent.

Matrix results of a very similar nature to Theorem 2 have been established in [6] – [11] and [73]. The matrix results of these papers have been used to treat a wide variety of topics in both measure theory and functional analysis. Much of the content of these papers will be treated in chapters 4, 5, 8 and 9.

It should be pointed out that the functional analysis text of

E. Pap ([56]) uses the Antosik-Mikusinski Diagonal Theorem in a systematic manner to treat a variety of classical topics in functional analysis and in this sense is very much in the spirit of these notes except that we systematically employ the Basic Matrix Lemma.

3. $\mathcal{K}$ Convergence

In this section we introduce the notion of a $\mathcal{K}$ convergent sequence. This notion was introduced by P. Antosik in [6] and was further explored in [7] - [11]; further applications to the Uniform Boundedness Principle and bilinear maps are given in [14] and [75]. The " $\mathcal{K}$ " in the description below is in honor of the members of the Katowice Branch of the Mathematics Institute of the Polish Academy of Sciences who have extensively studied and developed many of the results pertaining to $\mathcal{K}$ convergent sequences.

As a historical note, it should be pointed out that S. Mazur and W. Orlicz introduced a concept very closely related to that of a $\mathcal{K}$ convergent sequence in [52], Axiom II, p. 169. They essentially introduced the notion of a $\mathcal{K}$ (metric linear) space which is defined below and noted that the classical Uniform Boundedness Principle holds in such spaces. A. Alexiewicz also studied consequences of this notion in convergence spaces ([1] axiom $A_2^{\cdot}$, p. 203). It should also be noted however that the notion of a $\mathcal{K}$ convergent sequence and that of a $\mathcal{K}$ bounded set permits the formulation of versions of the Uniform Boundedness Principle in arbitrary metric linear spaces (Theorem 4.2 below) in contrast to the situation encountered in the classical Uniform Boundedness Principle.

<u>Definition 1</u>. Let (E,τ) be a topological group. A sequence $\{x_i\}$ in E is a τ- $\mathcal{K}$ convergent sequence if each subsequence of $\{x_i\}$ has a subsequence $\{x_{i_k}\}$ such that the series $\sum_k x_{i_k}$ is τ-convergent to an element $x \in E$.

If the topology τ is understood, we drop the τ in the description of τ- $\mathcal{K}$ convergence.

Note that any τ- $\mathcal{K}$ convergent sequence $\{x_i\}$ is τ-convergent to 0 by the Urysohn property, i.e., any subsequence of $\{x_i\}$ has a subsequence which converges to 0. In complete spaces the converse holds.

Notice in the Basic Matrix Theorem 2.2, assumption (II) implies that the rows of the matrix are $\mathcal{K}$ convergent (in some uniform sense). This is the reason for the terminology: $\mathcal{K}$ matrix.

Example 2. Let E be a complete normed group and $\{x_i\}$ converge to 0 in E. Then any subsequence of $\{x_i\}$ has a subsequence $\{x_{i_k}\}$ such that $\sum_k |x_{i_k}| < \infty$. The completeness implies that the series Σx_{i_k} converges in E. Thus, in complete spaces a sequence is $\mathcal{K}$ convergent iff it converges to 0.

In general the statement in Example 2 is false as the following example shows.

Example 3. Let c_{oo} be the vector space of all real sequences $\{t_j\}$ such that $t_j = 0$ eventually. Equip c_{oo} with the sup-norm. Let e_k be the sequence in c_{oo} which has a 1 in the k^{th} coordinate and 0 elsewhere. Consider the sequence $\{(1/j)e_j\}$ in c_{oo}. This sequence converges to 0 in c_{oo} but no subseries of the series $\Sigma(1/j)e_j$ converges to an element of c_{oo}. That is, this sequence converges to 0 but is not $\mathcal{K}$ convergent.

Examples 2 and 3 might suggest that a (normed) space is complete iff it has the property that every sequence which converges to 0 is $\mathcal{K}$ convergent. There are, however, normed spaces which have this property but are not complete. A topological group which has the property that any sequence which converges to 0 is $\mathcal{K}$ convergent is called a $\mathcal{K}$ space. Klis ([45]) has given an example of a normed