Proceedings
of the
STEKLOV INSTITUTE
OF MATHEMATICS

1981 · ISSUE 1 of 4 · ISSN 0081-5438

Approximation of Functions by Polynomials and Splines

Translation of
ТРУДЫ
ордена Ленина
МАТЕМАТИЧЕСКОГО ИНСТИТУТА
имени В. А. СТЕКЛОВА
Том 145 (1980)

AMERICAN MATHEMATICAL SOCIETY · PROVIDENCE · RHODE ISLAND

Proceedings
of the
STEKLOV INSTITUTE
OF MATHEMATICS

1981, ISSUE 1

Approximation of Functions by Polynomials and Splines

Edited by

S. B. Stečkin

Translation of
ТРУДЫ
ордена Ленина
МАТЕМАТИЧЕСКОГО ИНСТИТУТА
имени В. А. СТЕКЛОВА

Том 145 (1980)

АКАДЕМИЯ НАУК

СОЮЗА СОВЕТСКИХ СОЦИАЛИСТИЧЕСКИХ РЕСПУБЛИК

ТРУДЫ

ордена Ленина
МАТЕМАТИЧЕСКОГО ИНСТИТУТА
имени В. А. СТЕКЛОВА

CXLV

ПРИБЛИЖЕНИЕ ФУНКЦИЙ ПОЛИНОМАМИ И СПЛАЙНАМИ

С. Б. СТЕЧКИН (S. B. Stečkin, Editor of this issue)

Ответственный редактор (Editor-in-chief)
академик С. М. НИКОЛЬСКИЙ (S. M. Nikol'skiĭ)

Заместитель ответственного редактора (Assistant to the editor-in-chief)
доктор физ.-матем. наук Е. А. ВОЛКОВ (E. A. Volkov)

Library of Congress Cataloging in Publication Data

Main entry under title:

Approximation of functions by polynomials and splines.

(Proceedings of the Steklov Institute of Mathematics ISSN 0081-5438; 1981, issue 1)

Translation of: Priblizhenie funkt͡siĭ polinomami i splaĭnami.

Includes bibliography.

1. Approximation theory—Addresses, essays, lectures. 2. Polynomials—Addresses, essays, lectures. 3. Spline theory—Addresses, essays, lectures. I. Stechkin, S. B. II. Series.

QA1.A413 t. 145, etc. [QA221] 510s [511'.4] 81-2805
ISBN 0-8218-3049-X AACR2

1981 subscription consists of 4 issues, published quarterly by American Mathematical Society. Subscription price of 1981 is list $210.00, member $105.00. Previously translated Russian Numbers are sold as separate books. Write to the Society for prices.

Subscription and orders should be addressed to American Mathematical Society, P. O. Box 1571, Annex Station, Providence, R. I. 02901. *All orders must be accompanied by payment.* Other correspondence should be addressed to P. O. Box 6248, Providence, R. I. 02940.

Proceedings of the Steklov Institute of Mathematics (ISSN 0081–5438) is published quarterly by the American Mathematical Society at 201 Charles Street, Providence, Rhode Island 02904. Controlled Circulation postage paid at Providence, Rhode Island 02940. Postmaster: Send address changes to Proceedings of the Steklov Institute of Mathematics, American Mathematical Society, P. O. Box 6248, Providence, RI 02940.

March 1981

1980 *Mathematics Subject Classification.* Primary 41A15, 41A17, 41A35, 41A50, 41A65, 42A05, 42A20, 42C15; Secondary 41A10, 41A17, 41A35, 41A46, 41A65, 42A10, 42C05.

ABSTRACT. This collection includes papers devoted to the approximation of functions by polynomials and splines. The behavior of upper bounds on the deviation of classes of differentiable functions from their Fourier sums is studied, as well as the deviation from the partial Fourier sums in orthogonal polynomials. The problem of reconstructing functions on the basis of incomplete information is considered, and approximation of differentiable functions by splines with fixed as well as variable knots is investigated in weighted L_p-spaces. The problem of approximating an unbounded operator by bounded ones also receives attention. Finally, a solution (uniform relative to the order of differentiation) of the problem of the norm of the derivatives of a polynomial in the L-metric is given, and extremal properties of polynomials are studied.

Editor's note. In order to economize on production costs, displayed formulas in this translation are photographically reproduced from the Russian original. In addition to the intrinsic appearance of the formulas themselves, this means that symbols implemented in the text are not always exact replicas of their counterparts in displays.

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS IN THE ACADEMY OF SCIENCES OF THE USSR

TABLE OF CONTENTS

RUSSIAN TABLE OF CONTENTS*

* The American Mathematical Society scheme for transliteration of Cyrillic may be found at the end of index issues of *Mathematical Reviews.*

Труды Матем. Инст. Стеклов.
145(1980)

Proceedings of the Steklov
Institute of Mathematics
1981, Issue 1

APPROXIMATION OF OPERATORS OF CONVOLUTION TYPE BY BOUNDED LINEAR OPERATORS

UDC517.5

V. V. ARESTOV

ABSTRACT. In this article, the quantity

$$E(H, N) = \inf_{\|T\|_{L_r}^{L_s} \leq N} \sup_{x \in H, \|Bx\|_p \leq 1} \|Ax - Tx\|_q$$

is investigated for linear (unbounded) operators A and B in the spaces $L_\gamma = L_\gamma(R^m)$. Let S be the set of infinitely differentiable, rapidly decreasing functions, $W_{r,p}(B)$ the Banach space of functions with norm $\|x\| = \|x\|_r + \|Bx\|_p$, $\dot{W}_{r,p}(B)$ the closure of S in $W_{r,p}(B)$, and $\Pi(r, s)$ the space of multipliers from L_r to L_s. Introduce the norm $\|x\|_{r,s} = \sup\{(\theta, x): \theta \in \Pi(r, s), \|\theta\| < 1\}$ in S. It is shown that if A and B are translation invariant operators, $AS \subset S$, $BS \subset S$, and BS is dense in L_p, then

$$E(\dot{W}_{r,p}(B), N) = \inf_{\theta \in \Pi(r, s), \|\theta\| \leq N} \sup_{x \in S, \|Bx\|_{p,q} \leq 1} \{Ax(0) - (\theta, x)\}.$$

If S is dense in $W_{r,p}(B)$ then in this relation $\dot{W}_{r,p}(B)$ can be replaced by $W_{r,p}(B)$. As an example, the quantity $E(W_{r,p}(B), N)$ is calculated, and the extremal operator is determined when $A = d^k/dt^k$ and $B = d^n/dt^n$, where $0 < k < n$, $n > 3$, $1 < r = s < \infty$, $p = q = 2$ and $m = 1$.

Bibliography: 17 titles.

1. Let $L_\gamma = L_\gamma(R^m)$, $m \geqslant 1$, $1 \leqslant \gamma \leqslant \infty$, be the space of functions x measurable on R^m with finite norm

$$\|x\|_\gamma = \left\{ \int\limits_{R^m} |x(t)|^\gamma \, dt \right\}^{1/\gamma},$$

for $\gamma < \infty$, and the space C_0 of functions continuous on R^m and having zero limit when $|t| \to \infty$ for $\gamma = \infty$. Denote by $\mathfrak{L}_r^s$ the set of bounded linear operators T from L_r to L_s, and by $\mathfrak{L}_r^s(N)$ the set of operators $T \in \mathfrak{L}_r^s$ with $\|T\| \leqslant N$. Let $\mathfrak{X}$ be the set of functions measurable and locally summable on R^m, and let A and B be linear operators from $\mathfrak{X}$ to $\mathfrak{X}$ with a common domain $\mathfrak{D}$. Put $Q = Q_r^p(B) = \{x \in \mathfrak{D} \cap L_r: \|Bx\|_p \leqslant 1\}$. The quantity

$$U(T) = \sup_{x \in Q} \|Ax - Tx\|_q, \quad T \in \mathfrak{L}_r^s, \tag{1.1}$$

1980 *Mathematics Subject Classification*. Primary 41A35, 41A50; Secondary 41A65.

is the deviation of the operator T from the operator A on the class Q, in the metric of L_q. The question of properties, and especially estimates, of $U(T)$ arises in many problems.

This paper is concerned with a problem of S. B. Stečkin [1] about the best approximation of the operator A on Q by (bounded linear) operators of the class $\mathfrak{L}_r^s(N)$, i.e. with investigation of the quantity

$$E(N) = E(N; A, B; r, s; p, q) = \inf_{T \in \mathfrak{L}_r^s(N)} U(T). \tag{1.2}$$

This problem was considered by S. B. Stečkin, Ju. N. Subbotin, L. V. Taĭkov, V. N. Gabušin, V. N. Strahov, Jaak Peetre and others (see [1]–[6] and the references cited there). It is known that (1.2) is connected with other extremal problems. For example, for $q = s$ the quantity $E(N)$ gives an upper estimate for the modulus of continuity of A on the class Q; if A and B are differential operators on the real line then the best constant in inequalities between norms of functions and their derivatives can be estimated from above by $E(N)$ [1].

In this paper problem (1.2) is considered for operators A and B invariant under translation, which are close to differential operators with constant coefficients. It is shown that (under certain supplementary conditions) (1.2) leads to the problem of approximation of functionals in the space of smooth functions where the norm is induced by the class of multipliers from L_r to L_s. In case of differential operators on the line this allows us to express $E(N)$ by the best constants in inequalities between norms of derivatives of functions, in the spaces considered.

For $h \in R^m$, define in $\mathfrak{X}$ the translation operator τ_h and a similar operator σ_h by the formulas $(\tau_h x)(t) = x(t - h)$ and $(\sigma_h x)(t) = x(h - t)$. We assume that the operators A and B on $\mathfrak{D}$ are invariant with respect to translation, i.e. $\tau_h \mathfrak{D} = \mathfrak{D}$ and $A\tau_h = \tau_h A$, $B\tau_h = \tau_h B$ on $\mathfrak{D}$ for arbitrary $h \in R^m$. Denote by $\mathfrak{T}_r^s$ the set of operators $T \in \mathfrak{L}_r^s$ invariant with respect to translation on L_r, and let $\mathfrak{T}_r^s(N) = \mathfrak{T}_r^s \cap \mathfrak{L}_r^s(N)$. In [6] it is proved that under these assumptions

$$E(N) = \inf_{T \in \mathfrak{T}_r^s(N)} U(T). \tag{1.3}$$

Some properties of the quantity (1.3) are also observed in [6]. Similar problems are considered in [7] for the periodic case. For $s < r$ or $q < p$, problem (1.3) degenerates; namely, (cf. [6]) if $s < r$ then $E(N) = E(0)$, and when $q < p$ the condition $E(N) < \infty$ implies $E(N) = 0$. Therefore in what follows we assume that $s \geqslant r$ and $q \geqslant p$.

Let S be the space of rapidly decreasing, infinitely differentiable functions on R^m, and S' the corresponding dual space of generalized functions (see, for example, [8]–[10]). The value of the functional $\theta \in S'$ for the function $x \in S$ will be denoted by (θ, x). The Fourier transform $\tilde{x}$ of the function $x \in S$ (or even $x \in L_1$) is defined by

$$\tilde{x}(t) = \int x(\eta) e^{-2\pi i \langle \eta, t \rangle} d\eta;$$

here and in what follows in integrals taken over R^m the domain of integration will not be indicated. The inverse Fourier transform is determined by

$$\hat{x}(t) = \int x(\eta) e^{2\pi i \langle \eta, t \rangle} d\eta.$$

The Fourier transform of a functional $\theta \in S'$ is again a functional $\tilde{\theta} \in S'$ for which $(\tilde{\theta}, x) = (\theta, \hat{x})$. If $\theta \in L_\gamma$, $1 \leq \gamma \leq 2$, then $\tilde{\theta} \in L_{\gamma'}$, where $1/\gamma + 1/\gamma' = 1$. The function $y(t) = (\theta, \sigma_t x)$ will be called the *convolution* $\theta * x$ of the elements $\theta \in S'$ and $x \in S$; when θ is an ordinary function we have

$$(\theta * x)(t) = \int \theta(\eta)\, x(t - \eta)\, d\eta.$$

For $\theta \in S'$ we denote by $\bar{\theta}$ the functional determined by the formula $(\bar{\theta}, x) = (\theta, \sigma_0 x) = (\theta, x(-t))$, $x \in S$.

Let $b \in X'$. Define the operator B on S and formally the adjoint operator B^* by

$$Bx = b * x, \qquad B^* x = \bar{b} * x, \qquad x \in S.$$

Prescribe the condition

$$BS \subset S \tag{1.4}$$

on b, i.e. $b * x \in S$ for all $x \in S$; then obviously $B^* S \subset S$. Now we extend B to a wider set according to the scheme of Sobolev. Denote by $\Sigma = \Sigma(R^m)$ the set of those measurable, locally integrable functions x for which there exists a number $M = M(x)$ with the property

$$\int |x(t)| (1 + |t|^2)^{-M} dt < \infty.$$

Clearly, $\Sigma \subset S'$. For the pair of functions $x, y \in \Sigma$ we assume that x belongs to the domain $\mathfrak{D}(B)$ of B, and $y = Bx$ provided

$$\int x B^* \varphi\, dt = \int y \varphi\, dt \qquad \forall \varphi \in S.$$

Define the sets

$$W_p = W_p(B) = \{x \in \mathfrak{D}(B) : Bx \in L_p\},$$
$$W_{r,p} = W_{r,p}(B) = \{x \in L_r \cap \mathfrak{D}(B) : Bx \in L_p\}.$$

Using (1.4) and the fact that the spaces L_r and L_p are complete, we can easily see that $W_{r,p}(B)$ is a Banach space with respect to the norm

$$\|x\| = \|x\|_{W_{r,p}(B)} = \|x\|_r + \|Bx\|_p.$$

Let a be another element of S' with the property $a * s \in S$ for $x \in S$. In the same way, we define the operator A with domain $\mathfrak{D}(A)$.

Set

$$Q = Q_{r,p}(B) = \{x \in W_{r,p}(B) : \|Bx\|_p \leq 1\}.$$

For $W_{r,p}(B) \subset \mathfrak{D}(A)$, we define $U(T)$ for an operator $T \in \mathfrak{L}_r^s$ by

$$U(T) = \sup_{x \in Q_{r,p}(B)} \|Ax - Tx\|_q,$$

and for $W_{r,p}(B) \not\subset \mathfrak{D}(A)$ we take $U(T) = \infty$. Finally, let

$$E(N) = E(N; A, B; r, s; p, q) = \inf_{T \in \mathfrak{L}_r^s(N)} U(T). \tag{1.5}$$

In the sequel, this quantity will be investigated. The operators A and B are invariant under translation; hence relation (1.3) holds for problem (1.5).

It is known (see, for example, [8] or [10]) that if $s < r$ and $T \in \mathfrak{T}_r^s$ then $T \equiv 0$, and if $s \geqslant r$ then for S' the operator $T \in \mathfrak{T}_r^s$ is of convolution type with element $\theta = \theta_T \in S'$:

$$Tx = \theta * x, \quad x \in S. \tag{1.6}$$

The set $\Pi(r, s) = \{\theta_T\colon T \in \mathfrak{T}_r^s\}$ is a Banach space with norm $\|\theta_T\|_{\Pi(r,s)} = \|T\|_{L_r}^{L_s}$.

Choose ρ so that $1/\rho = 1 - (1/r - 1/s)$; $1 \leqslant r \leqslant s \leqslant \infty$ implies $1 \leqslant \rho \leqslant \infty$. For $\theta \in L_\rho$ and $x \in L_r$ the inequality

$$\|\theta * x\|_s \leqslant c(r, s)\|\theta\|_\rho \|x\|_r$$

holds [11], where

$$c(r, s) = A_r A_\rho A_s \leqslant 1, \quad A_\gamma = (\gamma^{1/\gamma}\gamma'^{1/\gamma'})^{1/2}, \quad 1/\gamma + 1/\gamma' = 1.$$

Hence

$$L_\rho \subset \Pi(r, s), \quad \frac{1}{\rho} = 1 - \left(\frac{1}{r} - \frac{1}{s}\right) \tag{1.7}$$

and if $\theta \in L_\rho$ then

$$\|\theta\|_{\Pi(r,s)} \leqslant c(r, s)\|\theta\|_\rho. \tag{1.8}$$

Denote by g_α that function for which $\tilde{g}_\alpha(t) = (1 + 4\pi^2|t|^2)^{-\alpha/2}$, $\alpha > 0$. It is known (see, for example, [12], Chapter V, §3) that $g_\alpha \in L_1(R^m)$ and $g_\alpha(t) = \mathcal{O}(e^{-c|t|})$ as $|t| \to \infty$ for some $c > 0$. For sufficiently large α we also have $g_\alpha \in L_{s'}$; for example, if $\alpha > m$ then $g_\alpha \in L_1 \cap C_0$. Define the convolution operator G_α on S by $\widetilde{G_\alpha x(t)} = (1 + 4\pi^2|t|)^{\alpha/2}\tilde{x}(t)$. We have $G_\alpha S \subset S$ and $x = g_\alpha * G_\alpha x$, $x \in S$. We show that if $\theta \in S'$ and $x \in S$, then

$$(\theta, x) = \int g_\alpha(h)(\theta, \tau_h G_\alpha x)\, dh, \quad x \in S. \tag{1.9}$$

It suffices to prove that for an arbitrary function $\varphi \in S$ the relation

$$\int \varphi(h)(\theta, \tau_h x)\, dt = \int \varphi(u) \int g_\alpha(h - u)(\theta, \tau_h G_\alpha x)\, dh du \tag{1.10}$$

holds. Putting $y = G_\alpha x$, we obtain (see, for example, [8], Theorem 3.13, or [9], Theorem 7.19)

$$(\varphi(t), (\theta, \tau_t x)) = (\theta, \varphi * x) = (\theta, \varphi * g_\alpha * y)$$

$$= (\varphi * g_\alpha, \theta * \bar{y}) = \int \varphi(u) \int g_\alpha(h - u)(\theta * \bar{y})(h)\, dh du,$$

which verifies (1.10).

Define the functional $\nu = \nu_{r,s}$ on S by

$$\nu(x) = \sup_{\|\theta\|_{\Pi(r,s)} \leqslant 1} (\theta, x). \tag{1.11}$$

(1.9) yields

$$\nu(x) \leqslant \|g_\alpha\|_{s'} \|G_\alpha x\|_r. \tag{1.12}$$

Using (1.7) and (1.8), we get the estimate

$$\|x\|_\gamma \leqslant c(r, s)\nu(x), \quad 1/\gamma = 1/r - 1/s.$$

Now it is easy to see that ν is a norm on S; in what follows we will denote it by $\| \ \|_{r,s}$. Thus the last inequality takes the form

$$\|x\|_\gamma \leqslant c(r, s)\|x\|_{r,s}, \quad x \in S, \quad 1/\gamma = 1/r - 1/s. \tag{1.13}$$

The norm $\|x\|_{p,q}$, $x \in S$, is defined the same way.

For $\theta \in \Pi(r, s)$ set

$$j(\theta) = \sup_{x\in S,\ \|Bx\|_{p,q}\leqslant 1} \{(\bar{a}, x) - (\bar{\theta}, x)\} \tag{1.14}$$

and introduce

$$\varkappa(N) = \inf_{\|\theta\|_{\Pi(r,s)}\leqslant N} j(\theta). \tag{1.15}$$

We will show below that (under certain additional conditions with respect to the operator B) the quantities (1.5) and (1.15) are the same.

Let us list some well-known properties of the space $\Pi(r, s)$ and, as a consequence, of the norm $\nu_{r,s}$. It is known (see, for example, [8] or [10]) that

$$\Pi(r, s) = \Pi(s', r') \text{ and } \|\theta\|_{\Pi(r,s)} = \|\theta\|_{\Pi(s',r')}, \quad \theta \in \Pi(r, s). \tag{1.16}$$

Therefore

$$\|x\|_{r,s} = \|x\|_{s',r'}, \quad x \in S. \tag{1.17}$$

Moreover, by the Riesz-Thorin interpolation theorem and (1.16) we have the embedding (see [10])

$$\Pi(r, s) \subset \Pi(\alpha, \beta), \quad \frac{1}{\alpha} = \frac{1-t}{r} + \frac{t}{s'}, \quad \frac{1}{\beta} = \frac{1-t}{s} + \frac{t}{r'}, \quad 0 \leqslant t \leqslant 1,$$

and $\|\theta\|_{\Pi(\alpha,\beta)} \leqslant \|\theta\|_{\Pi(r,s)}$. This implies

$$\|x\|_{r,s} \leqslant \|x\|_{\alpha,\beta}, \quad x \in S. \tag{1.18}$$

In particular, if p is between r and r' then

$$\|x\|_{r,r} \leqslant \|x\|_{p,p}, \quad x \in S. \tag{1.19}$$

The structure of the space $\Pi(2, 2)$ and $\Pi(r, \infty)$ is known [8], [10]; namely,

$$\Pi(2, 2) = \hat{L}_\infty = \{\theta : \tilde{\theta} \in L_\infty\},$$
$$\Pi(r, \infty) = L_{r'} \text{ for } 1 \leqslant r < \infty, \ \Pi(\infty, \infty) = \mathcal{V}, \tag{1.20}$$

where $\mathcal{V}$ is the space of Borel measures on R^m. Hence

$$\begin{aligned} &\|x\|_{r,\infty} = \|x\|_{1,r'} = \|x\|_r, \quad 1 \leqslant r < \infty, \\ &\|x\|_{1,1} = \|x\|_{\infty,\infty} = \|x\|_C, \\ &\|x\|_{2,2} = \|\hat{x}\|_1. \end{aligned} \tag{1.21}$$

(1.19) and (1.21) yield

$$\|x\|_C = \|x\|_{\infty,\infty} \leqslant \|x\|_{r,r} \leqslant \|x\|_{2,2} = \|\hat{x}\|_1, \quad x \in S. \tag{1.22}$$

Finally, we mention the inequality

$$\|x * y\|_{r,s} \leqslant \|y\|_{s'} \|x\|_{r'}, \quad x, y \in S, \tag{1.23}$$

which easily follows from (1.11) and the definition of the class $\Pi(r, s)$.

2. Let $H \subset W_{r,p}(B)$, and denote

$$u(H, \theta) = \sup_{x \in H \cap Q_{r,p}(B)} \|Ax - \theta * x\|_q, \quad \theta \in \Pi(r, s),$$
$$U(H, T) = \sup_{x \in H \cap Q_{r,p}(B)} \|Ax - Tx\|_q, \quad T \in \mathfrak{L}_r^s, \tag{2.1}$$
$$E(H, N) = \inf_{\|T\| < N} U(H, T).$$

If H is invariant with respect to translation, then by (1.3) and (1.6)

$$E(H, N) = \inf_{\|\theta\| \leqslant N} u(H, \theta). \tag{2.2}$$

In particular, when $H = S$, $u(S, \theta)$ will be denoted by $u(\theta)$.

Lemma 2.1. *If the set $Y = BS$ is dense in L_p and $u(\theta) < \infty$ for the functional $\theta \in \Pi(r, s)$, then the representation*

$$Ax = \theta * x + f * Bx \tag{2.3}$$

holds on S, where

$$f = f_\theta \in \Pi(p, q), \quad \|f\|_{\Pi(p,q)} = u(\theta). \tag{2.4}$$

Proof. By Lemma 2.1 of [6] we have the representation

$$Ax - \theta * x = FBx$$

on S, where F is a bounded linear operator from Y (with L_p norm) to L_q, $\|F\|_Y^{L_q} = u(\theta)$, and F is invariant under translation. Since $\overline{Y} = L_p$, F can be continuously extended to all of L_p. Moreover, $F \in \mathfrak{T}_p^q$, and consequently there is an element $f \in \Pi(p, q)$ such that

$$Fy = f * y, \quad y \in S,$$
$$\|F\|_Y^{L_q} = \|F\|_{L_p}^{L_q} = \|f\|_{\Pi(p,q)} = u(\theta).$$

Thus Lemma 2.1 is proved.

Denote by $\overset{\circ}{W}_{r,p} = \overset{\circ}{W}_{r,p}(B)$ the closure of S with respect to norm of $W_{r,p}(B)$. Clearly

$$S \subset \overset{\circ}{W}_{r,p}(B) \subset W_{r,p}(B).$$

Lemma 2.2. *If BS is dense in L_p and there exists a functional $\theta \in \Pi(r, s)$ such that $u(\theta) < \infty$, then*

$$\overset{\circ}{W}_{r,p}(B) \subset \mathscr{D}(A), \quad A\overset{\circ}{W}_{r,p}(B) \subset L_s + L_q, \tag{2.5}$$
$$u(\overset{\circ}{W}_{r,p}(B), \theta) = u(\theta). \tag{2.6}$$

Moreover, the representation (2.3) *with property* (2.4) *holds on $\overset{\circ}{W}_{r,p}(B)$.*

PROOF. By Lemma 2.1, we have the representation (2.3) on S. Let $t \in R^m$ and $\varphi \in S$. Substituting the function $\sigma_t\varphi$ in (2.3), we obtain

$$A^*\varphi = \bar{\theta} * \varphi + \bar{f} * B^*\varphi, \quad \varphi \in S. \tag{2.7}$$

Let $x \in \mathring{W}_{r,p}$ and

$$g = \theta * x + f * Bx. \tag{2.8}$$

Then $g \in L_s + L_q$. We show that $g = Ax$, i.e.

$$(g, \varphi) = (x, A^*\varphi), \quad \varphi \in S. \tag{2.9}$$

Choose a sequence of functions $x_k \in S$ which converges to x in $W_{r,p}$. Then we may assert that

$$(g, \varphi) = \lim_{k\to\infty} \{(\varphi, \theta * x_k) + (\varphi, f * Bx_k)\} = \lim_{k\to\infty} \{(x_k, \bar{\theta} * \varphi) + (x_k, \bar{f} * B^*\varphi)\}.$$

Using (2.7) and $AS \subset S$, we get

$$(g, \varphi) = \lim_{k\to\infty} (x_k, A^*\varphi) = (x, A^*\varphi).$$

Thus (2.9) is proved, and hence (2.3)–(2.5) follow. $S \subset \mathring{W}_{r,p}(B)$ implies $u(\theta) \leqslant u(\mathring{W}_{r,p}, \theta)$. On the other hand, the representation (2.3)–(2.4) yields the opposite inequality. The proof of the lemma is complete.

LEMMA 2.3. *If* $\overline{BS} = L_p$, *then*

$$u(\theta) = j(\theta) \quad \forall \theta \in \Pi(r, s). \tag{2.10}$$

PROOF. Assume $u(\theta) < \infty$. Then by Lemma 2.1 we have $(\bar{a} - \bar{\theta}, x) = (\bar{f}, Bx)$, $x \in S$, and $j(\theta) \leqslant \|\bar{f}\|_{\Pi(p,q)} = \|f\|_{\Pi(p,q)} = u(\theta)$. Suppose now $j(\theta) < \infty$, and let

$$x \in S, \quad \varphi \in S, \quad \|\varphi\|_{q'} \leqslant 1, \quad \|b * x\|_p \leqslant 1.$$

Put $y = \varphi * x$. We have $b * y = \varphi * b * x$, and, by (1.23), $\|b * y\|_{p,q} \leqslant \|\varphi\|_{q'}\|b * x\|_p \leqslant 1$. Therefore $|(\bar{\varphi}, (a - \theta) * x)| = |(\bar{a} - \bar{\theta}, y)| \leqslant j(\theta)$. Hence $(a - \theta) * x \in L_q$ and $\|(a - \theta) * x\|_q \leqslant j(\theta)$. Consequently $u(\theta) \leqslant j(\theta)$. Thus if one of the quantities $u(\theta)$ or $j(\theta)$ is finite then so is the other, and they are equal. The lemma is proved.

THEOREM 2.1. *If* $AS \subset S$, $BS \subset S$ *and the set* BS *is dense in* L_p, *then*

$$E(\mathring{W}_{r,p}(B), N) = E(S, N) = \varkappa(N). \tag{2.11}$$

PROOF. Evidently, $E(S, N) \leqslant E(\mathring{W}_{r,p}, N)$. The sets S and $\mathring{W}_{r,p}$ are invariant under translation; therefore we may use (2.2). Applying (2.2) and Lemma 2.2, we get the first relation in (2.11). The second one follows by Lemma 2.3.

In general, the set S is not dense in L_p. Let, say, $B = \Delta + I$, where Δ is the Laplace operator and I is the identity operator. For this operator the homogeneous equation $B^*z = Bz = 0$ has the solution

$$z(t) = |t|^{1-m/2} \mathcal{J}_{m/2-1}(|t|), \tag{2.12}$$

where $\mathcal{J}_\mu$ is the Bessel function. It is easy to show that $z \in L_\gamma$ provided that $\gamma > 2m/(m-1) = \gamma_0$. Consequently BS is not dense in L_p for $p < \gamma_0' = 2m/(m+1)$.

We may state the following sufficient condition for the density of BS in L_p:

If $\tilde{b} \in C^\infty \cap \Sigma$, $\tilde{b}$ is different from zero a.e. and $2 \leqslant p \leqslant \infty$, then $BS \subset S$ and BS is dense in L_p.

Indeed, the embedding $BS \subset S$ is obvious. Assume that BS is not dense in L_p. This is equivalent to the following: there exists a functional $g \in L_p^*$, $g \not\equiv 0$, such that

$$(g, B\varphi) = 0 \quad \forall \varphi \in S; \tag{2.13}$$

since $2 \leqslant p \leqslant \infty$, $\tilde{g}$ is an ordinary function from L_p. Moreover, (2.13) implies $\tilde{g}\tilde{b} = 0$ a.e.; hence $\tilde{g} = 0$ a.e. Consequently $g = 0$.

In connection with Theorem 2.1, the question of coincidence of the quantities $\kappa(N)$ and $E(W_{r,p}(B), N)$ arises. For this it is sufficient to have the relation

$$W_{r,p}(B) = \mathring{W}_{r,p}(B), \tag{2.14}$$

i.e. the density of S in $W_{r,p}(B)$. In general, the set S is not dense in $W_{r,p}(B)$. As an example, take an operator B with the following properties: the characteristic function g of B belongs to L_γ, $1/\gamma = 1 - (1/p - 1/r)$, and the homogeneous equation $Bz = 0$ has a solution $z \in L_r$ different from zero. Then we have $x = g * Bx$ for $x \in S$, and hence the inequality

$$\|x\|_r \leqslant \|g\|_\gamma \|Bx\|_p \tag{2.15}$$

holds on S, and consequently on $W_{r,p}(B)$ as well. Clearly $z \in W_{r,p}(B)$, while (2.15) does not hold for z and therefore $z \notin \mathring{W}_{r,p}(B)$, i.e. (2.14) does not hold. The differential operator $B = \Delta(\Delta + I)$ with

$$r > \frac{2m}{m-1}, \qquad \frac{1}{p} - \frac{1}{r} > \frac{m+1}{2m}, \qquad m > 1, \tag{2.16}$$

has the properties required above. For this operator the homogeneous equation is satisfied by the function (2.12); the characteristic function can also be easily written in an explicit form. In this way, if the parameters satisfy condition (2.16) then S is not dense in $W_{r,p}(B)$. The following sufficient conditions can be stated for (2.14):

1. *If $p \geqslant r$, then* (2.14) *holds.*

Indeed, let $x \in W_{r,p}(B)$ and $\varphi, \eta \in S$. Then

$$\begin{gathered} \|x - \varphi * \eta\| \leqslant \|x - x * \eta\| + \|(x - \varphi) * \eta\|, \\ \|(x - \varphi) * \eta\| \leqslant \|x - \varphi\|_r \left(\|\eta\|_1 + \|B\eta\|_\gamma\right), \end{gathered} \tag{2.17}$$

where $1/\gamma = 1 - (1/r - 1/p)$ and $\|z\| = \|z\|_{W_{r,p}(B)} = \|z\|_r + \|Bz\|_p$. The first two terms on the right of (2.17) can be made arbitrarily small by a properly chosen η and φ, respectively. Consequently, $x \in \bar{S} = \mathring{W}_{r,p}(B)$.

2. *If $p \leqslant r$ and B is such that*

$$\inf_{\eta \in S} \left(\|x - x * B\eta\|_r + \|y - y * B\eta\|_p\right) = 0 \quad \forall x \in L_r,\ y \in L_p, \tag{2.18}$$

then (2.14) *holds.*

Indeed, let $x \in W_{r,p}(B)$ and $\varphi, \eta \in S$. Then

$$\|x-\varphi*\eta\| \leqslant \|x-x*B\eta\|+\|x*B\eta-\varphi*\eta\|.$$

But since $x * B\eta = Bx * \eta$ we get

$$\|x * B\eta - \varphi * \eta\| = \|(Bx - \varphi) * \eta\| = \|(Bx - \varphi) * \eta\|_r + \|(Bx - \varphi) * B\eta\|_p$$
$$\leqslant \|Bx - \varphi\|_p(\|\eta\|_\delta + \|B\eta\|_1),$$

where $1/\delta = 1 - (1/p - 1/r)$. Now take an $\varepsilon > 0$. Using (2.18) for the pair of functions x and $y = Bx$, we can choose an η such that $\|x - x * B\eta\| < \varepsilon/2$. Furthermore, take a φ satisfying the condition $\|Bx - \varphi\|_p(\|\eta\|_\delta + \|B\eta\|_1) < \varepsilon/2$. Then we will have $\|x - \varphi * \eta\| < \varepsilon$, and hence $x \in \bar{S}$.

3. *If the operator B has the property*

$$h \in C_0, \quad B^* h = 0 \Rightarrow h \equiv 0, \tag{2.19}$$

then, for $r \geqslant p > 1$, (2.18), *and hence* (2.14), *holds.*

Indeed, denote by $X = (L_r, L_p)$ the Cartesian product of the spaces L_r and L_p with norm $\|(x, y)\| = \|x\|_r + \|y\|_p$, $(x, y) \in X$. Let $(x, y) \in X$ and $Y = \{(x * \beta\eta, y * B\eta), \eta \in S\}$. Evidently, Y is a linear subspace in X. Condition (2.18) means that (x, y) belongs to the closure $\bar{Y}$ of Y in X. By the Hahn-Banach theorem, for this it is sufficient (and necessary) that for all functionals $g \in X^*$ the condition $g(Y) = 0$ imply $g(x, y) = 0$. If $g \in X^*$, then $g = (g_1, g_2)$, where $g_1 \in L_r^*$, $g_2 \in L_p^*$ and $g(\xi, \zeta) = (g_1, \xi) + (g_2, \zeta)$, $(\xi, \zeta) \in X$. The condition $g(Y) = 0$ can be written in the form

$$(g_1, x*B\eta)+(g_2, y*B\eta)=0 \quad \forall \eta \in S. \tag{2.20}$$

Set $h = \bar{g}_1 * x + \bar{g}_2 * y$. (2.20) means that $B^*h = 0$. Moreover, $r \geqslant p > 1$ implies $h \in C_0$. Thus, by (2.19), $h \equiv 0$. In particular, $h(0) = (g_1, x) + (g_2, y) = g(x, y) = 0$.

4. *If BS is dense in L_1, then clearly* (2.18), *and hence* (2.14), *holds.*

It is easy to see tht the density of BS in L_1 means

$$h \in C, \quad B^*h = 0 \Rightarrow h \equiv 0. \tag{2.21}$$

5. *If $1/p - 1/r \leqslant 1/m$, and B is a differential operator with constant coefficients, then S is dense in $W_{r,p}(B)$.*

It suffices to show this for $r > p$. Let $x \in W_{r,p}(B)$. If $\psi \in S$, then $y = \psi * x \in L_r \cap C_0^\infty$, and, by a proper choice of ψ, $\|x - y\|$ can be made arbitrarily small. Now denote by η a function of one variable with the properties $\eta \in S$, $\eta(\tau) = 0$ for $|\tau| \geqslant 2$ and $\eta(\tau) = 1$ for $|\tau| \leqslant 1$. For $h > 0$ define the function φ_h on R^m by $\varphi_h(t) = \eta(ht_1) \cdots \eta(ht_m)$, $t = (t_1, \ldots, t_m) \in R^m$. Let $y_h = y\varphi_h$; then evidently $y_h \in S$. We show that $\|y - y_h\| \to 0$ for $h \to 0$. Clearly

$$\|y-y_h\|_r+\|(1-\varphi_h)By\|_p \to 0, \quad h \to 0. \tag{2.22}$$

Further, for the function $By_h = By\varphi_h$ we have the representation

$$By_h = \varphi_h By + \sum_{1 \leqslant |\nu| \leqslant M} D^\nu \varphi_h B_\nu y, \tag{2.23}$$

where B_ν are certain differential operators and M is the order of the operator B. Let $\Omega(h) = \{t = (t_1, \ldots, t_m) \in R^m\colon\ 1 \leqslant h|t_k| \leqslant 2,\ k = 1, \ldots, m\}$. If $t \notin \Omega(h)$, then $D^\nu\varphi_h(t) = 0$ for $|\nu| \geqslant 1$. Therefore by Hölder's inequality we obtain

$$\|D^\nu\varphi_h B_\nu y\|_p \leqslant \|D^\nu\varphi_h\|_\gamma \|B_\nu y\|_{L_r(\Omega(h))},$$

where $\gamma = rp/(p-r)$. Moreover, $\|D^\nu\varphi_h\|_\gamma = h^{|\nu|-m/\gamma}\|D^\nu\varphi_1\|_\gamma$. But if $1/p - 1/r \leqslant 1/m$ and $|\nu| \geqslant 1$, then $|\nu| - m/\gamma \geqslant 0$ and hence $\|D^\nu\varphi_h B_\nu y\|_p \to 0$ as $h \to 0$. Now (2.22) and (2.23) imply $\|y - y_h\| \to 0$ as $h \to 0$. Thus $x \in \bar{S}$.

Note that in fact we have proved that if at least one of the above conditions 1–5 holds, then the set K of compactly supported functions from S is dense in $W_{r,p}(B)$.

COROLLARY 2.1. *If $AS \subset S$, $BS \subset S$, BS is dense in L_p and at least one of the above conditions 1–5 holds, then*

$$E(W_{r,p}(B),\ N) = E(S,\ N) = \varkappa(N). \tag{2.24}$$

For $\delta \geqslant 0$ we define the class

$$S(\delta) = S^{p,q}_{r,s}(\delta) = \{x \in S : \|x\|_{r,s} \leqslant \delta,\ \|Bx\|_{p,q} \leqslant 1\}$$

and let

$$\lambda(\delta) = \sup_{x \in S(\delta)} Ax(0) = \sup_{x \in S(\delta)} \|Ax\|_C, \tag{2.25}$$

$$\Lambda(N) = \sup_{\delta>0}\{\lambda(\delta) - N\delta\} = \sup_{x \in S,\ \|Bx\|_{p,q}\leqslant 1}\{\|Ax\|_C - N\|x\|_{r,s}\}. \tag{2.26}$$

Applying Lemma 1 of V. N. Gabušin [3] to the (functional) problem (1.15), we obtain

$$\varkappa(N) = \Lambda(N). \tag{2.27}$$

Hence it follows that if the conditions of Theorem 2.1 hold, then

$$E(\mathring{W}_{r,p}(B),\ N) = \Lambda(N); \tag{2.28}$$

furthermore, if at least one of the conditions 1–5 holds as well, then

$$E(W_{r,p}(B),\ N) = \Lambda(N). \tag{2.29}$$

3. In this section we make the previous results more exact for the operators $A = d^k/dt^k$ and $B = d^n/dt^n$ $(0 \leqslant k < n)$ in the spaces $L_\gamma = L_\gamma(-\infty, \infty)$ on the real line $(m = 1)$. In this case the set $\mathfrak{D}(B) = \mathfrak{D}_n$ consists of functions $x \in \Sigma$ such that $x^{(n-1)}$ is locally absolutely continuous and $x^{(n)} \in \Sigma$. Let

$$W^n_{r,p} = \{x \in L_r \cap \mathscr{D}_n : x^{(n)} \in L_p\}. \tag{3.1}$$

In this case

$$E(N) = \inf_{\|T\|_{L_r}^{L_q} \leqslant N}\ \sup_{\substack{x \in W^n_{r,p}\\ \|x^{(n)}\|_p \leqslant 1}} \|x^{(k)} - Tx\|_q, \tag{3.2}$$

$$\varkappa(N) = \inf_{\|\theta\|_{\Pi(r,s)}\leqslant N} \sup_{x\in S,\, \|x^{(n)}\|_{p,q}\leqslant 1} \{x^{(k)}(0) - (\theta, x)\},$$

$$\lambda(\delta) = \sup_{x\in S(\delta)} x^{(k)}(0) = \sup_{x\in S(\delta)} \|x^{(k)}\|_C,$$

$$S(\delta) = \{x\in S : \|x\|_{r,s}\leqslant \delta,\ \|x^{(n)}\|_{p,q}\leqslant 1\},$$

$$\Lambda(N) = \sup_{\delta>0}\{\lambda(\delta) - N\delta\} = \sup_{x\in S,\, \|x^{(n)}\|_{p,q}\leqslant 1} \{\|x^{(k)}\|_C - N\|x\|_{r,s}\}. \tag{3.3}$$

As before, we assume here that $s \geqslant r$ and $q \geqslant p$.

THEOREM 3.1. *Let $s \geqslant r$ and $q \geqslant p$, and let*

$$n + \frac{1}{q} - \frac{1}{p} + \frac{1}{r} - \frac{1}{s} > 0. \tag{3.4}$$

Then

$$K = \lambda(1) < \infty, \tag{3.5}$$

$$\lambda(\delta) = K\delta^{\alpha}, \qquad \alpha = \frac{n - k + 1/q - 1/p}{n + 1/q - 1/p + 1/r - 1/s}. \tag{3.6}$$

If, in addition, $p > 1$ and $s > r$ for $k = 0$, then

$$E(N) = \varkappa(N) = \Lambda(N) = \beta\alpha^{\alpha/\beta}K^{1/\beta}N^{-\alpha/\beta}, \qquad \beta = 1 - \alpha. \tag{3.7}$$

PROOF. It is easy to construct compactly supported functions $\theta, f \in L_\infty$ such that in $\mathfrak{D}_n$ the representation $x^{(k)} = \theta * x + f * x^{(n)}$ holds (see [13]). Hence $\kappa(\|\theta\|_{\Pi(r,s)}) \leqslant \|f\|_{\Pi(p,q)}$, and consequently there is an N such that $\kappa(N) < \infty$. Clearly $\lambda(\delta) \leqslant N\delta + \kappa(N)$ for arbitrary N and δ. In particular, we have (3.5).

Now we prove (3.6). Let $\theta \in \Pi(r, s)$, $h > 0$ and θ_h a functional defined by $(\theta_h, x) = (\theta, x_h)$, where $x_h(t) = x(ht)$, $x \in S$. It is easily seen that $\theta_h \in \Pi(r, s)$, $\|\theta_h\| = h^{1/s-1/r}\|\theta\|$ and the correspondence $\theta \to \theta_h$ is a one-to-one mapping of $\Pi(r, s)$ into itself. Hence if $x \in S$, then $x_h \in S$ and $\|x_h\|_{r,s} = h^{1/s-1/r}\|x\|_{r,s}$. Let $x \in S$, $c > 0$ and $h > 0$. Then

$$\|cx_h\|_{r,s} = ch^{1/s-1/r}\|x\|_{r,s}, \qquad cx_h^{(k)}(0) = ch^{(k)}x^{(k)}(0),$$
$$\|cx_h^{(n)}\|_{p,q} = ch^{n+1/q-1/p}\|x^{(n)}\|_{p,q}. \tag{3.8}$$

(3.4) and (3.8) imply that the parameters c and h can be chosen such that

$$ch^{1/s-1/r} = \delta, \qquad ch^{n+1/q-1/p} = 1; \tag{3.9}$$

moreover

$$ch^k = \delta^\alpha, \tag{3.10}$$

where α is defined by (3.6). If c and h satisfy (3.9), then the correspondence $x \to cx_h$ is a one-to-one mapping of $S(1)$ into $S(\delta)$. This and (3.10) yield (3.6).

If $\varphi \in C_0$ and $\varphi^{(n)} = 0$, then evidently $\varphi \equiv 0$. Therefore the operator d^n/dt^n satisfies (2.19). By (2.24) and (2.27) (for $p > 1$) we have $E(N) = \kappa(N) = \Lambda(N)$. Substituting the value of $\lambda(\delta)$ in (3.6) into (2.26), we obtain (3.7). Q.E.D.