1961 Brandeis Summer Institute

LECTURES IN THEORETICAL PHYSICS

K. W. Ford

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Foreword

The lecture notes contained in this volume appear with the kind permission of the lecturers. The notes are based directly on the lectures and are intended to be informal presentations, not formal monographs. The lecturers have been cooperative in checking over the work of the notetakers, but have not undertaken significant revision or expansion, so that the responsibility for errors or for misplaced emphasis lies with the notetakers.

Notes based on the lectures of Professor Roy Glauber (statistical physics) and of Professor Max Krook (plasma physics), originally planned for this volume, will appear in the next volume of Brandeis Summer Institute Lecture Notes, scheduled for publication in the fall of 1962.

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K. W. FORD
Director
1961 Brandeis Summer Institute
in Theoretical Physics

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POLARIZATION PHENOMENA IN BETA AND GAMMA EMISSION

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I. INTRODUCTION

Since the subject of polarization phenomena is rather wide, we shall have to be very selective in such a short series of lectures. We shall mainly be concerned with polarization phenomena associated with the emission and absorption of radiation in parity conserving and non-conserving interactions. We shall primarily be interested in cases where the energy of the radiation involved is low so that its wavelength is much greater than the dimensions of the radiating system. The retardation expansion is therefore meaningful and the fields of the emitted radiation often carry off the minimum angular momentum consistent with the conservation laws. These transitions usually take place between states of well defined angular momentum and parity and the radiation itself always carries off a well defined angular momentum and parity, even though the interaction itself may or may not conserve parity. For the parity conserving situation our remarks will apply chiefly to the emission of photons, for the parity non-conserving situation, our remarks will be applied to beta decay.

The techniques we shall be using were developed mainly by Wigner and Racah, and are of sufficient generality so that they can be used in elementary particle physics, nuclear structure and in the many body problem.

General References:

1. M. E. Rose: Elementary Theory of Angular Momentum (referred to as ETAM).
2. Rotenberg et. al: 3j and 6j Symbols.
3. Edmonds: Angular Momentum and Quantum Mechanics.
4. Gelfand and Sapiro: American Mathematical Society Translations, Series 2, Vol. 3, 207.
5. U. Fano: National Bureau of Standards Report 1214.

2. GENERAL DEFINITION AND PROPERTIES OF ANGULAR MOMENTUM OPERATORS

2-1 Definition of Angular Momentum Operators

Consider a point $\vec{x}$ in real 3-dimensional Euclidean space which undergoes a transformation to $\vec{x}'$ under a rotation R of the coordinate system.

$$\vec{x}' = R\,\vec{x} \tag{2.1}$$

R is a 3 x 3 real orthogonal matrix. The set of matrices R constitute a group under ordinary matrix multiplication, the three dimensional rotation group R_3. This group is a 3 parameter compact Lie Group. Clearly the matrices R themselves form a representation of R_3. However, we are interested in another representation of R_3. Consider a general one component field $\psi(\vec{x})$ which is classical in the sense that it is to be interpreted as assigning a number to every point in space. Then the rotation R induces a change in $\psi \rightarrow \psi'$ such that

$$\psi'(\vec{x}') = \psi(\vec{x}) \tag{2.2}$$

where $\vec{x}'$ is related to $\vec{x}$ by (2.1). The transformed field ψ' is related to ψ by a linear unitary transformation which we denote by T(R):

$$[T(R)\,\psi]\,(\vec{x}) = \psi(R^{-1}\vec{x}) \tag{2.3}$$

T(R) operates in the space of the $\psi(\vec{x})$, generally a complex infinite dimensional space, in contradistinction to R which operates in a real 3-dimensional space. It can be shown that the operators T(R) constitute a representation of R_3. It is this representation which we shall use to define the angular momentum operators.

We specialize to the case of an infinitesimal rotation by an angle θ around an arbitrary axis $\vec{n}$. This rotation can be reproduced by three successive rotations around the three coordinate axes. To first order, we can write

$$\left[T(R_{\vec{n},\theta})\,\psi\right](\vec{x}) = \left[\prod_{k=1}^{3} (1 + X_k)\,\psi\right](\vec{x}) \tag{2.4}$$

where the X_k are certain infinitesimal operators. To first order,

$$\left[T(R_{\vec{n},\theta})\, \psi \right](\vec{x}) = \left[(1 + \sum_{k=1}^{3} X_k)\, \psi \right](\vec{x}) = \psi(\vec{x}) + \delta\psi(\vec{x}) \qquad (2.5)$$

We should really write

$X_k \equiv X_k(\vec{n}, \theta)$ with the X_k such that $\theta \rightarrow 0 \Rightarrow X_k \rightarrow 0$.

It is verified that the X_k satisfy

$$X_k = -(X_k^{-1}) \qquad (2.6)$$

$$X_k^{*} = -X_k \quad \text{(*denotes Hermitian conjugate)} \qquad (2.7)$$

We can make the parametric dependence of the X_k's explicit by writing

$$X_k = -i\, n_k\, J_k\, \theta \quad \text{(no summation)} \qquad (2.8)$$

$J_{1,2,3}$ are a set of linear hermitian operators independent of θ. [The definitions of hermiticity etc. are as in usual non-relativistic quantum mechanics.] For a finite rotation, it can be shown that

$$\left[T(R_{\vec{n},\theta})\, \psi \right](\vec{x}) = \left[e^{-i\vec{n}\cdot\vec{J}\,\theta} \right](\vec{x}) \qquad (2.9)$$

It will be noticed that the mapping $T(R) \rightarrow J$ in

$$T(R) = e^{i\,\vec{n}\cdot\vec{J}\,\theta}$$

is essentially the mapping from a Lie group to its generating Lie Algebra. The algebraic aspects of the theory of angular momentum operators thus correspond to properties of the Lie Algebra of R_3.

We define the Hermitian operator J_k (k=1, 2, 3) as the angular momentum operators of the system.

For an explicit form for the J_k in terms of differential operators consider an infinitesimal rotation θ around the x_3 axis; then

$$x_1' = x_1 + \theta x_2$$

$$x_2' = - \theta x_1 + x_2 \qquad (2.10)$$

$$x_3' = x_3$$

$$\therefore \quad [T(R)\psi](\vec{x'}) = \psi(\vec{x}) + \theta \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}\right) \psi(\vec{x})$$

$$= (1 - i J_3 \theta)\psi(\vec{x}) \text{ by definition}$$

$$\therefore \quad J_3 = - i (\vec{x} \times \vec{\nabla})_3 = L_3 \qquad (2.11)$$

where L_3 is the usual notation for the differential operator

$$- i (\vec{x} \times \vec{\nabla})_3)$$

If ψ is a multi-component wave function with components $\psi_\rho(\vec{x})$, $\rho = 1, 2, \ldots, n$ then it can be shown that

$$\vec{J}\psi_\rho(\vec{x}) = (\vec{L} \times \mathbb{1} + \vec{S})\psi_\rho(\vec{x})$$

where $\vec{S}$ is a certain set of three square matrices and $\mathbb{1}$ is the unit matrix of the same number of dimensions as $\vec{S}$. For a particle of spin s, $\vec{S}$ has dimension 2s + 1.

2-2 Commutation Relations

We discuss in brief an intuitive geometrical approach for the derivation of the commutation relations of the angular momentum operators J_k [for details see ETAM pp. 20-22] .

Consider a point P on the x-axis and apply two infinitesimal rotations $d\theta_x$ and $d\theta_y$ about the x- and y-axes respectively. We next interchange the order in which the rotations are applied and notice that the net difference between the two rotations is a rotation of magnitude $d\theta_x \; d\theta_y$ about the Z axis and we write:

$$e^{-id\theta_x J_x} e^{-id\theta_y J_y} - e^{-id\theta_y J_y} e^{-id\theta_x J_x}$$

$$= e^{-id\theta_x d\theta_y J_z} - 1$$

Expanding the exponentials up to second order in $d\theta_1$ and $d\theta_2$

$$[1 - id\theta_x J_x - \tfrac{1}{2}(d\theta_x)^2 J_x^2]\,[1 - id\theta_y J_y (d\theta_y)^2 J_y^2]$$

$$-[1 - id\theta_y J_y - \tfrac{1}{2}(d\theta_y)^2 J_y^2]\,[1 - id\theta_x J_x - \tfrac{1}{2}(d\theta_x)^2 J_x^2]$$

$$= 1 - id\theta_x d\theta_y J_z$$

Consequently,

$$[J_x , J_y] = i J_z \tag{2.12}$$

Equating higher order coefficients on both sides would give relations deducible from (2.12). This "proof" emphasises the fact that the commutation relations for the angular momentum operator are a direct consequence of the lack of commutation of rotations.

All the commutation relations of type (2.12) can be summarized as

$$[J_j , J_k] = i \,\epsilon_{jk\ell} J_\ell \tag{2.13}$$

(summation convention)

$\epsilon_{jk\ell}$ being the completely anti-symmetric third rank unit tensor.

2-3 The Total Angular Momentum Operator

Define next the operator $\vec{J}^2 = \sum_{k=1}^{3} J_k^2$. The eigenvalues of $\vec{J}^2$ give the total angular momentum of the system. We shall show that $[\vec{J}^2, T(R)] = 0$. This is an important result since it ensures that the result of a measurement of the total angular momentum is independent of the coordinate system in which the observation is made.

Thus it makes sense to speak of the spin of Li^7 as 3/2, for instance.

Now

$$\left[\vec{J}^2, J_i\right] = \left[\sum_{k=1}^{3} J_k^{\,2}, J_i\right] = \sum_k J_k \left[J_k, J_i\right] + \sum_k \left[J_k, J_i\right] J_k$$

$$= i \sum_{k,\ell} \epsilon_{k i \ell} (J_k J_\ell + J_\ell J_k) = 0$$

since the first factor is antisymmetric in k and ℓ and the second factor is symmetric in k and ℓ. $\therefore$ $\vec{J}^2$ commutes with all the functions of the J_k, in particular with T(R). Later we recognize that $\vec{J}^2$ is a scalar (i. e.a multiple of the unit matrix) when the commutation relations with the J_i follow at once. The eigenvalues of $\vec{J}^2$ are thus independent of rotations of the coordinate system. Further, we can diagonalize $\vec{J}^2$ and one of the J_k simultaneously and label eigenfunctions of the angular momentum operators with their eigenvalues. Conventionally, J_3 i.e.J_z is diagonalized with $\vec{J}^2$ corresponding to the choice,in older quantum theory, of z as the " axis of quantization".

The discussion so far has been purely geometrical. This emphasizes that many of the answers to questions relating to transitions in nuclei are purely kinematic having nothing to do with the dynamics of the system. This separation will be emphasized and made explicit when we discuss the Wigner-Eckart Theorem.

2-4 Eigenvalues of $\vec{J}^2$ and J_z

Let the normalized eigenfunctions ψ of $\vec{J}^2$ and J_z be labelled by j and m, two numbers corresponding to the eigenvalues η_j of $\vec{J}^2$ and m of J_z. The ψ_j^m are complete and orthonormal. By definition

$$\vec{J}^2 \psi_j^m = \eta_j \psi_j^m \qquad (2.14)$$

$$J_z \psi_j^m = m \psi_j^m \qquad (2.15)$$

$$\therefore (J_x^{\,2} + J_y^{\,2})\psi_j^m = (\vec{J}^2 - J_z^{\,2}) \psi_j^m = (\eta_j - m^2)\psi_j^m .$$

Since $(J_x^{\,2} + J_y^{\,2})$ is the sum of squares of Hermitian operators its eigenvalues are real and positive.

$$\therefore (\eta_j - m^2) \geqslant 0 .$$

$$\therefore -\sqrt{\eta_j} \leqslant m \leqslant \sqrt{\eta_j} \tag{2.16}$$

Let
$$J_{\pm} = J_x \pm i J_y \tag{2.17}$$

Also, let
$$(\phi_j^m)^{\pm} \equiv (J_1 \pm i J_2)\psi_j^m \tag{2.18}$$

Then $(\phi_j^m)^{\pm}$ is an eigenfunction of J_3 with eigenvalue $m \pm 1$; for

$$J_z (\phi_j^m)^{\pm} = J_{\pm} J_z \psi_j^m + [J_z, J_{\pm}] \psi_j^m$$

$$= m J_{\pm} \psi_j^m \pm J_{\pm} \psi_j^m$$

since direct evaluation gives

$$[J_z, J_{\pm}] = \pm J_{\pm}$$

$$\therefore J_z (\phi_j^m)^{\pm} = m J_{\pm} \psi_j^m \pm J_{\pm} \psi_j^m$$

$$= (m \pm 1) J_{\pm} \psi_j^m \quad = (m \pm 1) (\phi_j^m)^{\pm}$$

Also, $(\phi_j^m)^{\pm}$ is an eigenfunction of $\vec{J}^2$ with eigenvalue η_j, for

$$\vec{J}^2 (J_{\pm} \psi_j^m) = J_{\pm} \vec{J}^2 \psi_j^m + [\vec{J}^2, J_{\pm}] \psi_j^m = J_{\pm} \vec{J}^2 \psi_j^m$$

$$= \eta_j (\phi_j^m)^{\pm}$$

Thus one has

$$\vec{J}^2 (\phi_j^m)^{\pm} = \eta_j (\phi_j^m)^{\pm} \tag{2. 19}$$

$$J_z (\phi_j^m)^{\pm} = (m \pm 1)(\phi_j^m)^{\pm} \tag{2. 20}$$

These equations show that the functions $(\phi_j^m)^{\pm}$ must be proportional to the normalized eigenfunctions $\psi_j^{m \pm 1}$ and one may write

$$J_{\pm} \psi_j^m = (\phi_j^m)^{\pm} = \Gamma_{\pm} \psi_j^{m \pm 1} \tag{2. 21}$$

$\Gamma_{\pm}$ are constants of proportionality. Because of (2. 21) $J_{\pm}$ are sometimes called "raising" and "lowering" operators respectively. Since (2. 16) holds, the values of m for a given value of η_j are bounded above and below. Let the maximum and minimum values of m be m_2 and m_1 respectively for $\eta_j = \eta_j$. Then

$$J_+ \psi_j^{m_2} = 0, \quad \psi_j^{m_2} \neq 0$$

$$J_- \psi_j^{m_1} = 0, \quad \psi_j^{m_1} \neq 0 \tag{2. 22}$$

But

$$J_- J_+ = \vec{J}^2 - J_z^2 - J_z ;$$

$$J_+ J_- = \vec{J}^2 - J_z^2 + J_z$$

$\therefore$

$$J_- J_+ \psi_j^{m_2} = 0 = (\eta_j - m_2^2 - m_2) \psi_j^{m_2}$$

$$J_+ J_- \psi_j^{m_1} = 0 = (\eta_j - m_1^2 + m_1) \psi_j^{m_1}$$

$\therefore$

$$\left.\begin{aligned} \eta_j - m_2 (m_2+1) = 0 \\ \eta_j - m_1 (m_1-1) = 0 \end{aligned}\right\} \tag{2. 23}$$

Eliminating η_j,

$$m_2(m_2 + 1) = m_1 (m_1 - 1)$$

$\therefore$

$$(m_2 + m_1)(m_2 - m_1 + 1) = 0$$

By assumption $m_2 \geqslant m_1$; thus $m_1 = -m_2$. Since the successive values by m differ by unity, $(m_2 - m_1)$ must be a non-negative integer. We denote this integer by 2j; thus j can have possible values

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots .$$

and from $m_2 - m_1 = 2j$, $m_2 = j$, $m_1 = -j$

∴ For a given value of j, permissible values of m are

$$-j \leqslant m \leqslant j \; ; \; j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots . \tag{2.24}$$

There are therefore 2j + 1 permissible value of m for a given j. Inserting in either of (2.23)

$$\eta_j = j(j+1) \tag{2.25}$$

To summarize: $\vec{J}^2 \psi_j^m = j(j+1) \psi_j^m$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (2.26)

$$J_z \psi_j^m = m \psi_j^m, \quad m = j, j-1, \ldots, -j \tag{2.27}$$

2-5 Matrix Elements of J_x and J_y in a Representation with $\vec{J}^2$ and J_z Diagonal

From (2.21) $J_{\pm} \psi_j^m = \Gamma_{\pm} \psi_j^{m \pm 1}$

$J_{\pm}$ are clearly non-diagonal in this representation since they connect states with different m values. It is also clear that they have non-zero matrix elements only one step off the diagonal.

We have

$$(J_{\pm} \psi_j^m, J_{\pm} \psi_j^m) = |\Gamma_{\pm}|^2$$

$$\therefore \; (\psi_j^m, J_{\pm}^* J_{\pm} \psi_j^m) = |\Gamma_{\pm}|^2$$