VECTOR ANALYSIS

with Applications to Geometry and Physics

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Preface

THIS book has been written primarily to facilitate the grasp of both the fundamental concepts and the techniques of vector analysis by the beginning student. In addition to its use as a beginning text on vector analysis, it should serve as a reference for advanced students, especially in the engineering and physical science fields. It is hoped that the numerous examples and exercises will make the book valuable to a large number of engineers for self-study.

To carry out these objectives, the authors have included examples and exercises that illustrate many applications of the theory to geometry and physics. In addition to problems of the standard types, many problems have been included which bring out more difficult aspects of the subject. They serve the dual purpose of introducing the student to a deeper appreciation of the subject and of indicating many applications which might not be apparent on a first reading.

It is the belief of the authors that vector analysis should be considered as both a mathematical discipline and a language of physics. The close relation between these two branches of science that arises naturally is not difficult to see, but the ability to think in terms of vectors is an art that requires both insight and practice. It is the intent of the authors to present the material in such a way that both objectives will result from a close study of this book. Consequently, throughout the book chapters on applications are interspersed with the mathematical theory and development. Thus the study of applications, that is, the relations among vector analysis, geometry, and physics, is carried along with the mathematical theory. In this way, insight is developed, and the many problems provide the practice.

Many topics of an advanced nature necessarily were omitted. As a mathematical text it covers the fundamentals of vector analysis. The applications to geometry and physics are limited to those with with which most readers will be expected to have some familiarity.

The first part of the book is devoted to the algebra and calculus of vectors and vector functions, and some of the simpler applications to geometry and physics are treated. Later chapters are concerned with specialized topics and more advanced types of applications. Chapters 8 to 11 discuss rather thoroughly the subjects of differential geometry, harmonic functions, and electricity and magnetism. The final chapter on linear vector functions introduces dyadics and their applications to geometry and physics.

For a three-hour, one-semester course, the first seven chapters may prove to be sufficient. Material selected from Chapters 8 to 12 may supplement them.

For ease of reference, equations and theorems are numbered by sections. A reference to a numbered equation or theorem is given by Eq. (11.18–3) or Theorem 11.12–1, for example, which means that the equation will be found in Chapter 11, Section 18, the theorem in Chapter 11, Section 12. Sections and illustrative figures are numbered by chapters.

The authors wish to express their appreciation to Professors Henry Spragens and Carl Adams of the University of Louisville for their assistance in reviewing the manuscript and their many valuable suggestions.

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Contents

CHAPTER I

Vector Algebra

1.1. Definition of Vector

In physics we usually divide quantities into two classes. Some quantities, such as length and time, having only magnitude can be specified by a single real number which is called a scalar. For other quantities, such as displacement and velocity, it is necessary to specify both the magnitudes and directions to describe them completely. Quantities having only magnitude are called scalar quantities. Quantities having both magnitude and direction are called vector quantities.

A vector quantity can be represented by a directed line segment which, if it obeys the laws for scalar multiplication and for addition as described in Sect. 1.4 and Sect. 1.5 below, will be called a vector. The magnitude of the vector quantity will be represented by the length of the vector, the direction associated with the quantity will be represented by the direction of the vector (Fig. 1.1).

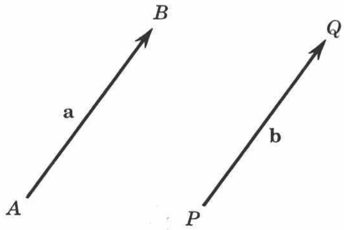

Fig. 1.1

For brevity we may refer to scalar quantities as scalars, and to vector quantities as vectors.

1.2. Equal Vectors

Two vectors are called equal if they have the same direction and length. Thus, in Fig. 1.1, the two vectors shown are equal. It is to be noted that the equality of vectors is independent of their positions in space.

1.3. Notation

We shall refer to the vectors in Fig. 1.1 as **AB** and **PQ**, the order of the letters indicating the initial point and the terminal point. They may also be indicated by a lower-case boldfaced letter, as **a**. The length of a vector (a scalar) will be denoted by the absolute value symbol; thus the length of **AB** is written $|\mathbf{AB}|$. When **a** is used to represent a vector, then the length of **a** may be written as $|\mathbf{a}|$ or simply as a. A vector of length zero will be denoted by **0** and is called the zero vector. The zero vector will be considered to have any direction. This convention will avoid the necessity of making many special statements for the exceptional cases involving the zero vector.

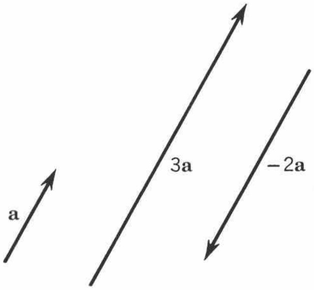

Fig. 1.2

1.4. Scalar Multiplication

The product of a scalar λ and a vector **a** (written $\lambda\mathbf{a}$) is a vector whose length is λa and whose direction is the same as that of **a** when $\lambda > 0$ and opposite to that of **a** when $\lambda < 0$. In Fig. 1.2 the vector **a** and the products 3**a** and −2**a** are shown. Clearly if **b** is a vector parallel to **a**, then **b** is a scalar multiple of **a**.

1.5. Vector Addition

The addition of vectors is performed by the "triangle" rule. The (vector) sum $\mathbf{a} + \mathbf{b}$ is a vector whose initial point is the initial point of **a** and whose terminal point is the terminal point of $\mathbf{b}'$ where $\mathbf{b}' = \mathbf{b}$, and $\mathbf{b}'$ is adjoined to **a** as shown in Fig. 1.3. From Fig. 1.4 it is evident

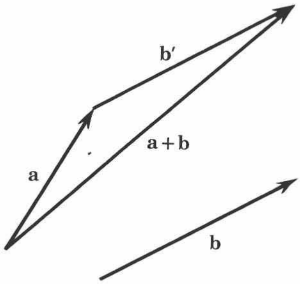

Fig. 1.3

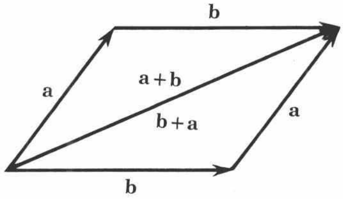

Fig. 1.4

that the vector sum $\mathbf{b} + \mathbf{a}$ equals the sum $\mathbf{a} + \mathbf{b}$. This is the commutative law of vector addition.

To add three vectors **a**, **b**, and **c**, let them be adjoined as in Fig. 1.5. It is easily seen that $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$, and we have the associative law for vector addition. Note that the three vectors need not lie in the same plane. The extension to any number of vectors is obvious.

Vectors also obey the distributive law with respect to multiplication by a scalar, that is,

$$\lambda\mathbf{a} + \lambda\mathbf{b} = \lambda(\mathbf{a} + \mathbf{b}).$$

In Fig. 1.6, $\triangle ABC$ is similar to $\triangle AB'C'$, since two corresponding sides are parallel and the angle at A is common. Then

$$\frac{|\mathbf{AC}'|}{|\mathbf{AC}|} = \frac{|\mathbf{AB}'|}{|\mathbf{AB}|} = \lambda\frac{|\mathbf{b}|}{|\mathbf{b}|} = \lambda.$$

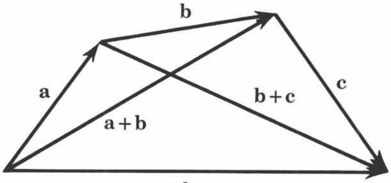

Fig. 1.5

Since $\mathbf{AC} = \mathbf{a} + \mathbf{b}$,

$$\frac{|\mathbf{AC}'|}{|\mathbf{AC}|} = \frac{|\mathbf{AC}'|}{|\mathbf{a} + \mathbf{b}|} = \lambda,$$

or

$$|\mathbf{AC}'| = \lambda(|\mathbf{a} + \mathbf{b}|).$$

The vectors $\mathbf{AC}'$ and $\mathbf{AC}$ have the same direction; hence,

$$\mathbf{AC}' = \lambda(\mathbf{a} + \mathbf{b}).$$

But

$$\mathbf{AC}' = \lambda\mathbf{b} + \mathbf{B}'\mathbf{C}' = \lambda\mathbf{b} + \lambda\mathbf{a}.$$

Therefore,

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}.$$

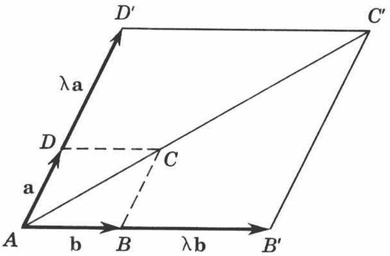

Fig. 1.6

The preceding discussion shows that vectors obey three familiar arithmetic laws, namely, the commutative and associative laws of addition, and the distributive law for scalar multiplication.

In the addition of vectors we made use of the fact that a vector can be replaced by an equal vector. If vectors **a** and **b** lie in the same plane, then we shall consider any vector $\mathbf{b}' = \mathbf{b}$ to be coplanar with **a**. It is a trivial consequence that any two vectors are coplanar. However, three or more vectors may be non-coplanar.

1.6. Linear Functions

Consider a vector **c** (Fig. 1.7). Any two vectors **a** and **b** whose vector sum is **c** are called vector components of **c**. Thus in Fig. 1.7 **a** and **b** are vector components of **c**. Also **a**′ and **b**′ are vector components of **c**.

The most important case of vector components is that in which the components are perpendicular. Thus in Fig. 1.8 **a** and **b** are perpendicular (plane) components of **c**. Then $c^2 = a^2 + b^2$. If **u** is a vector parallel to **a**, and **v** is a vector parallel to **b**, then

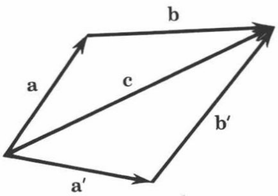

Fig. 1.7

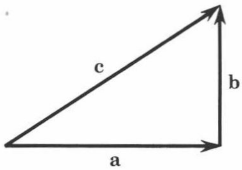

Fig. 1.8

$$\mathbf{a} = \lambda \mathbf{u}, \qquad \mathbf{b} = \mu \mathbf{v}$$

for proper choices of λ and μ. Thus

$$\mathbf{c} = \lambda \mathbf{u} + \mu \mathbf{v}.$$

This expresses **c** as a linear combination (or linear function) of **u** and **v**. In particular if **u** and **v** are unit vectors, that is, they have length 1, then the length of **c** is

$$c = \sqrt{\lambda^2 + \mu^2}.$$

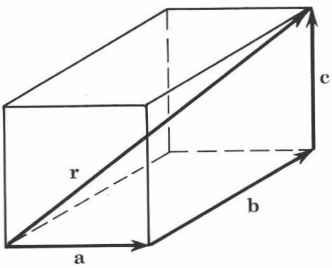

Fig. 1.9

The above may be extended to three components in space. Thus in Fig. 1.9

$$\mathbf{r} = \mathbf{a} + \mathbf{b} + \mathbf{c},$$

and if **a**, **b**, and **c** are pairwise perpendicular, they constitute three perpendicular components of **r**. If $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are unit vectors parallel to **a**, **b**, and **c**, respectively, then

$$\mathbf{a} = a\mathbf{u}_1, \qquad \mathbf{b} = b\mathbf{u}_2, \qquad \mathbf{c} = c\mathbf{u}_3,$$

and r, the length of **r**, is the diagonal of the parallelepiped. Hence

$$|\mathbf{r}| = r = \sqrt{a^2 + b^2 + c^2},$$

and **r** may be expressed as a linear combination of these unit vectors. Thus

$$\mathbf{r} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3.$$

1.7. Linear Dependence

Let **a**, **b**, and **c** be any vectors. If there are three scalars λ, μ, and ν at least one of which is not zero, and

$$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = \mathbf{0}, \tag{1.7-1}$$

then we say that **a**, **b**, and **c** are linearly dependent vectors. On the other hand, if λ, μ, and ν are all zero whenever Eq. (1.7-1) holds, then we say that the vectors are linearly independent.

The above statements may be extended to any number of vectors. For example, the vectors **a**, **a** − **b**, **b**, and **b** + **c** are linearly dependent since

$$1\mathbf{a} - 1(\mathbf{a} - \mathbf{b}) - 1\mathbf{b} + 0(\mathbf{b} + \mathbf{c}) = \mathbf{0},$$

and not all the coefficients are zero. *Note.* $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

EXAMPLES

1. *ABCD* is a parallelogram. Let **AB** = **a**, **AD** = **d** (Fig. 1.10). Express **BC**, **CD**, **CA**, and **BD** in terms of **a** and **d**.

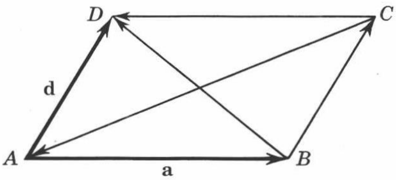

Fig. 1.10

Solution. Since **BC** is parallel to **AD** and oriented in the same direction, **BC** = **AD** = **d**. **CD** is parallel to **AB** but has an opposite orientation; hence, **CD** = −**AB** = −**a**. Now, **BC** = **d**, and it follows that **AC** = **a** + **d**. Hence **CA** = −**AC** = −**a** − **d**. Also, **AB** + **BD** = **AD**, whence **BD** = **AD** − **AB** = **d** − **a**.

2. If $\lambda\mathbf{a} + \mu\mathbf{b} = \mathbf{0}$ but not both λ and μ are zero, then **a** is parallel to **b**.

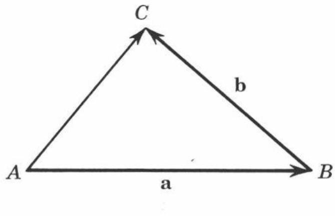

Fig. 1.11

Proof. If $\lambda \neq 0$, then $\mathbf{a} = (-\mu/\lambda)\,\mathbf{b}$; that is, **a** is a scalar multiple of **b** and is, therefore, parallel to **b**. *Note.* The statement may be made that "if **a** and **b** are linearly dependent, then they are parallel."

3. Show: (*a*) $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$;
(*b*) $|\mathbf{a} - \mathbf{b}| \geqslant \big||\mathbf{a}| - |\mathbf{b}|\big|$.

Proof of (*a*). In $\triangle ABC$ (Fig. 1.11) let **AB** = **a**, **BC** = **b**. Then **a** + **b** = **AC**. From plane geometry the length of one side of a triangle is less than or equal to the sum of the other two sides. Thus

$$|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|.$$

Proof of (b). We have

$$|\mathbf{a}| = |(\mathbf{a} - \mathbf{b}) + \mathbf{b}|.$$

From part (*a*)

$$|\mathbf{a} - \mathbf{b}| + |\mathbf{b}| \geqslant |\mathbf{a}|,$$

or

$$|\mathbf{a} - \mathbf{b}| \geqslant |\mathbf{a}| - |\mathbf{b}|.$$

If $|\mathbf{a}| > |\mathbf{b}|$, then $|\mathbf{a}| - |\mathbf{b}| = ||\mathbf{a}| - |\mathbf{b}||$ and the theorem follows. If $|\mathbf{b}| > |\mathbf{a}|$, the theorem may be proved by simply interchanging **a** and **b**.

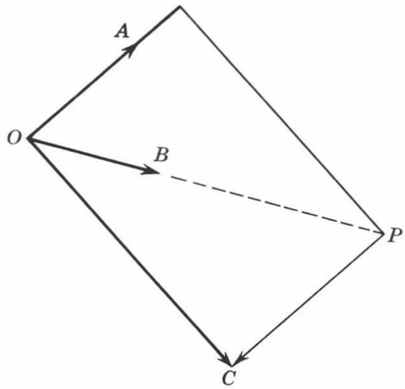

Fig. 1.12

4. If **OA** and **OB** are not parallel and not **0**, any vector **OC** in the plane of **OA** and **OB** can be expressed in the form

$$\lambda\mathbf{OA} + \mu\mathbf{OB}.$$

Proof. Through C draw a line parallel to **OA** intersecting the line OB at P (Fig. 1.12). Now $\mathbf{OP} = \mu\mathbf{OB}$ and $\mathbf{PC} = \lambda\mathbf{OA}$. Since $\mathbf{OC} = \mathbf{OP} + \mathbf{PC}$ we have $\mathbf{OC} = \mu\mathbf{OB} + \lambda\mathbf{OA}$.

5. If **a** and **b** are any two non-parallel non-zero vectors, then any vector **c** coplanar with both **a** and **b** can be written in the form

$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}.$$

Proof. If **a** and **b** issue from the same point, then Example 4 applies. If not, we may use a vector equal to **b** having the same initial point as **a**, and then use the results of Example 4.

6. If $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, then **c** is coplanar with **a** and **b**.

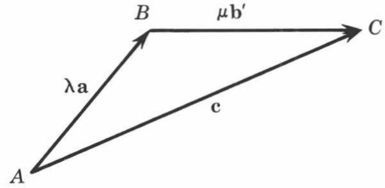

Fig. 1.13

Proof. We may use a vector $\mathbf{b}'$ equal to **b**, where $\mathbf{b}'$ has its initial point at the terminal point of **a**. Then $\lambda\mathbf{a} + \mu\mathbf{b} = \lambda\mathbf{a} + \mu\mathbf{b}' = \mathbf{c}$. From Fig. 1.13, **c**, or a vector equal to **c**, has two points in the plane of A, B, C, namely, A and C. Therefore **c** lies in the plane of $\lambda\mathbf{a}$ and $\mu\mathbf{b}'$. Hence, **c** is coplanar with **a** and **b**.

7. If **a** and **b** are not parallel and not zero vectors, then any vector **c** coplanar with **a** and **b** can be represented in one and only one way in the form

$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}.$$

Proof. From Example 4 it is possible to write $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$. Suppose now that there is a second such expression for $\mathbf{c}$, say $\mathbf{c} = \lambda'\mathbf{a} + \mu'\mathbf{b}$. Then

$$(\lambda - \lambda')\mathbf{a} + (\mu - \mu')\mathbf{b} = \mathbf{0}.$$

Now if either $\lambda - \lambda' \neq 0$ or $\mu - \mu' \neq 0$, then, by Example 2, $\mathbf{a}$ and $\mathbf{b}$ are parallel, contrary to hypothesis. Therefore, either $\lambda - \lambda' = 0$ or $\mu - \mu' = 0$. Suppose $\lambda - \lambda' = 0$. Then $(\mu - \mu')\mathbf{b} = \mathbf{0}$, and since $\mathbf{b} \neq \mathbf{0}$, it follows that $\mu - \mu' = 0$, or $\mu = \mu'$. Thus $\lambda = \lambda'$ and $\mu = \mu'$, and the representation of $\mathbf{c}$ is unique.

8. Let $\mathbf{a} = \mathbf{OA}$, $\mathbf{b} = \mathbf{OB}$, $\mathbf{c} = \mathbf{OC}$ (Fig. 1.14). If C lies on the line AB, then

$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b},$$

with

$$\lambda + \mu = 1.$$

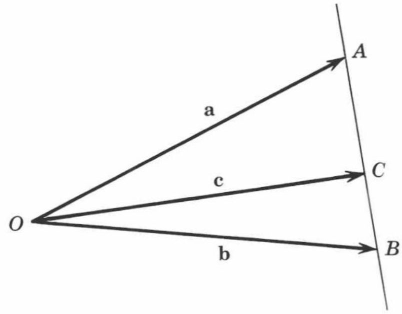

Fig. 1.14

Proof. $\mathbf{AB} + \mathbf{a} = \mathbf{b}$

or

$$\mathbf{AB} = \mathbf{b} - \mathbf{a},$$

and

$$\mathbf{AC} = \mathbf{c} - \mathbf{a}.$$

Since $\mathbf{AC}$ and $\mathbf{AB}$ are parallel, $\mathbf{AC} = \mu\mathbf{AB}$ for some μ; that is,

$$\mathbf{c} - \mathbf{a} = \mu(\mathbf{b} - \mathbf{a}).$$

On simplification, we have

$$\mathbf{c} = \mu\mathbf{b} + (1 - \mu)\mathbf{a}.$$

Let $1 - \mu = \lambda$. Then $\lambda + \mu = 1$ and

$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}.$$

Note.

$$\frac{|\mathbf{AC}|}{|\mathbf{CB}|} = \frac{\mu}{\lambda}.$$

Now

$$\frac{|\mathbf{AC}|}{|\mathbf{AB}|} = \mu \quad \text{and} \quad \mu = 1 - \lambda,$$

$$\frac{|\mathbf{AC}|}{|\mathbf{AC} + \mathbf{CB}|} = \mu$$

or, since $\mathbf{AC}$ and $\mathbf{AC} + \mathbf{CB}$ are in the same direction,

$$\mathbf{AC} = \mu\mathbf{AC} + \mu\mathbf{CB},$$

$$(1 - \mu)\mathbf{AC} = \mu\mathbf{CB}.$$

But

$$\lambda = 1 - \mu.$$

Therefore,

$$\lambda\mathbf{AC} = \mu\mathbf{CB}.$$

9. If **a**, **b**, and **c** have the same initial point and $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ with $\lambda + \mu = 1$, then the terminal points are collinear.

Proof. Since $\mu = 1 - \lambda$,

$$\begin{aligned} \mathbf{c} &= \lambda\mathbf{a} + (1 - \lambda)\mathbf{b} \\ &= \lambda(\mathbf{a} - \mathbf{b}) + \mathbf{b}, \\ \mathbf{c} - \mathbf{b} &= \lambda(\mathbf{a} - \mathbf{b}). \end{aligned}$$

Refer to Fig. 1.14. We see that

$$\begin{aligned} \mathbf{c} - \mathbf{b} &= \mathbf{BC}, \\ \mathbf{a} - \mathbf{b} &= \mathbf{BA}. \end{aligned}$$

Thus

$$\mathbf{BC} = \lambda\mathbf{BA},$$

and the points A, B, C are collinear.

10. Let $\mathbf{a} = \mathbf{OA}$, $\mathbf{b} = \mathbf{OB}$, $\mathbf{c} = \mathbf{OC}$, $\mathbf{d} = \mathbf{OD}$ with **a**, **b**, and **c** non-coplanar (Fig. 1.15); that is, the three vectors are not parallel to the same plane. If D lies in the plane of A, B, and C, then

$$\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} \text{ with } \lambda + \mu + \nu = 1.$$

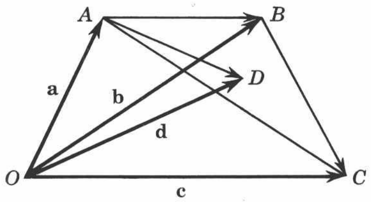

Fig. 1.15

Proof.

$$\begin{aligned} \mathbf{AB} &= \mathbf{b} - \mathbf{a}, \\ \mathbf{BC} &= \mathbf{c} - \mathbf{b}. \end{aligned}$$

Since **AD** lies in the plane of A, B, C it can be expressed as a linear combination of **AB** and **BC** (see Example 5). Then

$$\lambda'(\mathbf{b} - \mathbf{a}) + \mu'(\mathbf{c} - \mathbf{b}) = \mathbf{AD} = \mathbf{d} - \mathbf{a},$$

from which

$$\mathbf{d} = \mathbf{a}(1 - \lambda') + \mathbf{b}(\lambda' - \mu') + \mu'\mathbf{c}.$$

Set $1 - \lambda' = \lambda, \lambda' - \mu' = \mu, \mu' = \nu$. Then

$$\lambda + \mu + \nu = 1$$

and

$$\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}.$$

11. If **a**, **b**, **c**, and **d** have a common initial point O, and $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = \mathbf{d}$ with $\lambda + \mu + \nu = 1$, then the terminal points of **a**, **b**, **c**, and **d** are coplanar.

Proof. Since $\lambda + \mu + \nu = 1$, we may write $\nu = 1 - \lambda - \mu$. From Fig. 1.15, $\mathbf{AB} = \mathbf{b} - \mathbf{a}$, $\mathbf{BC} = \mathbf{c} - \mathbf{b}$, and $\mathbf{AD} = \mathbf{d} - \mathbf{a}$. Now

$$\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + (1 - \lambda - \mu)\mathbf{c},$$

and subtracting **a** from each side, we have

$$\begin{aligned}\mathbf{d} - \mathbf{a} &= (\lambda - 1)\mathbf{a} + \mu(\mathbf{b} - \mathbf{c}) + (1 - \lambda)\mathbf{c}\\ &= (1 - \lambda)(-\mathbf{a}) + (1 - \lambda)\mathbf{c} + \mu(\mathbf{b} - \mathbf{c}).\end{aligned}$$

Combining, we get

$$\mathbf{d} - \mathbf{a} = (1 - \lambda)(\mathbf{c} - \mathbf{a}) + \mu(\mathbf{b} - \mathbf{c}).$$

This says that $\mathbf{d} - \mathbf{a}$ is a linear combination of **AC** and **BC**, and therefore **AD** lies in the plane of the points A, B, and C.

12. Prove the following.

(1) If a subset of n vectors is linearly dependent, then the vectors are linearly dependent.

(2) If n vectors are linearly independent, then any subset of these n vectors is linearly independent.

(3) If n vectors are linearly dependent, $n > 1$, then at least one of the vectors is a linear combination of the remaining vectors.

(4) If n vectors are linearly independent, but $n + 1$ vectors are linearly dependent, then the $(n + 1)$st vector is a linear combination of the other n.

Proofs.

(1) Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_m$ constitute the subset. Then there are c_i $(i = 1, 2, \cdots, m)$ not all zero such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_m\mathbf{a}_m = \mathbf{0}.$$

Then

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_m\mathbf{a}_m + 0\,\mathbf{a}_{m+1} + \cdots + 0\,\mathbf{a}_n = \mathbf{0}.$$

Since not all the c's are zero, the n vectors are linearly dependent.

(2) If any subset of the n vectors were not linearly independent, then it would be possible to find c_i's not all different from zero, and proceeding as in (1), we could show that the n vectors are linearly dependent, contrary to hypothesis.

(3) Since n vectors are linearly dependent, we can write

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

where not all the c's are zero. Suppose $c_i \neq 0$. Then

$$\mathbf{a}_i = -\left[\frac{c_1}{c_i}\,\mathbf{a}_1 + \frac{c_2}{c_i}\,\mathbf{a}_2 + \cdots + \frac{c_n}{c_i}\,\mathbf{a}_n\right].$$

(4) Let $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ be n linearly independent vectors and $\mathbf{b}$ be the $(n + 1)$st vector. Then

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n + c_{n+1}\mathbf{b} = \mathbf{0}$$

and not all c's are zero. If $c_{n+1} = 0$, then

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

and not all these c's are zero which is contrary to the hypothesis that the n vectors are linearly independent. Therefore $c_{n+1} \neq 0$, and

$$\mathbf{b} = -\left[\frac{c_1}{c_{n+1}}\mathbf{a}_1 + \frac{c_2}{c_{n+1}}\mathbf{a}_2 + \cdots + \frac{c_n}{c_{n+1}}\mathbf{a}_n\right].$$

13. In a plane at most two vectors can be linearly independent.

Proof. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three coplanar vectors. Suppose $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Then $\mathbf{a}$ is not parallel to $\mathbf{b}$ since then $\mathbf{a}$ would be a scalar multiple of $\mathbf{b}$ and they would be linearly dependent. By Example 5 we may write

$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}.$$

Then

$$\lambda\mathbf{a} + \mu\mathbf{b} - 1\mathbf{c} = \mathbf{0}$$

with not all coefficients zero. Hence the vectors are linearly dependent.

14. If $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = \lambda\mathbf{m} + \mu\mathbf{n} + \nu\mathbf{p}$, and not all of λ, μ, ν are zero, then the vectors $\mathbf{a} - \mathbf{m}$, $\mathbf{b} - \mathbf{n}$, $\mathbf{c} - \mathbf{p}$ are linearly dependent.

Proof. Since $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = \lambda\mathbf{m} + \mu\mathbf{n} + \nu\mathbf{p}$, then

$$\lambda(\mathbf{a} - \mathbf{m}) + \mu(\mathbf{b} - \mathbf{n}) + \nu(\mathbf{c} - \mathbf{p}) = \mathbf{0}.$$

Since by hypothesis not all of λ, μ, ν are zero, it follows that these vectors are linearly dependent.

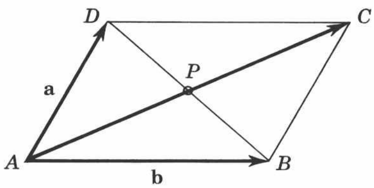

Fig. 1.16

15. Show that the diagonals of a parallelogram bisect each other.

Proof. In Fig. 1.16, P lies on BD and therefore,

$$\mathbf{AP} = \lambda\mathbf{a} + \mu\mathbf{b}$$

with

$$\lambda + \mu = 1.$$

Also

$$\mathbf{AP} = \nu\mathbf{AC} = \nu(\mathbf{a} + \mathbf{b});$$

therefore,

$$\lambda\mathbf{a} + \mu\mathbf{b} = \nu(\mathbf{a} + \mathbf{b})$$