Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Optimal Periodic Control

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Author

Fritz Colonius
Institut für Dynamische Systeme, Universität Bremen Fb 3
Postfach 330440, 2800 Bremen 33, Federal Republic of Germany

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To Didi Hinrichsen, who taught me
how to do mathematics

CONTENTS

CHAPTER I

INTRODUCTION

1. These notes are concerned with optimal periodic control for ordinary and functional differential equations of retarded type. In its simplest version this problem can be stated as follows:

Consider a controlled system

$$\dot{x}(t) = f(x(t),u(t)), \quad t \in R_+ := [0,\infty) \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$. Look for a τ-periodic control function u and a corresponding τ-periodic trajectory x such that the "average cost"

$$1/\tau \int_0^\tau g(x(t),u(t))dt \tag{2}$$

is minimized (or the "average output" is maximized).
If one adds the boundary condition

$$x(0) = x(\tau), \tag{3}$$

it is sufficient to consider x and u on the compact interval $T := [0,\tau]$ only: By (3), the periodic extensions of x and u to R_+ lead to an absolutely continuous solution x of (1) on R_+.

Hence the optimal periodic control problem formulated above (abbreviated as (OPC)) is intermediate between dynamic optimization problems on R_+ and the following static or steady state optimization problem (OSS) associated with (OPC):

Minimize the "instantaneous cost"

$$g(x,u)$$

where $x \in R^n$ is a steady state corresponding to a constant control $u \in R^m$, i.e. satisfies

$$0 = f(x,u).$$

In these notes we study the relation between (OPC) and (OSS). The more complicated relation between dynamic optimization problems on R_+ and periodic problems is not considered here (some results in this direction are contained in Leizarowitz [1985], Colonius/Sieveking [1987], Colonius/Kliemann [1986]).

The fundamental problem concerning the relation between (OPC) and (OSS) can be formulated as follows: Suppose that $(x^o,u^o) \in R^n \times R^m$ is an optimal solution of (the relatively simple, finite dimensional) problem (OSS). Can the average performance be improved (in every neighborhood of the constant functions $\bar{x}^o \equiv x^o$, $\bar{u}^o \equiv u^o$) by allowing for τ-periodic (x,u)? That is, exist τ-periodic (x,u) satisfying the constraints of (OPC) and

$$1/\tau \int_0^\tau g(x(t),u(t))dt < g(x^o,u^o) \text{ ?}$$

If this is the case, we call (x^o,u^o) *locally proper*.

Local properness can be tested by checking if $(\bar{x}^o,\bar{u}^o)$ satisfies necessary optimality conditions for a local solution of (OPC). If $(\bar{x}^o,\bar{u}^o)$ violates these conditions, (x^o,u^o) is locally proper. This problem becomes mathematically interesting, since first order necessary optimality conditions (for weak local) minima do *not* allow to discern steady states which are merely optimal among steady states from those which are also optimal among periodic solutions. We will prove various necessary optimality conditions for weak and strong local minima and local relaxed minima of (OPC) and develop corresponding tests for local properness. Furthermore we will relate local properness to dynamic properties of the system equation.

Other important aspects of optimal periodic control theory will not be discussed here. For existence results of optimal periodic solutions (of ordinary differential equations) we refer to Nistri [1983] and Gaines/Peterson [1983]; cp. also Miller/Michel [1980]. Numerical methods for the computation of optimal periodic solutions as well as sufficient optimality conditions are briefly reviewed in Section IX.2, below.

Actually, we consider more general system equations than (1), namely functional differential equation of retarded type

$$\dot{x}(t) = f(x_t,u(t)) \qquad (4)$$

where $f: C(-r,0;R^n) \times R^m \to R^n$, $r > 0$, and $x_t \in C(-r,0;R^n)$ is given by

$$x_t(s) := x(t+s), \quad s \in [-r,0];$$

this includes delay equations of the form

$$\dot{x}(t) = f(x(t),x(t-r),u(t)),$$

where $f: R^n \times R^n \times R^m \to R^n$.

For these equations the boundary condition (3) is not adequate, since the "state" of (4) (or (5)) at time t is given by the function segment $x_t \in C(-r,0;R^n)$.

Hence (3) has to be replaced by the (infinite dimensional) condition

$$x_0 = x_\tau. \tag{6}$$

Thus we have to use the full force of optimization theory in infinite dimensional spaces in order to treat the corresponding optimal periodic problem.

2. Optimal periodic control theory was first motivated by problems from chemical engineering. Sometimes "cycling" of a chemical reactor allows to increase the average output compared to steady state operation. Here (steady state) relaxed control played as particular role, see e.g. Horn/Bailey [1968].

Early work in the field is reviewed in Bailey [1973], see also the survey Matsubara/Nishimura/Watanabe/Onogi [1981]. Other recent work includes Watanabe/Onogi/Matsubara [1981], Watanabe/Kurimoto/Matsubara [1984], Schädlich/Hoffmann/Hofmann [1983].

Besides control of chemical reactors, flight performance optimization provides a second main source of motivation. Speyer [1973,1976] observed that sometimes steady state cruise is not fuel optimal. This led to the consideration of "chattering cruise" which is a (steady state) relaxed solution (apparently, a more complete problem description avoids chattering here: Houlihan/Cliff/Kelley [1982]); see also Gilbert/Lyons [1981], Speyer/Dannemiller/Walker [1985], Chuang/Speyer [1985], Sachs/Christodopulou [1986].

Diverse other reported applications of optimal periodic control include harvesting problems (Vincent/Lee/Goh [1977], Deklerk/Gatto [1981]; cp. also Brauer [1984], Brauer/Soudack [1984]), soaring of gliders (e.g. Dickmanns [1982]), vehicle cruise (Gilbert [1976]), maintenance problems (Khandelwal/Sharma/Ray [1979]) and dynamic pricing problems (Timonen/Hämäläinen [1979]).

Early contributions to the mathematical theory of optimal periodic control were given in Horn/Lin [1967], Markus [1973], Halanay [1974] and CIME Lecture Notes edited by Marzollo [1972] (including also annotations on the history and prehistory of optimal periodic control); see also the surveys Guarbadassi [1976], Guarbadassi/Locatelli/Rinaldi [1974], Noldus [1975].

Problems with discrete system equations are considered e.g. in Bittanti/

Fronza/Guarbadassi [1974,1976], Ortlieb [1980] and Valkó/Almasy [1982].

Periodic problems with delay equations are treated in Sincic/Bailey [1978] (motivated from chemical engineering) and in Li [1985], Li/Chow [1987] (for linear equations see also Barbu/Precupanu [1978] and DaPrato [1987]).

The best available survey on optimal control of functional differential equations is still Manitius [1976] (cp. also Banks/Manitius [1974], and Oguztöreli [1966], Warga [1972], Gabasov/Kirillova [1976,1981] and for contributions from the engineering side, Koivo/Koivo [1978], Marshall [1980], Malek-Zavarei/Jamshidi [1987]). Delay equations frequently occur in chemical engineering models and ecological problems; cp. e.g. Manitius [1974] for a discussion of various models.

3. These notes are structured as follows:
Chapter II collects results from general optimization theory needed in the sequel, in particular first and second order necessary optimality conditions for problems in Banach spaces. Although Hale's book [1977] is used as a reference text for functional differential equations, Chapter III includes a sketch of duality for linear time-varying functional differential equations based on a calculus of structural operators. This allows to avoid excessive use of the Unsymmetric Fubini Theorem (which is hidden now behind the properties of the structural operators). Furthermore, extendability to the product space is discussed. Chapter IV presents a global maximum principle for strong local minima including a "stopping condition" for determination of the optimal period length. The proof relies on Ekeland's Variational Principle. Chapters V and VI contain first and second order necessary optimality conditions for weak local minima and local relaxed minima, respectively. A remarkable observation here is that - under reasonable assumptions - every ordinary optimal solution is also optimal among relaxed solutions.

The last three chapters are devoted to a discussion of local properness. Chapter VII develops tests for local properness (in particular, a so-called Π-Test) which are based on the catalogue of necessary optimality conditions from Chapters IV-VI. The relation between necessary optimality conditions for the periodic and the steady state problems is discussed in detail.

In Chapter VIII, we relate local properness to dynamic properties of the system equation. We exhibit a scenario for local properness which is related to Hopf bifurcation. An example involving a retarded Liénard equation is worked out.

The final Chapter IX treats problems with ordinary differential equations. In particular, a Π-Test for problems with state constraints is proved. For a simple model of a "Continuous Flow Stirred Tank Reactor" (in Rutherford Aris' words "So sesquipedelian a style supplicates a sobriquet": we use CSTR) it is shown how local properness occurs near a Hopf bifurcation point.

In summary, our main results for optimal periodic control of functional differential equations are:

- Proof of a global maximum principle based on Ekeland's Variational Principle;
- A "stopping condition" for determination of the optimal period length;
- First and second order necessary optimality conditions for ordinary and relaxed problems with state and control constraints and isoperimetric constraints;
- Tests for local properness, in particular a Π-Test, based on the necessary optimality conditions;
- A scenario for local properness related to Hopf bifurcation;
- Discussion of two examples involving retarded Liênard equations and an ordinary differential model of a chemical reactor;

and finally

- a Π-Test for state constrained problems with ordinary differential equations.

We hope, that these results will help to renew interest in optimal periodic control theory. It is apparent from the literature cited above that a first interest in mathematical questions in this field had occurred at the beginning of the seventies. However, (1) there is continuing interest from chemical engineering and aerospace engineering; other applications, e.g. in ecology, are promising, too; (2) a further analysis of the relation between optimal periodic control and dynamic properties appears possible; (3) the results above show that periodic control of functional differential equations is much more well-behaved than control with fixed boundary values (in order to make this point clearer we have included in Chapters V and VI a discussion of fixed boundary value problems), and (4) some of the results derived here for retarded functional differential equations remain true for other infinite dimensional, in particular parabolic differential equations (cp. Colonius [1987]).

4. This research report is a revised version of my Habilitationschrift, Universität Bremen, Bremen 1986. The main revisions are (i) a sharpened and more general version of the second order necessary conditions in Section II.2 made possible by using some ideas from Werner [1984], (ii) a new proof of a stopping condition for determination of the optimal period length based on Ekeland's Variational Principle, now in Section IV.2; in this chapter, extendability to the product space is now assumed; (iii) a corrected version of a π-Test under state constraints in section IX.3.

The research reported here was performed during visits to Mathematisches Institut der Universität Graz (1983/84) and, as Visiting Assistant Professor, at Lefschetz Center for Dynamical Systems, Brown University, Providence, R.I. (1984/85). These visits were supported by a grant from Deutsche Forschungsgemeinschaft. It is a pleasure to thank Prof. F. Kappel, Universität Graz, and Prof. H.T. Banks, Brown University, for their invitations. Furthermore an invitation by the late Prof. G.S.S. Ludford, Cornell University, to take part in the Special Year on Reacting Flows was very helpful for an understanding of the CSTR problem. Prof. Matsubara, Nagoya University, draw my attention to the interesting problem of a π-Test under state constraints.

I am indebted to A.W. Manitius and D. Salamon for the permission to use some unpublished material in Chapter III. Furthermore, D. Hinrichsen and M. Brokate pointed out errors in the earlier version. Finally, I thank V. Landau, who typed the earlier version, and E. Sieber for their competent work.

5. Some remarks on the notation are in order: Standing hypotheses in a section or chapter are only repeated in statements of theorems. The end of a proof is marked by □.

For a set Q in a vector space we define the conical hull of Q with respect to $q^0 \in Q$ as

$$Q(q^0) := \{\alpha(q-q^0) : \alpha \geq 0,\ q \in Q\}.$$

The norm in a Banach space X is denoted by $|\cdot|_X$; where no confusion appears possible, we omit the index X. Furthermore, for $\rho > 0$, we let

$$X_\rho := \{x \in X : |x| \leq \rho\}.$$

The space of linear functionals on X is denoted by X', while X^* denotes the dual Banach space of bounded linear functionals on X. The dual of the space $C(a,b;R^n)$ of continuous function on $[a,b]$ with values in R^n is identified with the space $NBV(a,b;R^n)$ of normalized functions v of bounded variation, i.e. v is left continuous on (a,b) and $v(b) = 0$. Derivatives are denoted in various ways, as it appears most convenient in the respective context. Furthermore $R_+ := [0,\infty)$.

CHAPTER II

OPTIMIZATION THEORY

This chapter collects results from general optimization theory. For most of them proofs are available in books and hence omitted here. However, complete proofs for the second order necessary conditions in section 2 are included, since the specific results we need were not available in sufficient generality. Furthermore, second order conditions play a central role in optimal periodic control theory; hence completeness in the arguments appears adequate.

After the exposition of first and second order necessary optimality conditions in sections 1 and 2, section 3 indicates a result by A.V. Fiacco on smooth dependence of optimal solutions on a parameter and cites I. Ekeland's Variational Principle.

The main results of this chapter are Theorem 1.11, Corollary 2.12 and Corollary 3.7.

1. First Order Optimality Conditions

In this section we consider the following optimization problem in Banach spaces.

Problem 1.1 Minimize $G(x)$

s.t. $F(x) \in K, \quad x \in C.$

where $G: X \to R$, $F: X \to Y$, X,Y are Banach spaces, the set $C \subset X$ is closed and convex, and $K \subset Y$ is a closed and convex cone with vertex at the origin.

For a set Q in a Banach space X define the conical hull $Q(q^o)$ of Q with respect to $q^o \in Q$ by

$$Q(q^o) := \{\alpha(q-q^o) : \alpha \geq 0,\ q \in Q\}.$$

Observe that for a convex cone K with vertex at the origin and $y^o \in K$

$$K(y^o) = \{k-\alpha y^o : \alpha \geq 0,\ k \in K\}.$$

Frequently, we abbreviate

$$Q_\rho := Q \cap X_\rho, \qquad \rho > 0.$$

The following two theorems, a generalized open mapping theorem and first order necessary optimality conditions, go back to work by S.M. Robinson [1976] (cp. also Zowe/Kurcyusz [1979], Alt [1979]). A nice, self-contained treatment is given in the lecture notes by Werner [1984].

Theorem 1.2 Let X and Y be Banach spaces and $T: X \to Y$ be a bounded linear map. Suppose that Q is a closed and convex set in X and K is a closed and convex cone with vertex at the origin in Y. Then for $q^o \in Q$ and $y^o \in K$ the following two statements are equivalent:

(i) $Y = TQ(q^o) - K(y^o)$

(ii) $Y_\rho \subset T(Q-q^o)_1 - K(y^o)_1$ for some $\rho > 0$.

Proof: See Werner [1984, Theorem 5.2.3].

One obtains immediately the following corollary, Werner [1984, Corollary 5.2.4].

Corollary 1.3 Suppose the hypotheses of Theorem 1.2 are satisfied. Let

$$\rho_0 := \sup\{\rho > 0: Y_\rho \subset T(Q-q^o)_1 - K(y^o)_1\}.$$

Then for $L > 1/\rho_0$ and $y \in Y$ there exist

$$\begin{Bmatrix} x \\ z \end{Bmatrix} \in L|y| \begin{Bmatrix} (Q-q^o)_1 \\ K(y^o)_1 \end{Bmatrix} \quad \text{with} \quad y = Tx + z.$$

Theorem 1.4 Let x^o be a local minimum of Problem 1.1 and assume that the functional G is Fréchet differentiable at x^o and the map F is continuously Fréchet differentiable at x^o. If the constraint qualification

$$F'(x^o)C(x^o) - K(F(x^o)) = Y \tag{1.1}$$

holds, then there exists $y^* \in Y^*$ satisfying

(i) $y^*y \geq 0$ for all $y \in K$

(ii) $y^*F(x^o) = 0$

(iii) $[\lambda_0 G'(x^o) - y^*F'(x^o)]x \geq 0$ for all $x \in C(x^o)$.

Proof: See Werner [1984, Theorem 5.3.2].

Remark 1.5 It suffices, naturally, that F and G are defined in a neighborhood $\mathcal{O}$ of x^0. This, being true for all following necessary optimality conditions, is very convenient if F and G are implicitly defined only.

The following corollary slightly extends the result above.

Define, for $x \in X$, $\lambda = (\lambda_0, y^*) \in R \times Y^*$, the Lagrangean functional

$$L(x,\lambda) := \lambda_0 G(x) - y^* F(x). \tag{1.2}$$

Corollary 1.6 Let the assumptions of Theorem 1.4 be satisfied and assume that either $F'(x^0)C(x^0) - K(F(x^0))$ is not dense in Y or contains a subspace of finite codimension in Y. Then there exists $0 \neq \lambda = (\lambda_0, y^*) \in R \times Y^*$ satisfying

(i) $\lambda_0 \geq 0$, $y^*y \geq 0$ for all $y \in K$

(ii) $y^*F(x^0) = 0$

(iii) $\mathcal{D}_1 L(x^0,\lambda)x \geq 0$ for all $x \in C(x^0)$.

If the constraint qualification (1.1) is satisfied, then $\lambda_0 \neq 0$.
If (1.1) is supplemented by

$$c\ell[RF(x^0) + F'(x^0)N_X + N_Y] = Y, \tag{1.3}$$

where

$$N_Y = [-K(F(x^0))] \cap K(F(x^0))$$

and

$$N_X = [-C(x^0)] \cap C(x^0)$$

are the greatest linear subspaces contained in $K(F(x^0))$ and $C(x^0)$, respectively, then, for given λ_0, the conditions (i) - (iii) above determine y^* uniquely.

Proof: If (1.1) holds, the assertion follows by Theorem 1.4. If $F'(x^0)C(x^0) - K(F(x^0))$ is not dense in Y, the assertion follows by the Hahn-Banach Theorem (e.g. Klee [1969, 1.3]). Thus it remains to discuss the case where $F'(x^0)C(x^0) - K(F(x^0))$ contains a subspace N of finite codimension in Y. By a version of the Hahn-Banach Theorem (cp. e.g. Kirsch/Warth/Werner [1978, Satz 1.1.14]) there is $y' \in Y'$ with $y'y \geq 0$ for all $y \in K$ and

$$y'F'(x^0)x \geq 0 \quad \text{for all} \quad x \in C(x^0).$$

Let M be the linear span of $B := F'(x^0)C(x^0) - K(F(x^0))$. The subspaces M and N are closed in Y and the factor space M/N is finite dimensional. We denote by $\pi: M \to M/N$ the canonical (linear and bounded) projection. Thus

$$\pi y_1 = \pi y_2 \quad \text{iff} \quad y_1 - y_2 \in N.$$

If M is a proper subspace of Y, there exists $y^* \in Y^*$ satisfying the assertions with $\lambda_0 = 0$. Thus we may assume $M = Y$. Observe that πB is a convex subset of a finite dimensional space. Thus if 0 is a boundary point of B, there exists a bounded linear functional $\bar{y}$ on M/N with

$$\bar{y}\pi F'(x^0)x \le 0 \quad \text{for all} \quad x \in C(x^0)$$

$$\bar{y}\pi y \ge 0 \quad \text{for all} \quad y \in K$$

$$\bar{y}\pi F(x^0) = 0.$$

Hence the functional $\bar{y}\pi \in Y^*$ satisfies the assertions with $\lambda_0 = 0$. Now suppose that $0 \in \text{int } B$. Then

$$Y = M = F'(x^0)C(x^0) - K(F(x^0))$$

and hence (1.1) holds.

Finally, let (1.3) be satisfied and suppose $y_1^*, y_2^* \in Y^*$ satisfy (i) - (iii) with $\lambda_0 = 0$. Then

$$(y_1^* - y_2^*)[\alpha F(x^0) + F'(x^0)x + y] = 0 \quad \text{for all} \quad \alpha \in R,\ x \in N_X,\ y \in N_Y$$

and by (1.3) $y_1^* = y_2^*$. □

Remark 1.7 Zowe/Kurcyusz [1979], Kurcyusz [1973,1976], Penot [1982], and Brokate [1980] contain more information on condition (1.1), see also Theorem 1.18, below. Condition (1.3) is very restrictive if other than equality constraints are present. Hence, in this case, one has - in general - to live with non unique Lagrange multipliers (see also Lempio/ Zowe [1982]).

In the following problem, the cone constraint has a special structure which can be exploited.

Problem 1.8 Minimize $G(x)$

$$\text{s.t.} \quad F(x) = 0, \quad H(x) \in K, \quad x \in C,$$

where $G: X \to R$, $F: X \to Y$, $H: X \to Z$, X, Y and Z are Banach spaces, C is a closed and convex subset of X, and K is a closed and convex cone in Z with vertex at the origin and *non-empty interior*.

Note that Problem 1.8 is a special case of Problem 1.1 (with cone $\{0\} \times K \subset Y \times Z$). Frequently we will refer to the constraints $F(x) = 0$ and $H(x) \in K$ as the equality and the inequality constraint, respec-