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RECENT PROGRESS IN FOURIER ANALYSIS

Proceedings of the Seminar on Fourier Analysis held in El Escorial, Spain, June 30 - July 5, 1983

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CONTENTS

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FUNCTIONS OF L^p-BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

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The aim of this paper is to construct a functional calculus over an algebra of L^p-bounded pseudo-differential operators acting on functions defined on a compact manifold without boundary.

The operators we consider here depend on amplitudes or symbols with a finite number of derivatives, without any hypothesis of homogeneity. The manifolds where the operators act are also of class C^M for a suitable M. In this way is it possible to control the number of derivatives of f that we need in order to give meaning to $f(A)$, when A is a self-adjoint operator in that algebra.

Indeed, this program was carried out in [1] and [2] when $p = 2$. In [1] an algebra of pseudo-differential operators acting on functions defined in R^n is constructed. The main tool to do that is the sharp L^2 estimates obtained by R. Coifman and Y. Meyer in [3]. Then, functions of those operators are defined by means of the H. Weyl formula (see [4], for example). Since it seems not to be possible to obtain directly a polynomial estimate for the exponential $\exp(-2\pi itA)$ in terms of t, a roundabout argument is employed by introducing an adapted version of the characteristic operators defined by A. P. Calderón in [5].

All this machinery is extended in [2] to non-infinitely differentiable compact manifolds without boundary.

In order to get the L^p version of these results the first thing to do is to obtain the analogous of the algebra constructed in [1]. The main point is to observe that amplitudes in a subclass of $S^o_{1,1}$ give rise to operators on which the classical theory of Calderón and Zygmund works (see [6]). Unfortunately as far as I know, it is an open question to get in the euclidean case a non trivial estimate for the exponential $\exp(-2\pi itA)$. However, when the operators act on

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functions defined in compact manifolds, a suitable estimate can be
obtained and so, a non-infinitely differentiable functional calculus
runs.

Given $0 \leq \delta < 1$, $k = 1, 2, \ldots$, let

$$N = \begin{cases} k/1-\delta & \text{if this is an integer} \\ [k/1-\delta] + 1 & \text{if not} \end{cases}$$

We will consider operators K acting on S in the following way

$$Kf = \sum_{j=0}^{N-1} \int e^{-2\pi i x\xi} \, p_j(x,\xi) \hat{f}(\xi) d\xi + Rf$$

where

i) The function p_j belongs to the class S^j; that is to say,
p_j is a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n$; it has continuous
derivatives in the variable ξ up to the order $n+N+2-j$ and each
function $D_\xi^\gamma p_j$ has continuous derivatives in x,ξ up to the order
$2[n/2]+N+k+2-j$, satisfying

$$\sup_{\substack{x,\xi \, \in \, \mathbb{R}^n \\ \alpha, \, \beta, \, \gamma}} \frac{|D_s^\alpha D_\xi^\beta D_\xi^\gamma \, p_j(x,\xi)|}{(1+|\xi|)^{-j(1-\delta)+|\alpha|\delta-|\beta+\gamma|}} < \infty$$

ii) For $1 < p_0 \leq 2$ fixed, R is a linear and continuous opera-
tor from L^p into itself for $p_0 \leq p \leq p_0'$. Moreover, R and the
adjoint R^* are continuous from L^p into L_k^p, where L_k^p denotes
the Sobolev space of order k and p_0' is the conjugate exponent
of p_0.

Let M_k be the class of the operators K.

Now, let X be a differentiable compact manifold of dimension n
and class C^M, $M = 2[n/2]+n+2N+k+5$, without boundary; X has a
measure μ which in terms of any local coordinate system
$x = (x_1, \ldots, x_n)$ can be express as $G(x)dx_1 \ldots dx_n$, where $G > 0$
is a function of class C^{M-1}.

We will introduce the following notation.

Let U_1, U_2 be open bounded subsets of X or $\mathbb{R}^n$ let
$\phi : U_1 \rightarrow U_2$ be a diffeomorphism of class C^M; if f is a function
defined on the ambient space of U_2, $\phi^*(f)$ will denote the function

defined on the ambient space of U_1 which coincides with $f \circ \phi$ on U_1 and vanishes outside U_1. On the other hand, if A is an operator acting on functions defined on the ambient space of U_2, by $\phi^*(A)$ we denote the operator acting on functions defined on the ambient space of U_1 as

$$\phi^*(A)(f) = \phi^*[A(\phi^{-1*}(f))]$$

Now, we are ready to define classes of operators on X.

Given $1 < p_0 \leq 2$, R belongs to $R_k(X)$ if R is a linear continuous operator from $L^p(X)$ into itself for $p_0 \leq p \leq p_0'$ and R, R^* map continuously $L^p(X)$ into $L_k^p(X)$ for $p_0 \leq p \leq p_0'$.

$R_k(X)$ is a self-adjoint Banach algebra with the norm

$$|R|_{R_k} = |R|_{L^{p_0}, L_k^{p_0}} + |R|_{L^{p_0'}, L_k^{p_0'}} + |R^*|_{L^{p_0}, L_k^{p_0}} + |R^*|_{L^{p_0'}, L_k^{p_0'}}$$

Now, given $1 < p_0 \leq 2$, $M_k(X)$ is the class of linear continuous operators A from $L^p(X)$ into itself for $p_0 \leq p \leq p_0'$, which satisfy the following two conditions

   i) Given $\phi_1, \phi_2 \in C_0^M(X)$ with disjoint supports, the operator $\phi_1 A \phi_2$ belongs to $R_k(X)$. Here ϕ_1, ϕ_2 stand for the operators of multiplication by the function ϕ_1, ϕ_2, respectively.

   ii) Let $U \subset X$ be an open subset and let $\phi : U \to U_1$ be a diffeomorphism of class C^M extendable to a neighborhood of $\bar{U}$, where $U_1 \subset \mathbb{R}^n$. There exists an operator $A_1 \in M_k$ such that if $\phi_1, \phi_2 \in C_0^M(U)$,

$$\phi_1 A \phi_2 = \phi_1 \phi^*(A_1) \phi_2$$

$M_k(X)$ is a self-adjoint algebra and $R_k(X)$ is a two-sided ideal of $M_k(X)$; moreover, operators in $M_k(X)$ are continuous from $L_m^p(X)$ into itself for $p_0 \leq p \leq p_0'$, $0 \leq m \leq k$.

   It is possible to endow $M_k(X)$ with a complete norm. In order to avoid technical details, we will not precise the definition. With this norm $R_k(X)$ is continuously included in $M_k(X)$ and $M_k(X)$ is continuously included in $L(L_m^p(X))$, the space of linear and continuous operators from $L_m^p(X)$ into itself, for $p_0 \leq p \leq p_0'$, $0 \leq m \leq k$.

THEOREM 1. *Let* p_0, k *and* n *be such that* $1/p_0 - k/n \le 1/2$.

Given a self-adjoint operator $A \in M_k(X)$ *and a function* f *in the Sobolev space* L_s^2, *where* $s > 2\mu + 5/2$, $\mu = 2[n/2]+n+k+ + N(N+3)/2+4$, *the Bochner integral*

$$\int_{-\infty}^{\infty} e^{-\pi itA} \hat{f}(t)dt$$

belongs to $R_k(X)$ *and coincides with* $f(A)$ *calculated by means of the spectral formula in* $L(L^2(X))$.

Remarks:

a) It is possible to impose on f additional conditions under which the operator $f(A)$ belongs to $R_k(X)$.

b) When $p_0 = 2$ the above theorem remains true with $s > \mu + 3/2$.

c) The Weyl's formula also allows to define functions of a tuple of non-commuting self-adjoint operators.

We will include here the proof of the theorem 1 in a particular but significant case.

Suppose that $\delta = 0$, $k = 1$; it follows that $N = 1$. It is clear that theorem 1 can be deduced from a suitable estimate for $|\exp(-2\pi itA)|_{M_1(X)}$ in terms of $t \in \mathbb{R}$.

In order to get this estimate, some notations and results will be needed. We fix in X coordinate neighborhoods U_j, diffeomorphisms $\phi_j : U_j \to \phi_j(U_j)$ of class C^M, where $M = 2[n/2]+n+8$, functions $\theta_j \in C_0^M(U_j)$, $\theta_j \ge 0$ and a finite partition of unity $\{\eta_j\}$ of class C^M, such that $\text{supp}(\theta_j) \subset U_i$ whenever $\text{supp}(\theta_j) \cap \text{supp}(\theta_i) \ne \emptyset$; $\theta_i = 1$ in a neighborhood of $\text{supp}(\eta_j)$ if $j = i$ or if $\text{supp}(\eta_j) \cap \text{supp}(\eta_i) \ne \emptyset$.

Now, we define an space of symbols for operators in $M_1(X)$. More exactly, for each j we consider the restriction to $\phi_j(U_j)$ of a function $p^{(j)} \in S^0$. We define a norm of such a restriction as

$$|p^{(j)}| = \sup \frac{|D_x^\alpha D_\xi^\beta D_\xi^\gamma p^{(j)}(x,\xi)|}{(1 + |\xi|)^{-|\gamma+\beta|}}$$

where the supremum is taken over $x \in \phi_j(U_j)$, $\xi \in \mathbb{R}^n$, $|\xi| \le n+3$,

$|\alpha+\beta| \leq 2[n/2]+4$, j.

We note $\mathbb{N}_1(X)$ this space. With the pointwise multiplication

$$(p^{(j)}).(q^{(j)}) = (p^{(j)}q^{(j)})$$

as a product, $\mathbb{N}_1(X)$ becomes a commutative Banach algebra.

LEMMA. _Let_ $H = (p^{(j)})$ _be an element in_ $\mathbb{N}_1(X)$; _we suppose that each_ $p^{(j)}$ _is a real function. Then, if_ $t \in \mathbb{R}$,

$$|\exp(-2\pi itH)| \leq C(1 + |H|)(1 + |t|)|^\mu$$

where $C = C(X) > 0$, $\mu = 2[n/2]+n+7$.

Proof

Since $\mathbb{N}_1(X)$ is a Banach algebra, the exponential $\exp(-2\pi itH)$ is well defined; moreover it is equal to $\exp(-2\pi itp^{(j)})_j$. According to the norm that the space $\mathbb{N}_1(X)$ has, the conclusion follows.

Now, we will introduce the space $\mathbb{M}_1(X)$ in the following way An element K of $\mathbb{M}_1(X)$ is an operator R in $R_1(X)$ and a vector $(p^{(j)})$ in $\mathbb{N}_1(X)$ subject to the condition that if $U_i \cap U_j \neq \emptyset$ and $\phi_{ij} = \phi_j \circ \phi_i^{-1}$, then

$$(p^{(i)}) = \phi_{ij}^*(p^{(j)}) \quad \text{in} \quad \phi_i(U_i \cap U_j).$$

Such an element K will be denoted as $\{(p^{(j)})_j, R\}$.

We define a norm in $\mathbb{M}_1(X)$ as follows

$$|K|_{\mathbb{M}_1} = |(p^{(j)})| + |R|_{R_1}$$

Given $K \in \mathbb{M}_1(X)$ we define an operator $\Lambda(K)$ in the following way

$$\Lambda(K) = \sum_j \eta_j \phi_j^*(A_j)\theta_j + R$$

where

$$(A_j f)(x) = \begin{cases} \int e^{-2\pi i x\xi} p^{(j)}(x,\xi)\hat{f}(\xi)d\xi & \text{if} \quad x \in \phi_j(U_j) \\ 0 & \text{if not} \end{cases}$$

It can be proved that $\Lambda(K)$ belongs to $M_1(X)$. Moreover the linear map

$$IM_1(X) \xrightarrow{\;\Lambda\;} H_1(X)$$

$$K \longrightarrow \Lambda(K)$$

is into and continuous. Furthermore, if $A \in M_1(X)$ is self-adjoint, $A = \Lambda(K)$ for some $K = \{(p^{(j)}), R\}$, with $p^{(j)}$ real for all j.

It is possible to define a product in $IM_1(X)$ in such a way that $IM_1(X)$ becomes a Banach algebra and the map Λ above is a continuous homomorphism of algebras.

Finally, let us consider the maps

$$IM_1(X) \xrightarrow{\;\Omega\;} IN_1(X) \xrightarrow{\;\Omega_1\;} IM_1(X)$$

$$\{(p^{(j)}), R\} \longrightarrow (p^{(j)}) \longrightarrow \{(p^{(j)}), 0\}$$

Ω is a continuous homomorphism of algebras and the linear map Ω_1 is a right continuous inverse of Ω.

THEOREM 2. *Suppose that* $1/p_0 - 1/n \leq 1/2$.

Let $H = \{(p^{(j)}), R\}$ *be an element of* $IM_1(X)$ *such that* $\Lambda(H)$ *is a self-adjoint operator and the functions* $p^{(j)}$ *are real for all* j. *Then, if* $t \in \mathbb{R}$,

$$\left| \exp(-2\pi itH) \right|_{IM_1} \leq C\left[(1 + |H|_{IM_1})(1 + |t|)\right]^{2\mu+2}$$

where $C = C(X) > 0$, $\mu = 2[n/2]+n+7$.

Proof

According to the notations above, we set $A = \Lambda(H)$, $K = \Omega(H) = (p^{(j)}) \in \mathbb{N}_1(X)$.

We assert that

$$e^{-2\pi itH} - \Omega_1(e^{-2\pi itK})$$

it an element of the form $\{(0), R(t)\}$.

In fact, since Ω is a continuous homomorphism of algebras and Ω_1 is a right inverse of Ω, we have

$$\Omega\left[e^{-2\pi itH} - \Omega_1(e^{-2\pi itK})\right] = e^{-2\pi itK} - \Omega\Omega_1(e^{-2\pi itK}) = 0.$$

On the other hand, since Ω_1 is a continuous map, according to the lemma it suffices to estimate the norm of $\{(0),R(t)\}$ in $IM_1(X)$, which coincides with the norm of $R(t)$ in $R_1(X)$.

We have

$$R(t) = \Lambda\left[e^{-2\pi itH} - \Omega_1(e^{-2\pi itK})\right] =$$

$$= e^{-2\pi itA} - \Lambda\Omega_1(e^{-2\pi itK}).$$

If we denote with $\dot{R}(t)$ the derivative of $R(t)$ with respect to t, we get

$$\dot{R}(t) = e^{-2\pi itA}(-2\pi iA) - \Lambda\Omega_1(e^{-2\pi itK}(-2\pi iK)) =$$

$$= \left[e^{-2\pi itA} - \Lambda\Omega_1(e^{-2\pi itK})\right](-2\pi iA) + \Lambda\Omega_1(e^{-2\pi itK})(-2\pi iA) -$$

$$- \Lambda\Omega_1(e^{-2\pi itK}(-2\pi iK)) = R(t)(-2\pi iA) + B_1(t) \qquad (1)$$

Since

$$B_1(t) = \Lambda\left[\Omega_1(e^{-2\pi itK})(-2\pi iH) - \Omega_1(e^{-2\pi itK}(-2\pi iK))\right]$$

and

$$\Omega\left[\Omega_1(e^{-2\pi itK})(-2\pi iH) - \Omega_1(e^{-2\pi itK}(-2\pi iK))\right] = 0,$$

we deduce that $B_1(t)$ belongs to $R_1(X)$ for each t.

Thus,

$$|B_1(t)|_{R_1} = |\Omega_1(e^{-2\pi itK})(-2\pi iH) - \Omega_1(e^{-2\pi itK}(-2\pi iK))|_{IM_1}$$

$$\leq C\left[(1 + |H|_{IM_1})(1 + |t|)\right]^\mu$$

where $C = C(X) > 0$.

Since $R(0) = 0$, from (1) it follows that

$$R(t) = \int_0^t B_1(s)e^{-2\pi i(t-s)A} \, ds \qquad (2)$$

But we can also write

$$R(t) = (-2\pi iA)e^{-2\pi itA} - \Lambda\Omega_1(e^{-2\pi itK}(-2\pi iK))$$

or

$$R(t) = (-2\pi iA)R(t) + B_2(t)$$

where $B_2(t) \in R_1(X)$ for each t and

$$|B_2(t)|_1 \leq C[(1 + |H|_{M_1})(1 + |t|)]^\mu , \quad c = C(X) > 0$$

So,

$$R(t) = \int_0^t e^{-2\pi i(t-s)A} B_2(s)\,ds \tag{3}$$

or

$$R^*(t) = \int_0^t B_2^*(s) e^{2\pi i(t-s)A}\,ds \tag{4}$$

where $*$ denotes the adjoint.

Now, suppose we show that

$$|e^{-2\pi i(t-s)A}|_{L^{p_0},L^p} \leq C[(1 + |H|_{M_1})(1 + |t-s|)]^{\mu+1} \tag{5}$$

We will get the same estimate for

$$|e^{-2\pi i(t-s)A}|_{L^{p_0'},L^{p_0'}}$$

Thus, according to (2) and (4), we can deduce that

$$|R(t)|_{R_1} \leq C[(1 + |H|_{M_1})(1 + |t|)]^{2\mu+2}$$

So, it remains to prove (5).

From the definition of the operator $R(t)$, it is clear that it suffices to obtain the estimate

$$|R(t)|_{L^{p_0},L^{p_0}} \leq C[(1 + |H|_{M_1})(1 + |t|)]^{\mu+1}$$

But according to the hypothesis $1/p_0 - 1/n \leq 1/2$, the Sobolev immersion theorem provides the continuous inclusion of $L_1^{p_0}(X)$ into $L^2(X)$; moreover, since $p_0 \leq 2$, we also have a continuous injection from $L^2(X)$ into $L^{p_0}(X)$. Thus, the desired estimate follows from (3).

This completes the proof of the theorem 2.

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<center>

ON PROBLEMS RELATED TO THEOREMS A and B
WITH ESTIMATES

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Introduction

We are interested in theorems of type Cartan A and B, but with estimates on the growth of the functions.

Type A problems

a) The caracterisation of the zero set of a holomorphic function in a given class. G. Henkin and H. Skoda gave, independently, a complete answer for this problem in the case of the Nevanlinna class of the unit ball B of $\mathbb{C}^n$, [12], [18]; N. Varopoulos studied the case of the Hardy classes H^p of the unit ball [19].

All of them used the P. Lelong's method leading to solve a $\partial\bar{\partial}$ equation with estimates.

b) Another example of type A problem is the corona problem: let $f_1,\ldots,f_k$ be bounded holomorphic functions in the unit ball of $\mathbb{C}^n$ such that:

$$|f_1| +\ldots+ |f_k| \geq \delta > 0 \quad \text{in}\quad B;$$

are there $g_1,\ldots,g_k$ bounded holomorphic functions in B with: $f_1 g_1 +\ldots+ f_k g_k = 1$?

This problem was solved in 1962 by L. Carleson [9] in the case n = 1; L. Hormander [14] showed that the problem can be replaced by the problem of solving a $\bar{\partial}$ equation with bounded solution, and in 1979 T. Wolff gave a very simple proof of the Corona in one variable. We will show in §1, in generalizing the proof of Wolff to the unit ball of $\mathbb{C}^n$, that in fact it is a $\partial\bar{\partial}$ equation we are led to solve.   Unfortunately we have not the complete answer but the following:

Theorem [1]. *Let* $f = (f_1,\ldots,f_k)$, k *functions in the Hardy class*